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SIMPLE ANALYTICAL ALGORITHM FOR CONSTRUCTING THE ENVELOPE OF THE PROJECTILE TRAJECTORIES IN MIDAIR

Peter Chudinov,Vladimir Eltyshev, Yuri Barykin
Perm State Agricultural Academy, Petropavlovskayast., 23, Perm, 614990, Russia
chupet@mail.ru

ABSTRACT

A classic problem of the motion of a projectile thrown at an angle to the horizon is studied. The air drag force is taken into account as the quadratic resistance law. An analytic approach is used for the investigation. Simple analytical formulae are used for constructing the envelope of the family of the projectile trajectories. The equation of envelope is applied for the determination of the maximum range of flight. The motion of a baseball is presented as an example.

Keywords

Classical problem, projectile motion, quadratic drag force, analytic formulae, envelope.

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INTRODUCTION

The problem of the motion of a point mass (projectile) thrown at an angle to the horizon in midair has a long history. It is one of the great classical problems. The number of works devoted to this task is immense. It is a constituent of many introductory courses of physics. With zero air drag force, the analytic solution is well known. The trajectory of the point mass is a parabola. In situations of practical interest, such as throwing a ball, taking into account the impact of the medium the quadratic resistance law is usually used. In that case the mathematical complexity of the task strongly grows. The problem probably does not have an exact analytic solution. Therefore the attempts are being continued to construct approximate analytical solutions for this problem. In this paper an analytic approach is used for the investigation of the projectile motion in a medium with quadratic resistance. For the first time simple analytical formulae are used for constructing the envelope of the family of the point mass trajectories. The equation of envelope is applied for the determination of the maximum range of flight. The motion of a baseball is presented as an example. The proposed analytical solution differs from other solutions by simplicity of formulae, ease of use and high accuracy. All required parameters are determined directly from the initial conditions of projectile motion - the initial velocity and angle of throwing. The proposed formulae make it possible to study the motion of a projectile in a medium with the resistance in the way it is done for the case without drag. The object of the present work is to give simple formulae for the construction ofthe projectile trajectoriesunder the motion with quadratic air resistance. These formulae are available even for first-year undergraduates.

The problem of the motion of a projectile in midair arouses interest of authors as before [1–3]. For the construction of the analytical solutions various methods are used – both the traditional approaches [4–9], and the modern methods[10]. All proposed approximate analytical solutions are rather complicated and inconvenient for educationalpurposes. This is why the description of the projectile motion by means of a simple approximate analytical formulae under the quadratic air resistance is of great methodological and educationalimportance. In [11–13] comparatively simple approximate analytical formulae have been obtained to study the motion of the projectile in a medium with a quadratic drag force. In this article these formulae are used to solve the classical problem of maximizing the projectile distance with using the envelope. In papers [2,16] the equation of the envelope was used to solve problems of maximizing the range of the projectile only within parabolic theory.From now on the term “point mass” means the center of mass of a smooth spherical object of finite radius r and cross-sectional area $S = \pi r^2$. The conditions of applicability of the quadratic resistance law are deemed to be fulfilled, i.e. Reynolds number Re lies within $1\times10^3 < Re < 2\times10^5$. These values correspond to the projectile motion velocity , lying in the range between 0.25 m/s and 53 m/s.

EQUATIONS OF POINT MASS MOTION AND ANALYTICAL FORMULAE FOR BASIC PARAMETERS

Suppose that the force of gravity affects the point mass together with the force of air resistance ***R*** (Fig.1).Air resistance force is proportional to the square of the velocity of the point mass and is directed opposite the velocity vector. For the convenience of further calculations, the drag force will be written as $R = mgkV^2$. Here m is the mass of the projectile, g is the acceleration due to gravity, k isthe proportionality factor. Vector equation of the motion of the point mass has the form

$$m\mathbf{w} = m\mathbf{g} + \mathbf{R},$$

where***w*** – acceleration vector of the point mass. Differential equations of the motion, commonly used in ballistics, are as follows [14]

$$\frac{dV}{dt} = -g\sin\theta - gkV^2, \quad \frac{d\theta}{dt} = -\frac{g\cos\theta}{V}, \quad \frac{dx}{dt} = V\cos\theta, \quad \frac{dy}{dt} = V\sin\theta. \qquad (1)$$

Here V is the velocity of the point mass, $\theta$is the angle between the tangent to the trajectory of the point mass and the horizontal, x, y are the Cartesian coordinates of the point mass, $k$is

$$k = \frac{\rho_a c_d S}{2mg} = \frac{1}{V_{term}^2} = const,$$

$\rho_a$isthe air density, $c_d$isthe drag factor for a sphere,S is the cross-section area ofthe object,andV_{term} is the terminal velocity.The first two equations of the system (1) represent the projections of the vector equation of motion on the tangent and principal normal to the trajectory, the other two are kinematic relations connecting the projections of the velocity vector point mass on the axis x, y with derivatives of the coordinates.

