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Spectral relationships of the integral equation with logarithmic kernel in some different domains

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Abstract

In this work, the Fredholm integral equation (**FIE**) with logarithmic kernel is investigated from the contact problem in the plane theory of elasticity. Then, using potential theory method (**PTM**), the spectral relationships (**SRs**) of this integral equation are obtained in some different domains of the contact. Many special cases and new **SRs** are established and discussed from this work.

Keywords: Potential theory method, spectral relationships, Chebyshev polynomial (**CP**), elliptic integral.

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INTRODUCTION

Many problems in the half-plane of elasticity, deformation in physics and engineering are reduced to an integral equation of the first kind; see Popov [1], Covalence [2], and Alexandrov [3]. In other words, different methods are established for solving the FIE **with discontinuous kernel. These methods are: Cauchy method [4], potential theory method [5, 6], orthogonal polynomial methods [7, 8], Fourier transformation method [9, 10] and Krein's method [11, 12]. The** FIE **with logarithmic kernel is investigated from the contact problem in two dimensional problem, in the theory of elasticity from an infinite strip; occupying the region $0 \le y \le h$ made of material satisfies the stress relations, lies without fraction on a rigid support. A rigid rectangular stamp is impressed into the boundary of the strip $y = h$ by constant force p whose eccentricity of application is e . Then, using Airy function and Fourier integral forms, Abdou et al. in [12], represented the plane contact problem, in a half-plane, as a** FIE **of the first kind with logarithmic kernel.**

Here, the **PTM** is used to obtain a boundary value problem (**BVP**) in two dimensional domain. Moreover the properties of Chebyshev polynomials of the first, second kind and the elliptic Jacobi functions are used. Then, by considering the equivalence condition between the differential equation and the integral equation we obtain many different **SRs** inside and outside the domain of contact (domain of integration). The importance of using spectral relationships in contact problems in the theory of elasticity can be found in Refs. [13-17].

1. Fredholm integral equation:

Consider the following **FIE**:

$$\int_{-1}^{1} k\left(\frac{x-\xi}{\lambda}\right) q(\xi)\, d\xi = \pi\theta\big(\delta + \alpha x - f(x)\big) \tag{1}$$

where the kernel is defined by

$$k\left(\frac{x-\xi}{\lambda}\right) = \int_{0}^{\infty}\left(\frac{L(\omega)}{\omega}\right)\cos\left(\frac{x-\xi}{\lambda}\omega\right) d\omega, \quad L(\omega) = \frac{\cosh 2\omega + 1}{2\omega + \sinh 2\omega} \tag{2}$$

Here, x, ξ, ω are the dimensionless variables; while $\lambda = \dfrac{h}{a}$ is a dimensionless parameter characterizing the strip thickness. It should be noted that as $\lambda \to \infty$, the integral equation takes the form:

$$\int_{-1}^{1} q(y)\left[\ln\frac{1}{|x-y|} + d\right] dy = \pi g(x), \qquad \left(g(x) = \pi\theta\big(\delta + \alpha x - f(x)\big); d = \ln\frac{2}{a}\right) \tag{3}$$

In the remainder part of this paper, we will obtain the solution of the **FIE** (3) in the form of **SRs** in different domains and discuss it.

2. Solution of integral equation

Here, we use the **PTM** to obtain the **SRs** for the **FIE** of the first kind in different domains. For this purpose, consider the integral operator

$$K\phi = \int_{L}\left[\ln\frac{1}{|x-t|} + d\right]\phi(t)\, dt = f(x) \tag{4}$$

under the condition

$$\int_{L}\phi(t)\, dt = P \tag{5}$$

where, we will consider the following cases:

$$\begin{aligned} &(1) L = \{(x,y) \in L : |x| \le a, y = 0\}, \quad (2) L = \{(x,y) \in L : |x| \ge a, y = 0\} \\ &(3) L = \{(x,y) \in L : b \le |x| \le a, y = 0\} \end{aligned} \tag{6}$$

2.1. The first case (1) of equation (6):

Using the potential theory method, see Abdou et al. [18], we have

$$
\begin{aligned}
&\Delta W(x,y)=0 \quad ,\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2} \quad ,((x,y)\notin\{L\})\\
&W(x,y)\big|_{y=0}=f(x)+P(-\ln|x|+d) \qquad ,(x\in\{L\}) \qquad (7)\\
&W(x,y)=0 \quad as\ r\to\infty,\ r=\sqrt{x^2+y^2}
\end{aligned}
$$

with the equivalence condition

$$
-\pi\phi(x)=\lim_{y\to 0}\left\{\operatorname{sgn} y.\left[\frac{\partial W(x,y)}{\partial y}-\pi P\delta(x)\right]\right\} \quad ,(x\in\{L\}). \qquad (8)
$$

where $\delta(x)$ is the Dirac-delta function.

