

# Extended Jacobian Elliptic Function Expansion Method and Its Applications in Mathematical Physics

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## Abstract

In this work, an extended Jacobian elliptic function expansion method is proposed for constructing the exact solutions of nonlinear evolution equations. The validity and reliability of the method are tested by its applications to some nonlinear evolution equations which play an important role in mathematical physics.

## Keywords

Extended Jacobian elliptic function expansion method; The nonlinear Phi- Four equation; Traveling wave solutions.

## 1- Introduction

No one can deny the important role which played by the nonlinear partial differential equations in the description of many and a wide variety of phenomena not only in physical phenomena, but also in plasma, fluid mechanics, optical fibers, solid state physics, chemical kinetics and geochemistry phenomena. So that, during the past five decades, a lot of method was discovered by a diverse group of scientists to solve the nonlinear partial differential equations. Such methods are tanh - sech method [1]-[3], extended tanh - method [4]-[6], sine - cosine method [7]-[9], homogeneous balance method [10, 11], F-expansion method [12]-[14], exp-function method [15, 16], trigonometric function series method [17],  $\left(\frac{G'}{G}\right)$ -expansion method [18]-[21], the exp  $(-\varphi(\xi))$ -expansion method [22]-[24], extended Jacobi elliptic function method [25]-[28], the exp  $(-\varphi(\xi))$ -expansion method [26]-[28] and so on. The objective of this article is to apply an extended Jacobian elliptic function expansion for finding the exact traveling wave solution of the nonlinear Phi- Four equation which play an important role in mathematical physics.

The rest of this paper is organized as follows: In Section 2, we give the description of an extended Jacobian elliptic function expansion. In Section 3, we use this method to find the exact solutions of the nonlinear evolution equations pointed out above. In Section 5, we give the physical interpretations of the solutions. In Section 5, conclusions are given.

## 2- Description of method

Consider the following nonlinear evolution equation

$$P(u, u_t, u_x, u_{tt}, u_{xx}, \dots) = 0, \quad (2.1)$$

since,  $P$  is a polynomial in  $u(x, t)$  and its partial derivatives. In the following, we give the main steps of this method

**Step 1.** We use the traveling wave solution in the form

$$u(x, t) = u(\xi), \quad \xi = x - ct, \quad (2.2)$$

where  $c$  is a positive constant, to reduce Eq. (2.1) to the following ODE:

$$p(u, u', u'', u''', \dots) = 0, \quad (2.3)$$

where  $P$  is a polynomial in  $u(\xi)$  and its total derivatives, while  $\left(u' = \frac{du}{d\xi}\right)$ .

**Step 2.** Making good use of ten Jacobian elliptic functions we assume that (2.3) have the solutions in these forms:

$$u(\xi) = a_0 + \sum_{j=1}^N f_i^{j-1}(\xi) [a_j f_i(\xi) + b_j g(\xi)], \quad i = 1, 2, 3, \dots \quad (2.4)$$

With



$$\begin{aligned}f_1(\xi) &= sn \xi, \quad g_1(\xi) = cn \xi, \\f_2(\xi) &= sn \xi, \quad g_2(\xi) = dn \xi, \\f_3(\xi) &= ns \xi, \quad g_3(\xi) = cs \xi, \\f_4(\xi) &= ns \xi, \quad g_4(\xi) = ds \xi, \\f_5(\xi) &= sc \xi, \quad g_5(\xi) = nc \xi, \\f_6(\xi) &= sd \xi, \quad g_6(\xi) = nd \xi,\end{aligned}\tag{2.5}$$

Where  $sn \xi, cn \xi, dn \xi$  are the Jacobian elliptic sine function, the Jacobian elliptic cosine function and the Jacobian elliptic function of the third kind and other Jacobian functions which are denoted by Glaisher's symbols and are generated by these three kinds of functions, namely:

$$\begin{aligned}ns \xi &= \frac{1}{sn \xi}, \quad nc \xi = \frac{1}{cn \xi}, \quad nd \xi = \frac{1}{dn \xi}, \quad sc \xi = \frac{cn \xi}{sn \xi}, \\cs \xi &= \frac{sn \xi}{cn \xi}, \quad ds \xi = \frac{dn \xi}{sn \xi}, \quad sd \xi = \frac{sn \xi}{dn \xi},\end{aligned}\tag{2.6}$$

those have the relations

$$\begin{aligned}sn^2 + cn^2 &= 1, \quad dn^2 \xi + m^2 sn^2 \xi = 1, \quad ns^2 \xi = 1 + cs^2 \xi, \\ns^2 \xi &= m^2 + ds^2 \xi, \quad sc^2 \xi + 1 = nc^2 \xi, \quad m^2 sd^2 + 1 = nd^2 \xi,\end{aligned}\tag{2.7}$$

with the modulus  $m$  ( $0 < m < 1$ ). In addition we know that

$$\frac{d}{d\xi} sn \xi = cn \xi dn \xi, \quad \frac{d}{d\xi} cn \xi = -sn \xi dn \xi, \quad \frac{d}{d\xi} dn \xi = -m^2 sn \xi cn \xi.\tag{2.8}$$