The well-known solution of system (1) consists of an explicit analytical dependence of the velocity on the slope angle of the trajectory and three quadratures

$$V(\theta)=\frac{V_0\cos\theta_0}{\cos\theta\sqrt{1+kV_0^2\cos^2\theta_0\left(f(\theta_0)-f(\theta)\right)}},\quad f(\theta)=\frac{\sin\theta}{\cos^2\theta}+\ln tg\left(\frac{\theta}{2}+\frac{\pi}{4}\right),\quad (2)\quad t=t_0-\frac{1}{g}\int_{\theta_0}^{\theta}\frac{V}{\cos\theta}d\theta$$

$$,\quad x=x_0-\frac{1}{g}\int_{\theta_0}^{\theta}V^2d\theta,\ y=y_0-\frac{1}{g}\int_{\theta_0}^{\theta}V^2tg\theta d\theta. \quad (3)$$

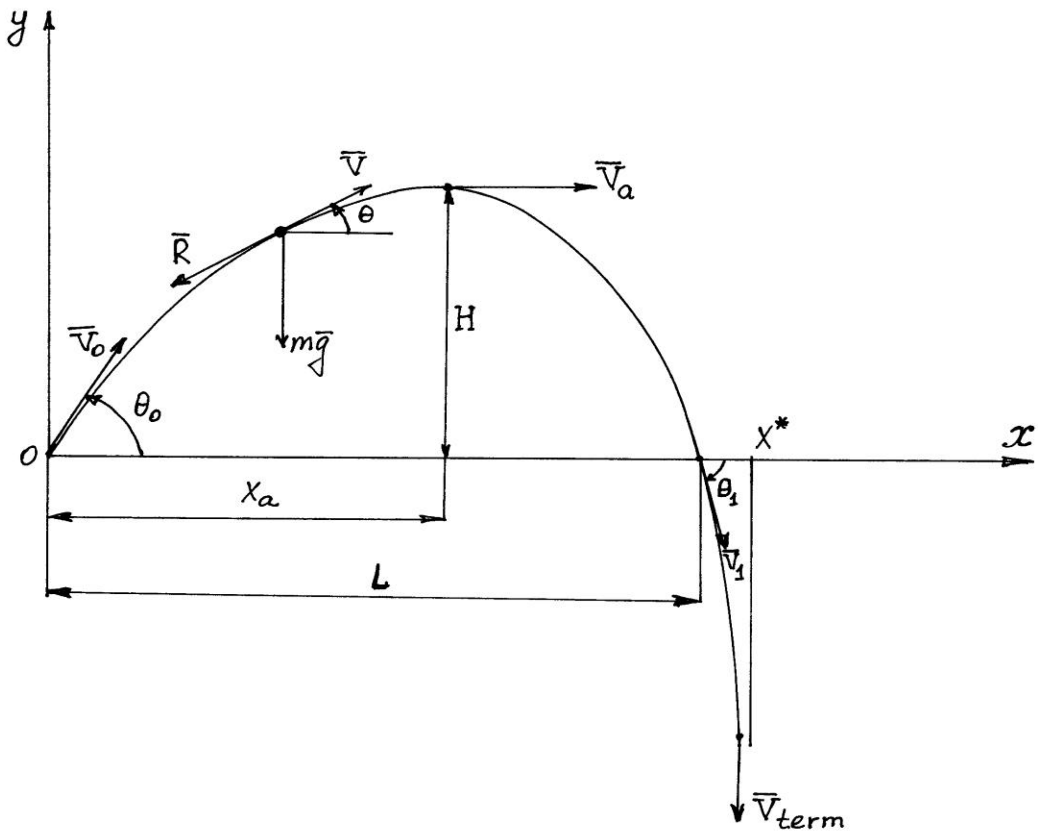

Figure 1.Basic motion parameters.