To obtain the solution of (7), we use the transformation mapping function:

$$
z=\frac{a}{2}\omega(\zeta)=\frac{a}{2}\left(\zeta+\zeta^{-1}\right), \qquad \zeta=\xi+i\eta=\rho e^{i\theta} \qquad (9)
$$

A useful method in engineering mathematics is using a conformal mapping to transform complicated region into a simpler one, for this reason, we use equation (9). The transformation (9) maps the region in $x-y$ plane into the region outside the unit circle γ, such that $\omega'(\zeta)$ does not vanish or becomes infinite outside the unit circle γ. The parametric equations of (9) are

$$
x=\frac{a}{2}\left(\rho+\frac{1}{\rho}\right)\cos\theta=\frac{a\xi}{2}\left(1+\frac{1}{\xi^2+\eta^2}\right), \quad y=\frac{a}{2}\left(\rho-\frac{1}{\rho}\right)\sin\theta=\frac{a\eta}{2}\left(1-\frac{1}{\xi^2+\eta^2}\right) \qquad (10)
$$

Using the transformation (9), we get

$$
\frac{1}{r}=\frac{2\rho}{a}\left(\rho^4+2\rho^2\cos 2\theta+1\right)^{-\frac{1}{2}}, \quad s.t.\ \frac{1}{r}=\frac{2\rho}{a}\ as\ \rho\to 0 \qquad (11)
$$

The mapping (9) maps the upper and lower of the interval $(x,y)\in[-a,a]$ into the lower and the upper of the semicircle $\rho=1$, respectively. Moreover the point $z=\infty$ will be mapped onto the point $\zeta=0$.

Using the transformation (9) in the **BVP** (7) we have

$$
\begin{aligned}
&\Delta W(\rho,\theta)=0 \quad ,\Delta=\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\theta^2} \qquad (\rho\le 1,-\pi\le\theta\le\pi)\\
&W(0,\theta)=0 \qquad (\rho=0) \qquad (12)\\
&W(1,\theta)=f_0(\theta)-P\left(\ln\left(\frac{2}{a}+d\right)\right) \qquad ,f_0(\theta)=f(a\cos\theta) \qquad ;(\rho=1)
\end{aligned}
$$

Also, the equivalence condition becomes

$$
\phi(a\cos\theta)=[\pi a\sin\theta]^{-1}\left[P+\frac{\partial W}{\partial\rho}\right]_{\rho=1} \qquad (13)
$$

Here, we assume

$$W(\rho,\theta)=W\left(\frac{a}{2}\left(\rho+\frac{1}{\rho}\right)\cos\theta,\frac{a}{2}\left(\rho-\frac{1}{\rho}\right)\sin\theta\right), \tag{14}$$

to solve the **BVP** of (12), we assume

$$W(\rho,\theta)=\sum_{n=0}^{\infty}X_n(\rho)\cos n\theta \tag{15}$$

Differentiating (15) with respect to ρ and θ, then introducing the result to satisfy the first equation of (12), and noting that, when $r\to\infty$, we have $\rho\to 0$. Therefore, the solution of the formula (12) can be adapted in the form

$$W(\rho,\theta)=\sum_{n=0}^{\infty}A_n\rho^n\cos n\theta \tag{16}$$

Where, with the aid of the second and third formulas of (12), we have

$$A_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}f_0(\theta)d\theta,\quad A_n=\frac{1}{2\pi}\int_{-\pi}^{\pi}f_0(\theta)\cos n\theta d\theta \tag{17}$$

Using the previous results in (13), we obtain

$$\phi(a\cos\theta)=\begin{cases}(\pi a\sin\theta)^{-1} & n=0,\quad 0<\theta<\pi\\ n\cos n\theta.(\pi a\sin\theta)^{-1} & n=1,2,\ldots\end{cases} \tag{18}$$

Introducing (18) into (4), and noting the definition of **CP** of the first kind, $T_n\left(\frac{x}{a}\right)=\cos n\theta$, we obtain

$$\int_{-a}^{a}\left[\ln\frac{1}{|x-t|}+d\right]\frac{T_n\left(\frac{t}{a}\right)}{\sqrt{a^2-t^2}}dt=\begin{cases}\pi\left(\ln\left(\frac{2}{a}+d\right)\right), & n=0\\ \frac{\pi}{n}T_n\left(\frac{x}{a}\right), & n\ge 1,|x|\le a.\end{cases} \tag{19}$$

The formula (19) represents the **SRs** for the **FIE** of the first kind with logarithmic kernel, where we assume the known function, in (4), $f(x)=T_n\left(\frac{x}{a}\right)$.

2.2. The second case (2) of equation (6):

In order to obtain the **SRs** of (19) for $|x|>a$, we obtain from (10) that $x>a$ for $\theta=0$ and $x<a$ for $\theta=\pi$, therefore, we have

$$\rho=\frac{|x|-\sqrt{x^2-a^2}}{a},\qquad (|x|>a) \tag{20}$$

Solving the **BVP** (12), under the condition (20), then using (17) we have the following **SRs**:

$$\frac{1}{\pi}\int_{-a}^{a}\left[\ln\frac{1}{|x-t|}+d\right]\frac{T_n\left(\frac{t}{a}\right)}{\sqrt{a^2-t^2}}dt=\begin{cases}\ln\left[2\frac{|x|-\sqrt{x^2-a^2}}{a^2}\right]+d & n=0\quad ,|x|>a\\ \frac{1}{n}\left[H(x)+(-1)^nH(-x)\right].\left[\frac{|x|-\sqrt{x^2-a^2}}{a^2}\right]^n & n=1,2,\ldots\end{cases} \tag{21}$$

Where, $H(x)$ is the Heaviside function, see Whittaker et al. [19].