The derivatives of other Jacobian elliptic functions are obtained by using Eq. (1.8). To balance the highest order linear term with nonlinear term we define the degree of  $u$  as  $D[u] = n$  which gives rise to the degrees of other expressions as

$$D\left[\frac{d^q u}{d\xi^q}\right] = n + q, \quad D\left[u^p \left(\frac{d^q u}{d\xi^q}\right)^s\right] = np + s(n + q).\tag{2.9}$$

According the rules, we can balance the highest order linear term and nonlinear term in Eq. (2.3) so that  $n$  in Eq. (2.4) can be determined. In addition we see that when  $(m \rightarrow 1)$ ,  $(sn \xi, cn \xi, dn \xi)$  degenerate as  $(\tanh \xi, \operatorname{sech} \xi, \operatorname{sech} \xi)$  respectively, while when therefore Eq. (2.5) degenerate as the following forms

$$\begin{aligned}u(\xi) &= a_0 + \sum_{j=1}^N \tanh^{j-1}(\xi) [a_j \tanh(\xi) + b_j \operatorname{sech}(\xi)], \\u(\xi) &= a_0 + \sum_{j=1}^N \coth^{j-1}(\xi) [a_j \coth(\xi) + b_j \operatorname{csch}(\xi)], \\u(\xi) &= a_0 + \sum_{j=1}^N \tanh^{j-1}(\xi) [a_j \tan(\xi) + b_j \sec(\xi)], \\u(\xi) &= a_0 + \sum_{j=1}^N \coth^{j-1}(\xi) [a_j \cot(\xi) + b_j \csc(\xi)],\end{aligned}\tag{2.10}$$

Therefore the extended Jacobian elliptic function expansion method is more general than sine-cosine method, the tan-function method and Jacobian elliptic function expansion method.

## Application

Here, we will apply, an extended Jacobian elliptic function expansion method described in Sec.2 to find the exact traveling wave solutions and the solitary wave solutions of the nonlinear Phi- Four equation [32, 33]. We consider the nonlinear Phi-Four equation.



$$u_{tt} - \alpha u_{xx} - u + u^3 = 0, \quad (3.1)$$

where  $\alpha$  and  $\beta$  are real constants. By using the wave transformation  $u(\xi) = u(x, t)$ , since

$(\xi = x - t)$ , we get:

$$(1 - \alpha)u'' - u + u^3 = 0. \quad (3.2)$$

Balancing between the highest order derivatives and nonlinear terms appearing in Eq. (3.2) ( $u^3$  and  $u'' \Rightarrow m = 1$ ). So that, by using Eq. (2.4) we get the formal solution of Eq. (3.2)

$$v(\xi) = a_0 + a_1 \operatorname{sn} \xi + b_1 \operatorname{cn} \xi. \quad (3.3)$$

Substituting Eq. (3.5) and its derivative into Eq. (3.2) and collecting all term with the same power of  $\operatorname{sn}^3, \operatorname{sn}^2 \operatorname{cn}, \operatorname{sn}^2, \operatorname{sn} \operatorname{cn}, \operatorname{sn}, \operatorname{cn}, \operatorname{sn}^0$  we get:

$$\begin{cases} (1 - \alpha)(2a_1 m^2) + (a_1^3 - 3a_1 b_1^2) = 0, \\ (1 - \alpha)(2m^2 b_1) + (3a_1^2 b_1 - b_1^3) = 0, \\ (3a_0 a_1^2 - 3a_0 b_1^2) = 0, \\ (6a_0 a_1 b_1) = 0, \\ (1 - \alpha)(-m^2 a_1 - a_1) - (a_1) + (3a_0^2 a_1 + 3a_1 b_1^2) = 0, \\ (1 - \alpha)(-b_1) - (b_1) + (3a_0^2 b_1 + b_1^3) = 0, \\ -(a_0) + (a_0^3 + 3a_0 b_1^2) = 0, \end{cases} \quad (3.4)$$