HereV_0 and$\theta_0$are the initial values of the velocity and the slope of the trajectory respectively, $t_0$is the initial value of the time, x_0, $y_0$are the initial values of the coordinates of the point mass (usually accepted $t_0=x_0=y_0=0$). The derivation of the formulae (2) is shown in the well-known monograph [15]. The integrals on the right-hand sides of formulae (3) cannot be expressed in terms of elementary functions. Hence, to determine the variables *t, x* and *y* we must either integrate system (1) numerically or evaluate the definite integrals (3). Formulae (2) of this solution will be used later.

Comparatively simple approximate analytical formulae for the main parameters of motion of the projectile are derived in [11, 13]. The four parameters correspond to the top of the trajectory, the four other parameters correspond to the point of drop. We will give a complete summary of the formulae for the maximum height of ascent of the point mass *H,* motion time *T,* the velocity at the trajectory apex $V_a = V(0)$, flight range *L,* the time of ascent t_a, the abscissa of the trajectory apex x_a , impact angle with respect to the horizontal θ_1 and the final velocity V_1(Fig. 1). These formulae are summarized in the right column of Table 1. In the left column of this Table 1 similar formulae of the parabolic theory are presented for comparison. With zero drag (k = 0), these formulae go over into the respective formulae of the point mass parabolic motion theory. All motion characteristics described by these formulae are functions of initial conditions of throwing V_0 , θ_0. Proposed formulae have a bounded region of application. We introduce the notation $p=kV_0^2$. The dimensionless parameter p has the following physical meaning – it is the ratio of air resistance to the weight of the projectile at the beginning of the movement. The main characteristics of the motion H, T, V_a, L, $x_a$have accuracy to within 2 - 3% for values of the launch angle, for initial velocity and for the parameter p from ranges

$$0°\le\theta_0\le 70°,\quad 0\le V_0\le 50 \text{ m/s},\quad 0\le p\le 1.5.$$

Table 1.Analytical formulae for the main parameters.

No drag($R = 0$)	Quadratic drag force ($R = mgkV^2$)
$H = \dfrac{V_0^2 \sin^2\theta_0}{2g}$	$H = \dfrac{V_0^2 \sin^2\theta_0}{g\left(2 + kV_0^2 \sin\theta_0\right)}$
$T = 2\dfrac{V_0 \sin\theta_0}{g} = 2\sqrt{\dfrac{2H}{g}}$	$T = 2\sqrt{\dfrac{2H}{g}}$
$V_a = V_0 \cos\theta_0$	$V_a = \dfrac{V_0 \cos\theta_0}{\sqrt{1 + kV_0^2\left(\sin\theta_0 + \cos^2\theta_0 \cdot \ln\tan\left(\dfrac{\theta_0}{2} + \dfrac{\pi}{4}\right)\right)}}$
$L = \dfrac{1}{g} V_0^2 \sin 2\theta_0 = V_a T$	$L = V_a T$
$t_a = \dfrac{V_0 \sin\theta_0}{g} = \dfrac{T}{2}$	$t_a = \dfrac{T - kHV_a}{2}$
$x_a = \dfrac{L}{2} = \sqrt{LH\cot\theta_0}$	$x_a = \sqrt{LH\cot\theta_0}$
$\theta_1 = -\theta_0 = -\arctan\left[\dfrac{LH}{(L - x_a)^2}\right]$	$\theta_1 = -\arctan\left[\dfrac{LH}{(L - x_a)^2}\right]$
$V_1 = V_0$	$V_1 = V(\theta_1)$

For a baseball the typical values of the drag force coefficient $k$are about 0.0005 ÷ 0.0006 s^2/m^2,maximal initial velocity is about 50m/s [1,9]. Therefore the proposed formulae are suitable for the qualitative and quantitative description of the baseball and other similar objects motion.