The conformal mapping (9), for $|x| \geq a$ maps the half plane, $y > 0$, into the semi-circle $\{\rho > 1, 0 < \theta < \pi\}$, and the half plane $y < 0$ into $\{\rho < 1, 0 < \theta < \pi\}$. Hence, we deduce that, outside the interval, the case for $x > a$ $(x < -a)$ are corresponding in ζ -plane to $\zeta > 1, (\zeta < -1)$, while inside the interval L the case for $x > a$ $(x < -a)$ is corresponding to $0 < \zeta < 1, (-1 < \zeta < 0)$. Therefore, the solution of the **BVP** (7) for $\eta > 0$ takes the form

$$W_0(\xi,\eta) = e^{-\lambda\eta}\begin{cases}\cos\lambda\xi \\ \sin\lambda\xi \qquad (\lambda > 0),\end{cases}$$
$$W_0(\xi,\eta) = W\left(\frac{a\xi}{2}\left(1 + \frac{1}{\xi^2+\eta^2}\right), \frac{a\xi}{2}\left(1 - \frac{1}{\xi^2-\eta^2}\right)\right), \quad (-\infty < \xi < \infty, \eta > 0). \tag{22}$$

Moreover, the term of equivalence relation of (8) $\frac{\partial W}{\partial y}$ will take the form

$$\lim_{y\to 0}\frac{\partial W}{\partial y} = \frac{2}{a}\frac{\xi^2}{|\xi^2 - 1|}\lim_{\eta\to 0}\frac{\partial W_0}{\partial \eta}, \qquad \left(x = \frac{a}{2}\left(\zeta + \frac{1}{\zeta}\right)\right) \tag{23}$$

Using the separation of variable method, with the aid of (22), in the **BVP** (7) and the equivalence relation (8), we can obtain

$$\frac{2}{\pi}\left(\int_{-\infty}^{-a} + \int_{a}^{\infty}\right)\ln\left|\frac{x}{x-t}\right|\omega_{\pm}(t).\begin{cases}\cos(\lambda\rho_{\pm}(t)) \\ t\sin(\lambda\rho_{\pm}(t))\end{cases}dt = \begin{cases}\cos(\lambda\rho_{\pm}(x)) \\ \operatorname{sgn}x\,\sin(\lambda\rho_{\pm}(x))\end{cases} \tag{24}$$

where

$$\omega_{\pm}(x) = \frac{|x| \pm \sqrt{x^2 - a^2}}{2\sqrt{x^2 - a^2}}, \quad \rho_{\pm}(x) = \frac{|x| \pm \sqrt{x^2 - a^2}}{a} \qquad (\lambda > 0, |x| > a) \tag{25}$$

The formula (24) leads us to assume the general integral operator

$$K\phi = \frac{2}{\pi}\left(\int_{-\infty}^{-a} + \int_{a}^{\infty}\right)\ln\left|\frac{x}{x-t}\right|\phi_{\pm}(t)dt \tag{26}$$

Hence, with the aid of (26), and for the interval $L = \{y = 0, -a < x < a\}$ in z - plane, which transformed into the semi-circle $\{\rho = 1, 0 < \theta < \pi\}$ in ζ - plane, we have the following **SRs**:

$$\frac{2}{\pi}\left(\int_{-\infty}^{-a} + \int_{a}^{\infty}\ln\left|\frac{x}{x-t}\right|\omega_{\pm}(t).\begin{cases}\cos(\lambda\rho_{\pm}(t)) \\ t\sin(\lambda\rho_{\pm}(t))\end{cases}dt\right) = \frac{a}{\lambda}e^{-\frac{\lambda}{a}\sqrt{a^2-x^2}}\begin{cases}\cos\left(\frac{\lambda x}{a}\right) \\ \sin\left(\frac{\lambda x}{a}\right), (-a < x < a, \lambda > 0).\end{cases} \tag{27}$$

2.3. The third case (3) of equation (6):

When $b \leq x \leq a$, we will seek the solution of the **BVP** (7) by using the transformation mapping

$$z = sn\left(\frac{K'}{\pi}\ln\zeta, k\right), \qquad \left(\zeta = \rho e^{i\theta}, \zeta = \xi + i\eta\right) \tag{28}$$

The transformation mapping (28), $z = x + iy$, is called the Jacobi elliptic transformation. The three basic functions of the elliptic functions are denoted $cn(u,k), du(u,k)$ and $sn(u,k)$ where k is known as the elliptic modulus. They arise from the inversion of the elliptic integral of the first kind

$$u = F(\phi,k) = \int_0^{\phi} \frac{dt}{\sqrt{1-k^2\sin^2 t}}, \qquad (0<k^2<1) \tag{29}$$

Where, $k = \mathrm{mod}\ u$ is the elliptic modulus and $\phi = am(u,k)$ is the Jacobi amplitude. The Jacobi elliptic function are periodic in $K(k)$ and $K^{/}(k)$, where $K(k)$ is the complete elliptic integral of the first kind, $K^{/}(k) = K(k^{/})$, and $k^{/} = \sqrt{1-k^2}$, is the complementary elliptic modulus (see Whittaker et al. [19]), where

$$K = K(k) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}, \qquad \left(k = \frac{b}{a}\right) \tag{30}$$

Also, the Jacobi elliptic integral of the first kind is defined

$$K^{/} = K(k^{/}) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^{/2}t^2)}} = \int_0^{\frac{1}{k}} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}, \quad \left(k^{/} = \sqrt{1-k^2}\right) \tag{31}$$

The transformation mapping (28) maps the region of the interval $(b \le |x| \le a, y = 0)$ in z -plane, $z = x + iy$, into the closed ring $\rho_0 \le \rho \le \rho_0^{-1}$, $\rho_0 \exp\left(-\pi \frac{k}{k^{/}}\right)$ in ζ – plane.