Solving above system by using maple 16, we get:

**Case1:**

$$\alpha = \frac{2(m^2 - 1)}{2m^2 - 1}, a_0 = a_1 = 0, b_1 = \pm m \sqrt{\frac{2}{2m^2 - 1}}.$$

Thus the exact traveling wave solution is

$$u(\xi) = \pm m \sqrt{\frac{2}{2m^2 - 1}} \operatorname{cn}(\xi). \quad (3.5)$$

When  $(m \rightarrow 1)$  we get:

$$u(\xi) = \pm \sqrt{2} \operatorname{sech}(\xi). \quad (3.6)$$

**Case2:**

$$\alpha = \frac{m^2 - 4}{m^2 - 2}, a_0 = 0, a_1 = \pm m \sqrt{\frac{1}{2 - m^2}}, b_1 = \pm m \sqrt{\frac{2}{2m^2 - 1}}.$$

Thus the exact traveling wave solution is

$$u(\xi) = \pm m \sqrt{\frac{1}{2 - m^2}} \operatorname{sn}(\xi) \pm m \sqrt{\frac{2}{2m^2 - 1}} \operatorname{cn}(\xi). \quad (3.7)$$

When  $(m \rightarrow 1)$  we get:

$$u(\xi) = \pm \tanh(\xi) \pm \sqrt{2} \operatorname{sech}(\xi). \quad (3.8)$$

**Case3:**

$$\alpha = \frac{2 + m^2}{1 + m^2}, a_0 = 0, a_1 = \pm m \sqrt{\frac{2}{m^2 + 1}}, b_1 = 0.$$

Thus the exact traveling wave solution is

$$u(\xi) = \pm m \sqrt{\frac{2}{m^2 + 1}} \operatorname{sn}(\xi). \quad (3.9)$$

When  $(m \rightarrow 1)$  we get:

$$u(\xi) = \pm \tanh(\xi). \quad (3.10)$$

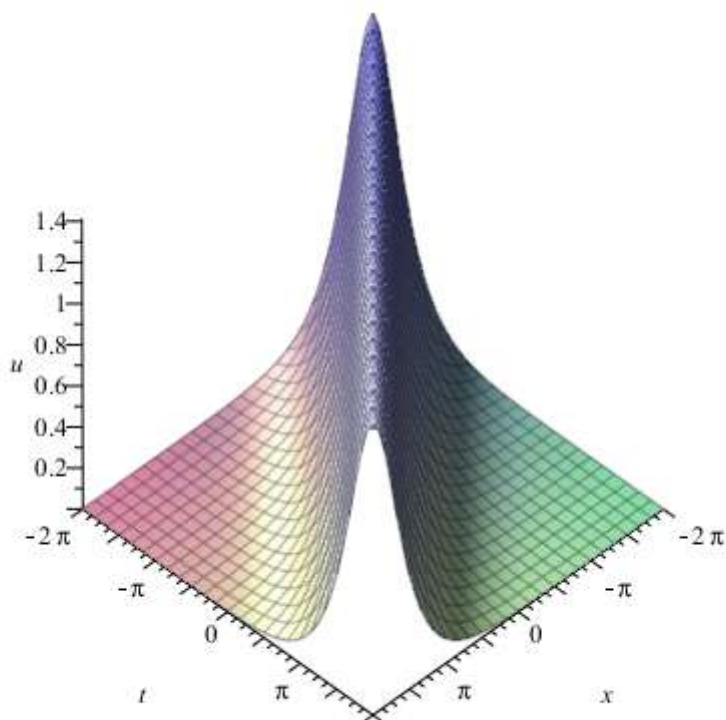


• **Note That:**

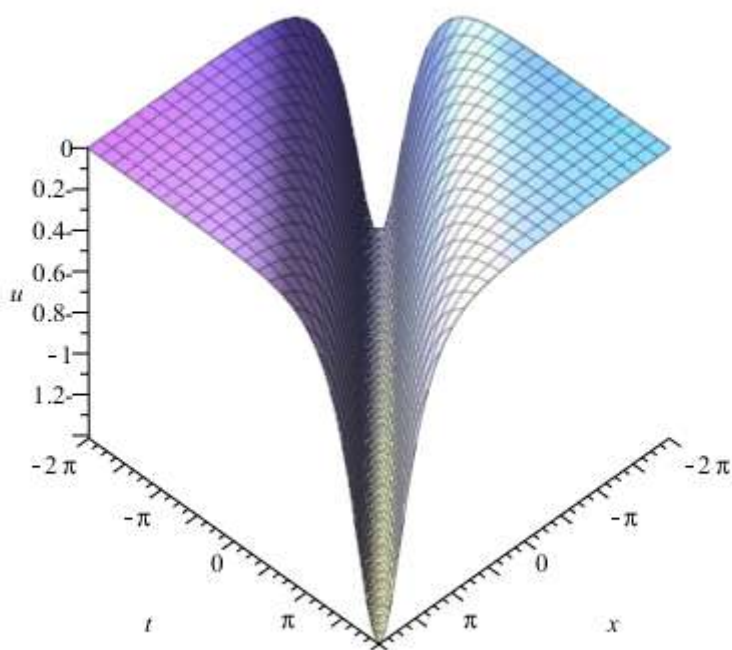
All the obtained results have been checked with Maple 16 by putting them back into the original equation and found correct.

**Figures:**

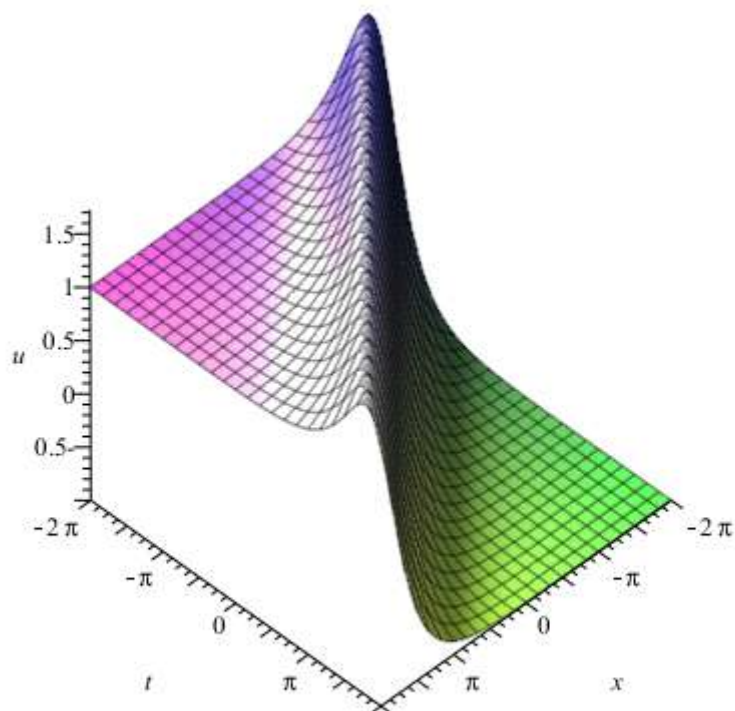
When  $(x = -2\pi..2\pi, t = -2\pi..2\pi)$