These formulae, in turn, make it possible to obtain a simple analytical formula for the main functional relationship of the problem $y(x)$ [11].In the absence of air resistance, the trajectory of a point mass is a parabola. The equation of the trajectory can be written in two forms. It can be written in terms of the initial conditions of throwing V_0 , θ_0 (first form). It can also be written in terms of the motion parameters H, L, x_a(second form)

$$y(x) = x \cdot \tan\theta_0 - \frac{gx^2}{2V_0^2 \cos^2\theta_0} = \frac{Hx(L - x)}{x_a^2}. \quad (4)$$

The trajectory is symmetric with respect to the maximum. When the point mass is under a drag force, the trajectory becomes asymmetrical. The top of the trajectory is shifted towards the point of incidence. In addition, a vertical asymptote appears near the trajectory. Taking these circumstances into account, the function $y(x)$ may be constructed using parameters H, L , $x_a$as [11]

$$y(x) = \frac{Hx(L - x)}{x_a^2 + (L - 2x_a)x} . \quad (5)$$

The constructed dependence$y(x)$ provides the shift of the apex of the trajectory to the right and has a vertical asymptote. In the case of no drag $L = 2x_a$and formula (5) goes over to formula (4). Wenotethe remarkable property of formula (5). We insert the exact values of the parameters L, H,x_a , obtained by numerical integration of system (1), into formula (5). Then the numerical trajectory and the analytical trajectory constructed by means offormula (5) are identical. This means that formula (5) approximates absolutely precisely projectile's trajectory which is numerically constructed using system (1) at any values of the initial conditions V_0 ,θ_0.

As an example of the use of the proposed formulaefor$H$, L, $x_a$from the Table 1 and of formula (5) we calculated the motion of a baseball with the following initial conditions

$$V_0 = 40 \text{ m/s}, \ \theta_0 = 45°, \ k = 0.000625 \text{ s}^2/\text{m}^2, \ g = 9.81 \text{ m/s}^2. \qquad (6)$$

An approximate trajectory is constructed. It is shown in Fig. 2 (dotted line). The thick solid line in Fig. 2 is obtained by numerical integration of system(1) with the aid of the 4-th order Runge-Kutta method. As it can be seen from Fig. 2, the analytical solution (formula (5)) and a numerical solution are almost the same. The dashed line in Fig. 2 is constructed in the absence of air resistance.

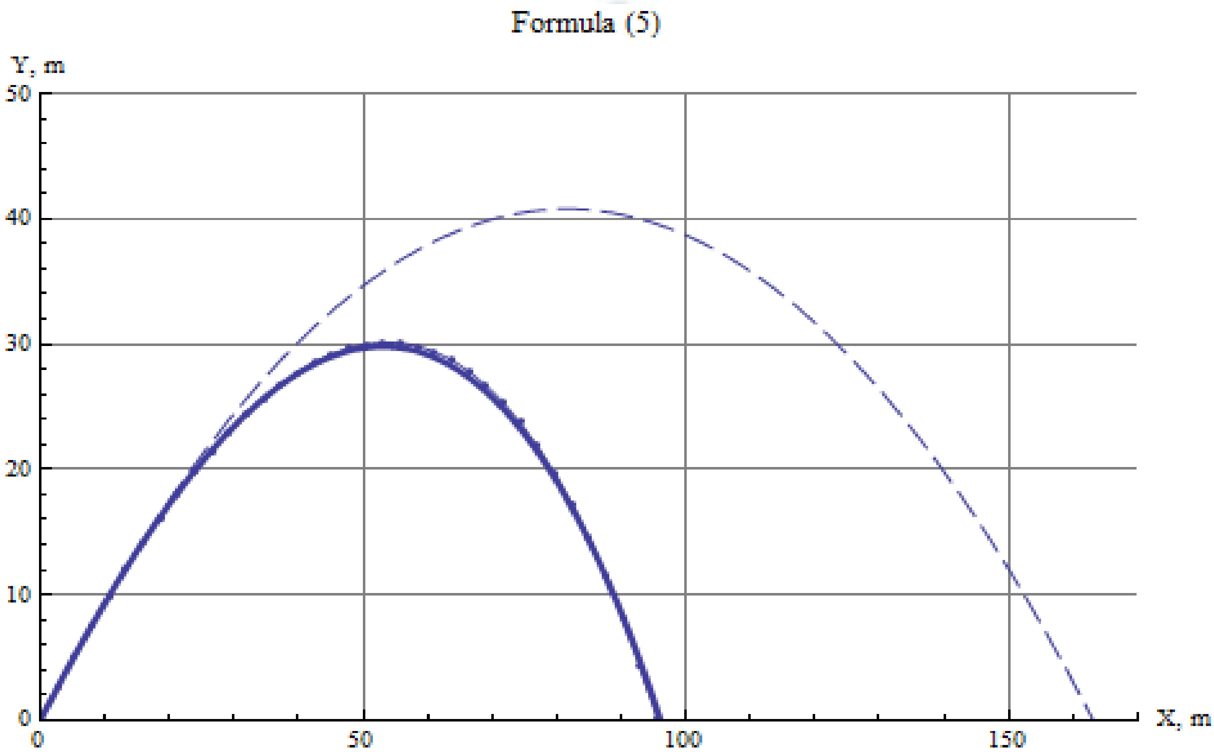

Figure 2.The graph of the trajectory *y=y(x).*

One of the most important aspects of the projectile motion problem is the determination of an optimum angle of throwing of a point mass which provides the maximum range. The equation for the optimum angle of throwing αin the case when the points of incidence and throwing are on the same horizontal is obtained in [12]:

$$\tan^2\alpha + \frac{p\sin\alpha}{4+4p\sin\alpha} = \frac{1+p\lambda}{1+p\left(\sin\alpha+\lambda\cos^2\alpha\right)}. \qquad (7)$$

Here $p = kV_0^2$, $\lambda(\alpha) = \ln\tan\left(\frac{\alpha}{2}+\frac{\pi}{4}\right)$. The dependence of the root $\alpha$of equation (7) on the parameter p is represented in Fig. 3. We use Fig.3for constructing the envelope.

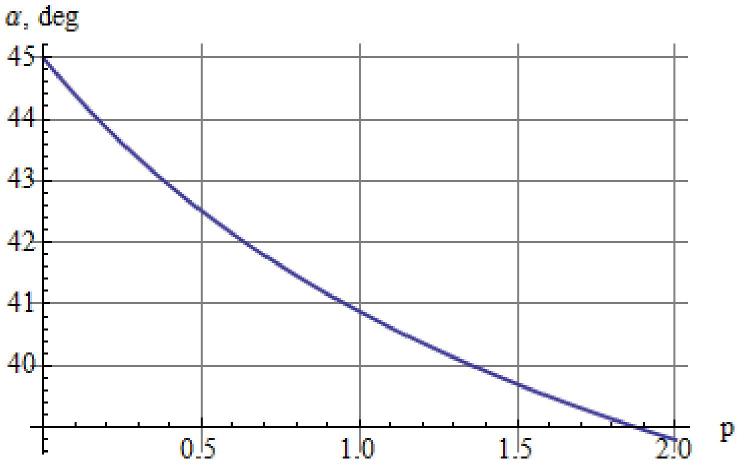

Figure 3. The graph of function $\alpha = \alpha(p)$.

THE EQUATION OF THE ENVELOPE IN MIDAIR

In the case of no drag the trajectory of a point mass is a parabola. For different angles of throwing under one and same initial velocity the projectile trajectories form a family of parabolas. The maximum range and the maximum height for limiting parabolas are given by formulae

$$L_{max} = \frac{V_0^2}{g}, \quad H_{max} = \frac{V_0^2}{2g}. \tag{8}$$

The envelope of this family is also a parabola, the equation of which is usually written as [16]

$$y(x) = \frac{V_0^2}{2g} - \frac{g}{2V_0^2} x^2. \tag{9}$$

Using (8), we will convert theequation(9) as

$$y(x) = \frac{H_{max}\left(L_{max}^2 - x^2\right)}{L_{max}^2}. \tag{10}$$

We will set up an analytical formula similar to (10)for the envelope of the projectiletrajectories taking into account the air drag force.Considering(10), we will write down an equation of the envelope as

$$y(x) = \frac{H_{max}\left(L_{max}^2 - x^2\right)}{L_{max}^2 - ax^2}. \tag{11}$$

Such structure of the equation takes into account the fact that the envelope has the maximum under $x = 0$. Besides, this function underthe conditions $0 < a < 1$has a vertical asymptote, as well as any projectile trajectory accounting the resistance of air. In (11)$H_{max}$is the maximum height, reached by the projectile when throwing with initial conditions V_0 , $\theta_0 = 90°$; $L_{max}$is the maximum range, reached when throwing a projectile with the initial velocity V_0 under some optimum angle$\theta_0 = \alpha$. In the parabolic theory an angle αis45° under any initial velocity . Taking into account the resistance of air, the optimum angle of throwing $\alpha$is less than 45°and depends on the value of parameter $p = kV_0^2$. The maximum heightH_{max}in the used notations is defined by formula [14]

$$H_{max} = \frac{1}{2gk}\ln\left(1 + kV_0^2\right). \tag{12}$$

For generation of the envelope it is required to construct the trajectory equation of the maximum distance. As it follows from formula (5), for the construction of the trajectory equation of the maximum distancethree parameters are required: *H*, L_{max}, x_a. We will calculate these parameters as follows. Under a given value of quantity $p = kV_0^2$ we will find the root αofthe equation (7) using the graph of the function $\alpha = \alpha(p)$. An angle αensures the maximum range of the flight. Substituting the initial conditions V_0, αin the formulae of the Table 1, we obtain the values $H(\alpha)$, $L(\alpha) = L_{max}$, $x_a(\alpha)$ for the maximum range trajectory.