Moreover, the transformation mapping (28) maps the region outside the plane $\mathrm{Im} z > 0$ to the region outside the semi-ring $\{\rho_0 \le \rho \le \rho_0^{-1}, 0 \le \theta \le \pi\}$, or inside the semi-ring $\{\rho_0 \le \rho \le \rho_0^{-1}, -\pi < \theta \le 0\}$. Also, the point $z = \infty$, will be mapped to $\zeta = -1$ in ζ –plane. Moreover, the points outside the interval $[b,a]$ will be mapped outside the ring $\rho = \rho_0^{-1}$ while outside the interval $[-a,-b]$ will be mapped inside the ring $\rho = \rho_0$.

Assume,

$$\omega = u + iv = \frac{K^{/}}{\pi}\ln\zeta, \qquad \{-K \le u \le K, -K^{/} \le v \le K^{/}\} \tag{32}$$

Hence, we have

$$u = \frac{K^{/}}{\pi}\ln\rho, \qquad v = \frac{K^{/}}{\pi}\theta \tag{33}$$

Using (32) and (33) in (28), with the help of properties of elliptic function, we can have the parametric equations

$$x = b\frac{sn\,u\ cn(iv)\,dn(iv)}{1-k^2 sn^2 u\, sn^2(iv)}, \quad y = -ib\frac{cn\,u\ dn(iv)\,sn(iv)}{1-k^2 sn^2 u\, sn^2(iv)} \tag{34}$$

The linear coordinate $u = -K$ will cover the interval $[-a,-b]$, while $u = K$ covers the interval $[a,b]$. For this, we have $sn(K,k) = 1$, and the first formula of (34), after using the properties of the elliptic functions, see Whittaker et al.[19], takes the form:

$$x = b\left[dn(v,k^{/})\right]^{-1}, \qquad (-K^{/} \le \theta \le K^{/}) \tag{35}$$

Also, the formula (35), with the properties of dn , can be adapted in the form

$$v=\int_1^{\frac{x}{b}}\frac{dt}{\sqrt{\left(t^2-1\right)\left(1-k^2t^2\right)}},\qquad \left(b\le x\le a\right) \tag{36}$$

The formula (36) is hold only for $b\le x\le a$ and when x is changed from b to a, v will be changed from 0 to K'. For this, we define

$$f_1(\theta)=f\left\{-b\left[dn\left(\frac{K'\theta}{\pi},k'\right)\right]^{-1}\right\},\quad f_2(\theta)=f\left\{b\left[dn\left(\frac{K'\theta}{\pi},k'\right)\right]^{-1}\right\},\quad \left(-\pi\le\theta\le\pi\right) \tag{37}$$

Where, $f_1(\theta)$ is defined inside $\rho=\rho_0$ and $f_2(\theta)$ outside $\rho=\rho_0^{-1}$.

After the above discussion, the **BVP** (7) can be modified as

$$\begin{gathered}\frac{\partial^2V}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial V}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2V}{\partial\theta^2}=0,\quad \left(\rho_0<\rho<\rho_0^{-1}\right),V\left(\rho,\theta\right)\Big|_{\rho=\rho_0}=f_1(\theta)-P\ln\left[b^{-1}dn\left(\frac{K\theta}{\pi},k'\right)\right],\\ V\left(\rho,\theta\right)\Big|_{\rho=\rho_0^{-1}}=f_2(\theta)-P\ln\left[b^{-1}dn\left(\frac{K\theta}{\pi},k'\right)\right],\quad V\left(1,\pm\pi\right)=0,\qquad \left(-\pi<\theta\le\pi\right).\end{gathered} \tag{38}$$

Where, $V\left(\rho,\theta\right)=W\left(x,y\right)$ and x,y are given by (34).

Using the chain rule, we can write the equivalence condition in the final form as

$$\phi(x)=\frac{a}{K'\sqrt{\left(a^2-x^2\right)\left(x^2-b^2\right)}}\left(\rho\frac{\partial V}{\partial\rho}\right)_{\rho=\rho_0^{-1}},\qquad \left(b<x<a\right) \tag{39}$$

Now, we assume the solution of (38) in the form

$$V\left(\rho,\theta\right)=\sum_{n=1}^{\infty}\left(A_n\rho^n+B_n\rho^{-n}\right)\cos n\theta+C\ln\rho+D,\quad \left(\rho_0<\rho<\rho_0^{-1},-\pi<\theta\le\pi\right) \tag{40}$$

Therefore, for determining the unknown constants $A_n, B_n, C,$ and $D,$ we assume

$$\begin{gathered}f_m(\theta)=f_0^{(m)}+\sum_{n=1}^{\infty}f_n^{(m)}\cos n\theta,\qquad \left(m=1,2\right),\\ \ln\left[dn\left(\frac{K\theta}{\pi},k'\right)\right]=\alpha_0+\sum_{n=1}^{\infty}\alpha_n\cos n\theta,\quad \left(-\pi<\theta<\pi\right).\end{gathered} \tag{41}$$