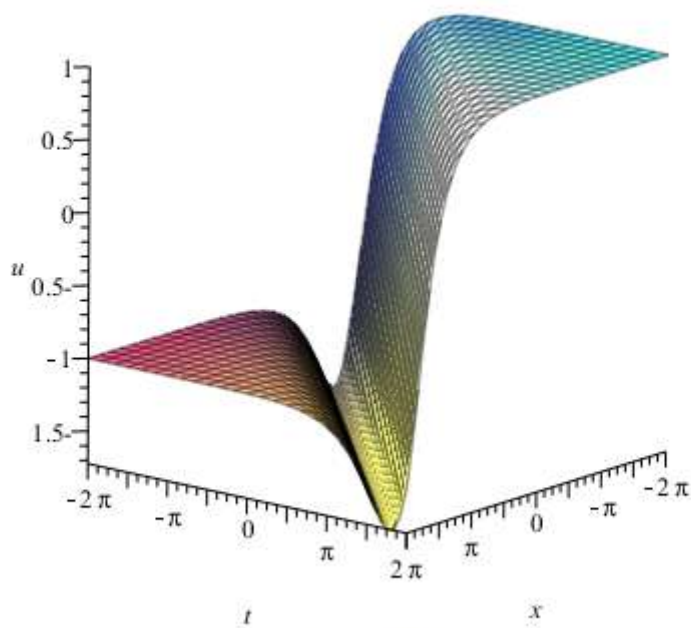
**Plot of (3.6)**



**Plot of (3.6)**

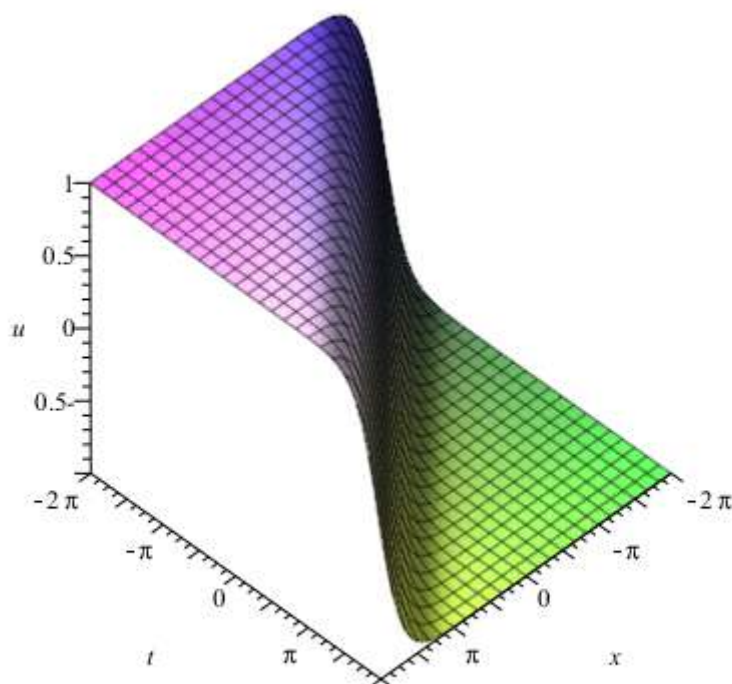


Plot of (3.8)

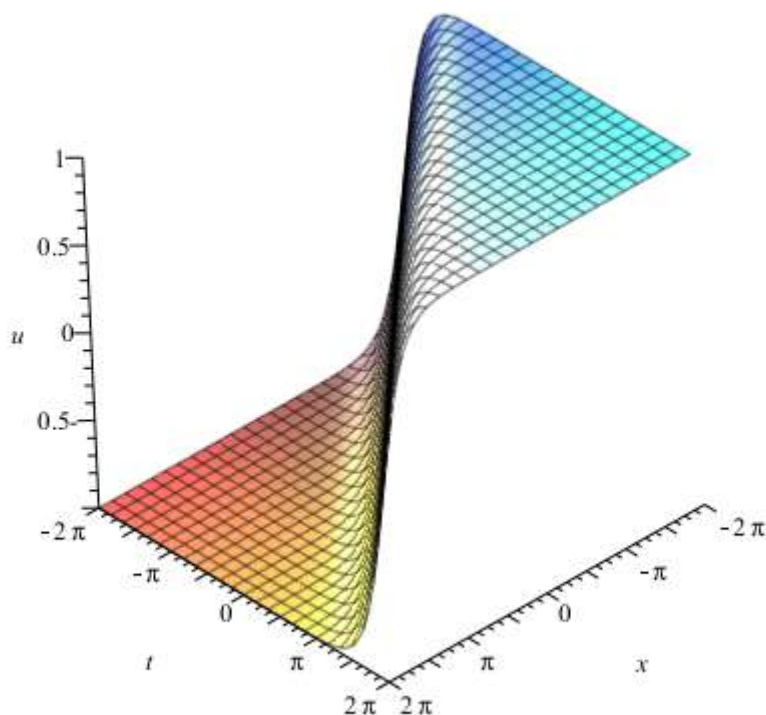


Plot of (3.8)





Plot of (3.10)



Plot of (3.10)

### 3- Conclusion

An extended Jacobian elliptic function expansion method has been applied in this paper to find the exact traveling wave solutions and then the solitary wave solutions of the nonlinear Phi- Four equation. Let us compare between our results obtained in the present article with the well-known results obtained by other authors using different methods as follows: Our results of the nonlinear Phi- Four equation are new and different from those obtained in [32,33], and figures show the solitary traveling wave solution of the nonlinear Phi- Four equation. We can conclude that an extended Jacobian elliptic



function expansion method is a very powerful and efficient technique in finding exact solutions for wide classes of nonlinear problems and can be applied to many other nonlinear evolution equations in mathematical physics. Another possible merit is that the reliability of the method and the reduction in the size of computational domain give this method a wider applicability

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This research received no specific grant from any funding agency in the public, commercial, or not-for-profit sectors. The author did not have any competing interests in this research.

#### 5- Author's contributions

All parts contained in the research carried out by the researcher through hard work and a review of the various references and contributions in the field of mathematics and the physical Applied

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