A choice of a positive factor $a$in(11) is sufficiently free. However it must satisfy the condition $a= 0$ in the absence of resistance ($k = 0$). We shall find this coefficient from the following considerations. We set the equal slopes of the tangents to envelope (11) and to the maximum range trajectory

$$y(x)=\frac{H(\alpha)x(L_{max}-x)}{x_a^2(\alpha)+(L_{max}-2x_a(\alpha))x}$$

in the spot of incidence$x = L_{max}$. It follows that parameter a is defined by the formula

$$a=1-\frac{2H_{max}}{H(\alpha)}\left(1-\frac{x_a(\alpha)}{L_{max}}\right)^2 . \qquad (13)$$

In the absence of air resistance parameter $a$vanishes (a=0).For typical values of the characteristics of a baseball and initial velocity of throwing the conditions $0 < a < 1$ are carried out.

The equation of the envelope can be used for the determination of the maximum range if the spot of falling lies above or below the spot of throwing. Let the spot of falling be on a horizontal straight line defined by the equation$y = y_1 = const$. We will substitute a value $y_1$in (11) and solve it for x. We obtain the formula

$$x_{max}=L_{max}\sqrt{\frac{H_{max}-y_1}{H_{max}-ay_1}} . \qquad (14)$$

Formula (14) allows us to find a maximum range under the given height of the spot of falling (point $B(x_{max}, y_1)$ in Fig.4). Another simple exampleof the use of the envelope is maximizing the vertical range. Let a projectile be thrown towards a vertical target wall at distance $x = x_1 = const$ from the spot of throwing (point $A(x_1, y_{max})$ in Fig.4). The y_{max}coordinate of the intersection of the envelope and the target wall is simply the y coordinate of the envelope at $x = x_1$.