From the second and third conditions of (38) and with the aid of (41), we have

$$\begin{gathered}A_n=\frac{\left(f_n^{(1)}-\alpha_nP\right)\rho_0^n-\left(f_n^{(2)}-\alpha_nP\right)\rho_0^{-n}}{\rho_0^{2n}-\rho_0^{-2n}},\quad B_n=\frac{\left(f_n^{(2)}-\alpha_nP\right)\rho_0^n-\left(f_n^{(1)}-\alpha_nP\right)\rho_0^{-n}}{\rho_0^{2n}-\rho_0^{-2n}},\left(n\ge1\right)\\ C=\left(f_0^{(1)}-f_0^{(2)}\right)\Big/2\ln\rho_0,\quad D=\frac{1}{2}\left(f_0^{(1)}-f_0^{(2)}\right)+P\ln b-\alpha_0P\end{gathered} \tag{42}$$

●Now we have the following two points of discussions:

(1) The first discuss for a symmetric case, we have $f_1(\theta)=f_2(\theta)=g(\theta)$, hence we get

$$f_n^{(1)}=f_n^{(2)}=g_n,\quad \left(n=0,1,2,\ldots\right);\quad C=0,\qquad D=g_0+P\left(\ln b-\alpha_0\right)$$

Therefore, rewrite (42) to take the form

$$V(\rho,\theta)=\frac{1}{2}\sum_{n=1}^{\infty}\frac{g_n-\alpha_n P}{ch\left(\frac{\pi nK}{K'}\right)}\left(\rho^n+\rho^{-n}\right)\cos n\theta+P\left(\ln b-\alpha_0\right)+g_0;\left(\rho_0<\rho<\rho_0^{-1},-\pi<\theta\le\pi\right). \quad (43)$$

With the aid of the last conditions of equation (38) and the formula (43), we can directly determine the value of P in the form:

$$P=\sum_{n=0}^{\infty}\frac{(-1)^n g_n}{ch\left(\frac{\pi nK}{K'}\right)}\left[\sum_{n=0}^{\infty}\frac{(-1)^n\alpha_n}{ch\left(\frac{\pi nK}{K'}\right)}-\ln b\right]^{-1}, \quad (chx=\cosh x). \quad (44)$$

The values of α_n can be obtained, after using the famous relation, see [19]

$$\ln\left[dn(u,k)\right]=-8\sum_{n=1}^{\infty}\frac{q^{2n-1}}{(2n-1)\left[1-q^{2(2n-1)}\right]}\sin^2\left[(2n-1)\frac{\pi u}{2K}\right], \quad \left(q=\exp\left(-\pi\frac{K'}{K}\right),\left|lm\left(\frac{\pi u}{2K}\right)\right|<\frac{\pi}{2}\frac{K'}{K}\right) \quad \text{to get}$$

$$\alpha_0=\ln\sqrt{k}, \quad \alpha_{2n}=0, \quad \alpha_{2n-1}=\frac{4\rho_0^{2n-1}}{(2n-1)\left[1-\rho_0^{2(2n-1)}\right]}; \quad (n\ge 1) \quad (45)$$

Finally, using (44) in (39), the potential function $\phi(x)$, becomes

$$\phi(x)=\frac{a}{K'\sqrt{(a^2-x^2)(x^2-b^2)}}\sum_{n=1}^{\infty}(g_n-\alpha_n P)\,n\tanh\left(\frac{\pi nK}{K'}\right)T_n(X),$$

$$X=\cos\theta, \qquad \theta=\frac{\pi}{K'}\int_1^{x/b}\frac{dt}{\sqrt{(t^2-1)(1-k^2t^2)}}, \quad (b<x<a). \quad (46)$$

where, the constant P is given by (44), and α's by (45), while the function $T_n(X)$ is the **CP** of the first kind.

(2) The second discuss for a skew symmetric case, we assume

$$f_n^{(1)}=-f_n^{(2)}=-h_n, \qquad (n=0,1,2,...). \quad (47)$$

In this case, the four constants of (42) and the constant P of (44) become

$$A_n=-B_n=\frac{h_n}{2\sinh\left(\frac{\pi nK}{K'}\right)},(n=1,2,...);D=0; \quad C=\frac{-h_0}{\ln\rho_0}=\frac{h_0K'}{\pi K}, \quad P=0. \quad (48)$$

Also the corresponding of the two formulas of (43) and (47), respectively, become

$$V(r,\theta)=\frac{1}{2}\sum_{n=1}^{\infty}\frac{h_n}{\sinh\left(\frac{\pi nK}{K'}\right)}\left(\rho-\rho^{-n}\right)\cos n\theta+\frac{h_0K'}{\pi K}\ln\rho, \quad \left(\rho_0<\rho<\rho_0^{-1},-\pi<\theta<\pi\right) \quad (49)$$

and

$$\phi(x)=\frac{a}{K'\sqrt{(a^2-x^2)(x^2-b^2)}}\left[\sum_{n=1}^{\infty}nh_n\coth\left(\frac{\pi nK}{K'}\right)T_n(X)+\frac{h_0K'}{\pi K}\right], \quad (b<x<a) \quad (50)$$

Assume, in (49) and (50) $h_m=0,\ (m\ne n);h_n=1;(n=0,1,2,...)$, then, we can obtain the following **SRs**:

$$\int_b^a \ln\frac{x+t}{|x-t|}\frac{T_n(Y)\,dt}{\sqrt{(a^2-t^2)(t^2-b^2)}} = \lambda_n T_n(X), \quad Y=\cos\theta; \lambda_n = \frac{K'}{an}\tanh\left(\frac{\pi n K}{K'}\right), \quad (n\ge 1)$$
$$\theta = \frac{\pi}{K'}\int_1^{t/b}\frac{du}{\sqrt{(u^2-1)(1-k^2u^2)}}, (b<t<a); \quad \lambda_0=\frac{\pi K}{a}, (n=0). \tag{51}$$