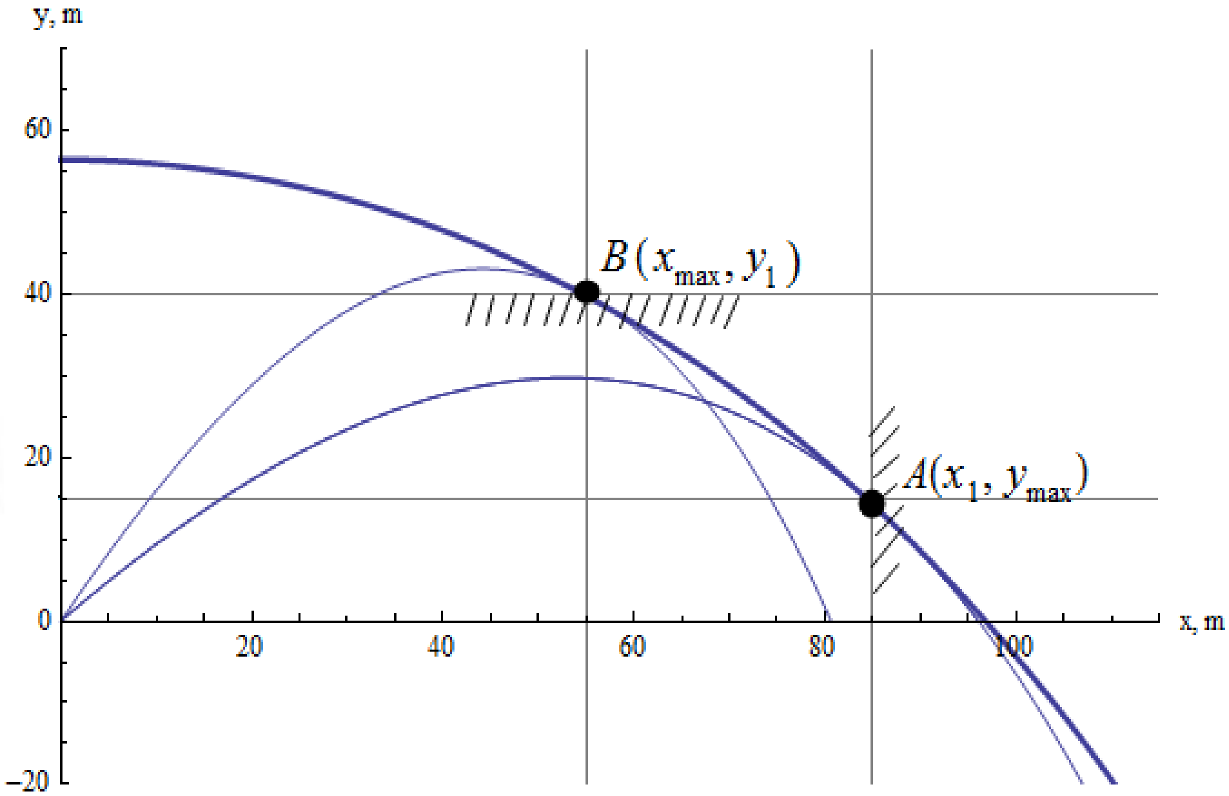

Figure 4. Maximizing the horizontal x_{max} and vertical y_{max} ranges of a projectile using the envelope.

Thus, it is required to use the following seven step algorithm to find the maximum range of the projectilex_{max}at a given height of the point of falling $y_1 = const$.

Step 1: $p = kV_0^2, \quad \alpha = \alpha(p)$

Step2: $H_{max} = \frac{1}{2gk}\ln(1+p)$

Step 3: $$H(\alpha) = \frac{V_0^2 \sin^2\alpha}{g(2+p\sin\alpha)}$$

Step 4: $$L_{max} = L(\alpha) = \frac{V_0^2 \sin 2\alpha}{g\sqrt{(1+0.5p\sin\alpha)(1+p(\sin\alpha+\cos^2\alpha\cdot\ln\tan(0.5\alpha+0.25\pi)))}}$$

Step 5: $$x_a(\alpha) = \sqrt{L(\alpha)H(\alpha)\cot\alpha}$$

Step 6: $$a = 1 - \frac{2H_{max}}{H(\alpha)}\left(1-\frac{x_a(\alpha)}{L_{max}}\right)^2$$

Step 7: $$x_{max} = L_{max}\sqrt{\frac{H_{max}-y_1}{H_{max}-ay_1}}$$

Calculations based on the present formulae can be done even on a standard calculator. But this algorithm does not determine the optimal throwing angle θ_0 for maximum distance x_{max}. To calculate this angle it is necessary to find the root of equation (5)

$$f(\theta_0) = y_1\left(x_a^2 + (L-2x_a)x_{max}\right) - Hx_{max}(L-x_{max}) = 0 \qquad (15)$$

This task can be solved using computer algebra systems *Maple*or *Mathematica*.

THE RESULTS OF THE CALCULATIONS

As an example we will consider the moving of a baseball with the resistance factor [1] $k = \frac{1}{V_{term}^2} = \frac{1}{40^2} = 0.000625$ s^2/m^2.

Other parameters of motion are given by values

g = 9.81 m/s^2, V_0= 40 m/s, y_1 = ±20, ±40m.

At specified values k and V_0 the value of non-dimensional parameter is $p = kV_0^2 = 1$. Under a given value p = 1we will find the root $\alpha$of the equation (7) using the graph of the function $\alpha = \alpha(p)$ (Fig. 3). This angle ensures the maximum range: α= 40.8°. Substituting values k and V_0in formula (12), we getH_{max} = 56.5 m. Substituting in the formulae of the Table 1 the initial conditions V_0 = 40, $\theta_0 = \alpha$= 40.8°, we find the meanings

$H(\alpha)$= 26.2 m, $L(\alpha)=L_{max}$= 96.6 m, $x_a(\alpha)$ = 54.2 m.

According to formula (13) the factor$a$ is$a$ =0.17. The graph of the envelope is plotted in Fig.5 together with the family of trajectories. The envelope is shown by a thick line in Fig.5. We note that family of trajectories is received by means of numerical integration of the system (1). A standard fourth-order Runge-Kutta method was used. The dashed line in Fig. 5 is the envelopeaccording to formula (9) with the same initial velocity in the absence of air resistance.