3. Special and new relations of SRs:

Many spectral relationships, which have many applications in astrophysics, mathematical engineering and contact problems in the theory of elasticity, can be derived and established from this work:

(I) the integral operator $K\phi = \frac{1}{\pi}\int_{-\alpha}^{\alpha}\ln\frac{1}{2\left|\sin\frac{\xi-\eta}{2}\right|}\phi(\eta)\,d\eta$ can be established if we consider, in (4), the following two cases:

$$n=2m, \quad \frac{x}{a}=\frac{\sin(\xi/2)}{\sin(\alpha/2)}, \frac{t}{a}=\frac{\sin(\eta/2)}{\sin(\alpha/2)};$$
$$n=2m+1, \quad \frac{x}{a}=\frac{\tan(\xi/2)}{\tan(\alpha/2)}, \quad \frac{t}{a}=\frac{\tan(\eta/2)}{\tan(\alpha/2)}; (-\alpha<\xi,\eta<\alpha, \alpha<\pi, m=0,1,2,...). \tag{52}$$

We have, directly the following:

$$\frac{1}{\pi}\int_{-\alpha}^{\alpha}\left[\ln\frac{1}{2\left|\sin\frac{\xi-\eta}{2}\right|}+d\right]\frac{T_{2m}\left(\frac{\sin(\eta/2)}{\sin(\alpha/2)}\right)\cos(\eta/2)\,d\eta}{\sqrt{2(\cos\eta-\cos\alpha)}} = \mu_{2m}T_{2m}\left(\frac{\sin(\xi/2)}{\sin(\alpha/2)}\right), \tag{53}$$

$$\frac{1}{\pi}\int_{-\alpha}^{\alpha}\left[\ln\frac{1}{2\left|\sin\frac{\xi-\eta}{2}\right|}+d\right]\frac{T_{2m+1}\left(\frac{\tan(\eta/2)}{\tan(\alpha/2)}\right)\sec(\eta/2)\,d\eta}{\sqrt{2(\cos\eta-\cos\alpha)}} = \mu_{2m+1}T_{2m+1}\left(\frac{\tan(\xi/2)}{\tan(\alpha/2)}\right), \tag{54}$$

where,

$$\mu_{2m} = \begin{cases}(2m)^{-1} & m\ge 1\\ \left[-\ln\sin\left(\frac{\alpha}{2}\right)+d\right] & m=0\end{cases}; \quad \mu_{2m+1}=(2m+1)^{-1}, (m\ge 0)$$

In this case, the orthogonal relation will take respectively the forms

$$\int_{-\alpha}^{\alpha}T_{2m}\left(\frac{\sin(\eta/2)}{\sin(\alpha/2)}\right)T_{2P}\left(\frac{\sin(\eta/2)}{\sin(\alpha/2)}\right)\frac{\cos(\eta/2)\,d\eta}{\sqrt{2(\cos\eta-\cos\alpha)}} = \begin{cases}0 & m\ne P\\ \frac{\pi}{2} & m=P; (m,p=0,1,2...)\\ \pi & m=P=0\end{cases} \tag{55}$$

$$\int_{-\alpha}^{\alpha} T_{2m+1}\left(\frac{\tan\left(\eta/2\right)}{\tan\left(\alpha/2\right)}\right) T_{2P+1}\left(\frac{\tan\left(\eta/2\right)}{\tan\left(\alpha/2\right)}\right) \frac{\sec\left(\eta/2\right)d\eta}{\sqrt{2(\cos\eta-\cos\alpha)}} = \begin{cases} 0 & m \neq P \\ \frac{\pi}{2}\sec\left(\frac{\alpha}{2}\right), & m = P \end{cases} \quad (56)$$

Differentiating (4) with respect to x , we get

$$\frac{1}{\pi}\int_{-a}^{a} \frac{T_n\left(t/a\right)}{t-x}\frac{dt}{\sqrt{a^2-t^2}} = a^{-1}U_{n-1}\left(x/a\right), \qquad (n \geq 1). \quad (57)$$

where, $U_n(x)$ is the **CP** of the second kind.

Let, $n = 0$, in (57), we have

$$\frac{1}{\pi}\int_{-a}^{a} \frac{1}{t-x}\frac{dt}{\sqrt{a^2-t^2}} = 0 \quad (58)$$

The value of the integral (58) has many important applications in the contact problem when the kernel takes a Cauchy form.

Also, in ξ − plane we follow

$$\frac{1}{\pi}\int_{-\alpha}^{\alpha}\cot\left(\frac{\eta-\xi}{2}\right)\frac{T_n\left(\frac{\tan\left(\eta/2\right)}{\tan\left(\alpha/2\right)}\right)\cos\left(\eta/2\right)d\eta}{\sqrt{2(\cos\eta-\cos\alpha)}} = \begin{cases} 0 & (n=0), \quad (|\xi|<\alpha) \\ \mathrm{cosec}\left(\alpha/2\right)U_{2m-1}\left(\frac{\tan\left(\xi/2\right)}{\tan\left(\alpha/2\right)}\right), & (n = 2m \quad ,m = 1,2,...) \\ \mathrm{cosec}\left(\alpha/2\right)U_{2m-1}\left(\frac{\tan\left(\xi/2\right)}{\tan\left(\alpha/2\right)}\right) + (-1)^m \frac{\sin\left(\alpha/2\right)}{1+\cos\left(\alpha/2\right)}\left[\tan\left(\alpha/4\right)\right]^{2m-2}, & (n = 2m-1), \end{cases} \quad (59)$$