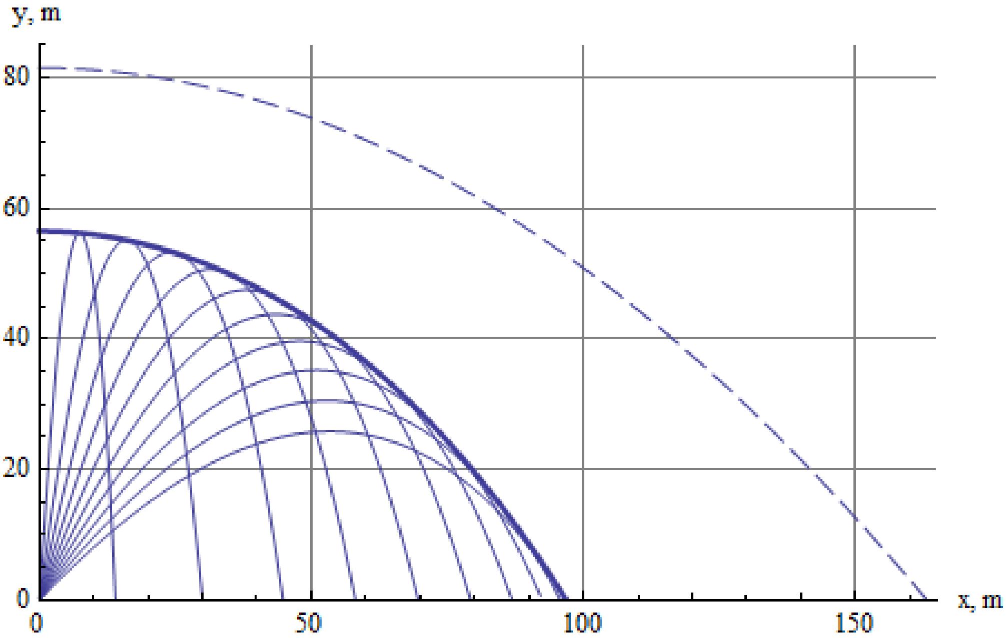

Figure 5.The family of projectile trajectories and the envelope of this family.

The results of calculations using formula (14) are presented in Table 2.The second column of Table 2 contains range values calculated analytically by formula (14). The third column of Table 2 contains range values from the integration of the equations of system (1). The fourth column presents the error of the calculation of the range in the percentage. The error does not exceed 2 %. Formula (14) gives a sufficiently exact value of the maximum distance in a wide range of the height of the drop point. The results of calculations using formula (15) with the help of *Mathematica* are presented in Table 3.

Table 2. Maximum range x_{max} at different heights of the point of the falling y_1.

y_1 (m)	Analyticalvalue x_{max}, (m)	Numericalvaluex_{max}, (m)	Error (%)
40	55.7	54.8	1.6
20	80.1	79.6	0.6
0	96.6	96.8	–0.2
–20	109.2	110.1	–0.8
–40	119.3	121.0	–1.4

Table 3. Optimal angle θ_0^{opt} at different heights of the point of the fallingy_1.

y_1, (m)	Analyticalvalue x_{max}, (m)	Root of the equation (15) θ_0^{opt}, (deg)	Numericalvalue θ_0^{opt}, (deg)	Error (%)
40	55.7	59.4°	60°	– 1.0
20	80.1	47.4°	47.8°	– 0.8
0	96.6	41.4°	40.8°	1.5

The second column of Table 3 contains range values calculated analytically by formula (14). The third column of Table 3 contains optimal angle values calculated using equation (15). The fourth column of Table 3 contains optimal angle values from the integration of the equations of system (1). The fifth column presents the error of the calculation of the optimal angle in the percentage. The error does not exceed 2 %. Formula (15) gives a sufficiently exact value of the optimal angle in a wide range of the height of the drop point.

CONCLUSION

The proposed approach based on the use of analytic formulae makes it possible to simplify significantly a qualitative analysis of the motion of a projectile with the air drag taken into account. All basic parameters of motion and various problems of optimization are described by simple analytical formulae containing elementary functions.Moreover, numerical values of the sought variables are determined with an acceptable accuracy. It can be implemented even on a standard calculator. Lately some authors [17–19] have used the Lambert W function to study the projectile motion with resistance. But this relatively "new" function is not available on a calculator.Special algorithms are required to compute this function. Thus, proposed formulae make it possible to study projectile motion with quadratic drag force even for first-year undergraduates. In conclusion, we hope that efforts to obtain an analytical solution to this problem will be continued and will get new exact and efficient solutions.

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