$$\frac{1}{\pi}\int_{-\alpha}^{\alpha}\cot\left(\frac{\eta-\xi}{2}\right)\frac{T_n\left(\frac{\tan\left(\eta/2\right)}{\tan\left(\alpha/2\right)}\right)\cos\left(\eta/2\right)d\eta}{\sqrt{2(\cos\eta-\cos\alpha)}} = \begin{cases} \left(\alpha/2\right)\sec^2\left(\xi/2\right)U_{n-1}\left(\frac{\tan\left(\xi/2\right)}{\tan\left(\alpha/2\right)}\right), & (n \geq 1) \\ \sec\left(\frac{\alpha}{2}\right)\tan\left(\xi/2\right), \quad n = 0, & |\xi| < \alpha \end{cases} \quad (60)$$

Also, the orthogonal relation for the **CPs** of the second kind takes the form

$$\int_{-\alpha}^{\alpha} U_{n-1}\left(\frac{\sin\left(\eta/2\right)}{\sin\left(\alpha/2\right)}\right) U_{m-1}\left(\frac{\tan\left(\eta/2\right)}{\tan\left(\alpha/2\right)}\right)\cos\left(\eta/2\right)\sqrt{2(\cos\eta-\cos\alpha)}\,d\eta = \begin{cases} 0 & n \neq m \\ 2\pi\sin^2\left(\frac{\alpha}{2}\right); \; n = m, & n,m = 1,2,... \end{cases} \quad (61)$$

and

$$\int_{-\alpha}^{\alpha} U_{n-1}\left(\frac{\tan\left(\eta/2\right)}{\tan\left(\alpha/2\right)}\right) U_{m-1}\left(\frac{\tan\left(\eta/2\right)}{\tan\left(\alpha/2\right)}\right)\sec^3\left(\eta/2\right)\sqrt{2(\cos\eta-\cos\alpha)}\,d\eta = \begin{cases} 0, & n \neq k \\ 2\pi\cos\left(\frac{\alpha}{2}\right)\tan^2\left(\frac{\alpha}{2}\right); & n = k \end{cases} \quad (62)$$

(II) For the integral operator $K\phi=\frac{2}{\pi}\left(\int_{-\infty}^{-a}+\int_{a}^{\infty}\right)\ln\left|\frac{x}{x-t}\right|\phi(t)dt$ we write:

$$\cos\left(\lambda\rho_{\pm}(x)\right)=\cos\left(\lambda\,\mathrm{sgn}\,x\,\rho_{\pm}(x)\right)=\mathrm{Re}\exp\left(i\lambda\,\mathrm{sgn}\,x\,\rho_{\pm}(x)\right)$$
$$\mathrm{sgn}\,x\,\sin\left(\lambda\rho_{\pm}(x)\right)=\sin\left(\lambda\,\mathrm{sgn}\,x\,\rho_{\pm}(x)\right)=\mathrm{Im}\exp\left(i\lambda\,\mathrm{sgn}\,x\,\rho_{\pm}(x)\right);\left(i=\sqrt{-1}\right) \qquad (63)$$

for this, we have

$$\frac{2}{\pi}\left(\int_{-\infty}^{-a}+\int_{a}^{\infty}\right)\ln\left|\frac{x}{x-t}\right|\omega_{\pm}(t)\exp\left(i\lambda\,\mathrm{sgn}t\,\rho_{\pm}(t)\right)dt=\frac{a}{|\lambda|}\exp\left(i\lambda\,\mathrm{sgn}x\,\rho_{\pm}(x)\right),(-\infty<\lambda<0,x<-a;0<\lambda<\infty,x>a),$$

$$\omega_{\pm}(x)=\frac{x\pm\sqrt{x^2-a^2}}{2\sqrt{x^2-a^2}};\quad \rho_{\pm}(x)=\frac{x\pm\sqrt{x^2-a^2}}{a},\qquad (x>a) \qquad (64)$$

also, we obtain

$$\frac{2}{\pi}\int_{a}^{\infty}\ln\frac{x^2}{|x^2-t^2|}\breve{\omega}_{\pm}(t)\cos\left(\lambda\breve{\rho}_{\pm}(t)\right)dt=\frac{a}{2}\cos\left(\lambda\breve{\rho}_{\pm}(x)\right),\quad(\lambda>0,x>a);$$
$$\frac{2}{\pi}\int_{a}^{\infty}\ln\frac{x+t}{|x-t|}\breve{\omega}_{\pm}(t)\sin\left(\lambda\breve{\rho}_{\pm}(t)\right)dt=\frac{a}{2}\sin\left(\lambda\breve{\rho}_{\pm}(x)\right) \qquad (65)$$

where, in (65), we assumed

$$\breve{\omega}_{\pm}(x)=\frac{x\pm\sqrt{x^2-a^2}}{2\sqrt{x^2-a^2}},\ \breve{\rho}_{\pm}(x)=\frac{x\pm\sqrt{x^2-a^2}}{a};\ (x>a)$$

If we differentiate (65) with respect to x we obtain many **SRs** of Cauchy kernel,

$$\frac{2}{\pi}\int_{a}^{\infty}\frac{t}{t^2-x^2}\breve{\omega}_{\pm}(t)\sin\left(\lambda\breve{\rho}_{\pm}(t)\right)dt=\pm\breve{\omega}_{\pm}\cos\left(\lambda\breve{\rho}_{\pm}(x)\right),\qquad(\lambda>0,x>a) \qquad (66)$$

Another **SR** if we assume, in (65), that $x=ae^{\xi/2},\quad t=ae^{\eta/2};(0<\xi,\eta<\infty)$, hence, after some algebra, we get

$$\frac{1}{\pi}\int_{0}^{\infty}\ln\left|\coth\frac{\xi-\eta}{4}\right|\left[\frac{2e^{\eta/2}X(\eta)\left(X(\eta)\pm e^{\eta/4}\right)+1}{2X(\eta)\left(e^{\eta/4}\pm X(\eta)\right)}\right]\sin\left(\lambda\left(e^{\eta/4}\pm X(\eta)\right)\right)d\eta=\frac{1}{\lambda}\left(\lambda\left(e^{\eta/4}\pm X(\xi)\right)\right),$$
$$\left(X(x)=\sqrt{2\sinh\frac{x}{2}},\ \lambda>0,\xi>0\right) \qquad (67)$$

The integral operator (II) with the **SRs** (65), (67) can be adapted, in the Fourier integral sine or cosine forms, as the following:

$$F_{\pm}(\lambda)=\frac{2}{a}\int_{a}^{\infty}f(x)\begin{Bmatrix}\cos\left(\lambda\breve{\rho}_{\pm}(x)\right)\\ \sin\left(\lambda\breve{\rho}_{\pm}(x)\right)\end{Bmatrix}\breve{\omega}_{\pm}(x)dx\qquad(x>a) \qquad (68)$$

and its inverse

$$f(x)=\frac{2}{a}\int_{a}^{\infty}F_{\pm}\begin{Bmatrix}\cos\left(\lambda\breve{\rho}_{\pm}(x)\right)\\ \sin\left(\lambda\breve{\rho}_{\pm}(x)\right)\end{Bmatrix}d\lambda\qquad(x>a) \qquad (69)$$

This leads us to deduce the following important relations

$$\frac{2}{a}\int_a^\infty F_+(\lambda)\begin{cases}\cos(\lambda\breve{\rho}_+(x))\\ \sin(\lambda\breve{\rho}_-(x))\end{cases}d\lambda=\int_a^\infty F_-(\lambda)\begin{cases}\cos(\lambda\breve{\rho}_+(x))\\ \sin(\lambda\breve{\rho}_-(x))\end{cases}d\lambda,\qquad (x>a) \tag{70}$$

The formulas (61) and (65)-(68) can be used with wide applications in the displacement problems of mechanics, see Aleksandrov et al.[6], Aleksandrov[13] and Abdou [16].

In the spectral relationships (51), we assume

$$a=e^{\alpha/2},\quad b=e^{-\alpha/2}\ (\alpha>0),\ x=e^{\xi/2},\ t=e^{\eta/2},\ \ (-\alpha<\xi,\eta<\alpha). \tag{71}$$

to obtain the following

$$\int_{-\alpha}^{\alpha}\ln\left|\coth\frac{\xi-\eta}{4}\right|\frac{T_n(Y)\,d\eta}{\sqrt{2(\cosh\alpha-\cosh\eta)}}=\lambda_n' T_n(X). \tag{72}$$

where,

$$X=\cos\theta,\ \theta=\frac{\pi e^{\alpha/2}}{k'}\int_{-\alpha}^{\xi}\frac{du}{\sqrt{2(\cosh\alpha-\cosh u)}},\qquad k'=k\left(\sqrt{1-e^{-2\alpha}}\right),\ \ Y=\cos\Phi,$$

$$\Phi=\frac{\pi e^{\alpha/2}}{k'}\int_{-\alpha}^{\eta}\frac{du}{\sqrt{2(\cosh\alpha-\cosh u)}},\quad \lambda_n'=\frac{2e^{-\alpha/2}k'}{n}\tanh\left(\frac{\pi nk\left(e^{-\alpha}\right)}{k'}\right),\quad (n\geq 1),\quad \lambda_0=2\pi e^{-\alpha/2}k\left(e^{-\alpha}\right) \tag{73}$$

Finally, for the interval $b<|x|<a$, after assuming

$$x=\left(\frac{a}{a^2-b^2}\right)\left(2\xi^2-a^2-b^2\right),\ t=\left(\frac{a}{a^2-b^2}\right)\left(2\eta^2-a^2-b^2\right),\ \ \left(|x|,|t|<a,\ \ b<|\xi|,|\eta|<a\right) \tag{74}$$

we can obtain

$$\frac{1}{\pi}\left(\int_{-a}^{-b}+\int_b^a\right)\ln\frac{1}{|\xi-\eta|}\cdot\frac{T_n\left(\frac{2\eta^2-b^2-a^2}{a^2-b^2}\right)|\eta|\,d\eta}{\sqrt{a^2-\eta^2}\left(\eta^2-b^2\right)}=\begin{cases}\frac{1}{2n}T_n\left(\frac{2\xi^2-b^2-a^2}{a^2-b^2}\right), & (n\geq 1),\\ \ln\frac{2}{\sqrt{a^2-b^2}}, & (n=0).\end{cases} \tag{75}$$

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