



The properties of Bessel-type fractional derivatives and integrals

V. R. Lakshmi Gorty
SVKM's NMIMS, MPSTME,
Vile Parle (W), Mumbai-400056, India
vr.lakshmigorty@nmims.edu

ABSTRACT

The fractional integrals of Bessel-type Fractional Integrals from left-sided and right-sided integrals of fractional order is established on finite and infinite interval of the real-line, half axis and real axis. The Bessel-type fractional derivatives are also established. The properties of Fractional derivatives and integrals are studied. The fractional derivatives of Bessel-type of fractional order on finite of the real-line are studied by graphical representation. Results are direct output of the computer algebra system coded from MATLAB R2011b.

Indexing terms/Keywords

Fractional derivatives, Bessel type functions, fractional order.

Academic Discipline And Sub-Disciplines

Integral transforms, fractional calculus.

SUBJECT CLASSIFICATION

30, 33, 35

1. INTRODUCTION

In [VI], Fractional integrals were defined for $\phi(x) \in L_1(a, b)$ by

$$(I_{a+}^{\alpha} \phi)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} \phi(t) dt, \quad x > a \quad (1.1)$$

$$(I_{b-}^{\alpha} \phi)(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^b (t-x)^{\alpha-1} \phi(t) dt, \quad x < b \quad (1.2)$$

where $\alpha > 0$ (α being the order). These integrals are also known as the Riemann-Liouville fractional integrals or the left-sided and right-sided fractional integrals, respectively. The integrals given in (1.1) and (1.2) are extensions to half and (or) whole axis finite interval $[a, b]$. These may be used on the half axis (a, ∞) and $(-\infty, b)$, respectively, subject to the variable limit of integration.

The Riemann-Liouville fractional derivatives are [IV]

$$\begin{aligned} (D_{a+}^{\alpha} y)(x) &= \left(\frac{d}{dx} \right)^n (I_{a+}^{n-\alpha} y)(x) \\ &= \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx} \right)^n \int_a^x \frac{y(t)}{(x-t)^{\alpha-n+1}} dt \quad (n = [\Re(\alpha) + 1; x > a]) \end{aligned} \quad (1.3)$$

and

$$\begin{aligned} (D_{b-}^{\alpha} y)(x) &= \left(-\frac{d}{dx} \right)^n (I_{b-}^{n-\alpha} y)(x) \\ &= \frac{1}{\Gamma(n-\alpha)} \left(-\frac{d}{dx} \right)^n \int_x^b \frac{y(t)}{(t-x)^{\alpha-n+1}} dt \quad (n = [\Re(\alpha) + 1; x < b]) \end{aligned} \quad (1.4)$$

In [II] authors introduced the Fractional Calculus and Fractional Differential Equations. In [III], new integral transform and associated distributions were introduced.



2. BESSEL-TYPE OF FRACTIONAL ORDER FUNCTION ON FINITE OR INFINITE INTERVAL OF THE HALF-AXIS

Let (a, b) ($0 \leq a < b \leq \infty$) be a finite or infinite interval of the half-axis \mathbb{R}^+ . Considering the left-sided and right-sided integrals of fractional order $\text{Re}(p) > 0; \text{Re}(q) > 0$ defined analogous as in [1]

$$(I_{a+}^p f)(x) = \frac{1}{\Gamma(p)} \int_a^x \{(x-t)^{p-1} t^p J_\nu(t)\} f(t) dt. \quad (a < x < b) \quad (2.1)$$

$$(I_{b-}^q f)(x) = \frac{1}{\Gamma(q)} \int_x^b \{(t-x)^{q-1} x^q J_\mu(t)\} f(t) dt. \quad (a < x < b) \quad (2.2)$$

respectively. When $a = 0$ and $b = \infty$, equation (1.1) and (1.2) are given by

$$(I_{0+}^p f)(x) = \frac{1}{\Gamma(p)} \int_0^x \{(x-t)^{p-1} t^p J_\nu(t)\} f(t) dt. \quad (x > 0) \quad (2.3)$$

$$(I_{0-}^q f)(x) = \frac{1}{\Gamma(q)} \int_x^\infty \{(t-x)^{q-1} x^q J_\mu(t)\} f(t) dt. \quad (x > 0) \quad (2.4)$$

The fractional integrals in more general than (1.3) and (1.4) can be written as

$$(I_{0+,\alpha}^p f)(x) = \frac{1}{\Gamma(p)} \int_0^x \{(x-t)^{p+\alpha-1} t^{p+\alpha} J_\nu(t)\} f(t) dt. \quad (x > 0) \quad (2.5)$$

$$(I_{0-,\beta}^q f)(x) = \frac{1}{\Gamma(q)} \int_x^\infty \{(t-x)^{q+\beta-1} x^{q+\beta} J_\mu(t)\} f(t) dt. \quad (x > 0) \quad (2.6)$$

Property 2.1 If $0 < a < b < \infty$ and $\text{Re}(\nu) > -1$, then from (2.3)

$$(I_{0+}^p ((x-t)^{1-p})) (x) = \frac{x^{\nu+1} {}_1F_2 \left[\frac{\nu+1}{2}, \left[\nu+1, \frac{\nu+3}{2} \right], \frac{-x^2}{4} \right]}{2^\nu \Gamma(\nu+2)}. \quad (2.7)$$

$$(D_{0+}^p ((x-t)^{1-p})) (x) = \frac{\nu J_\nu(t)}{t} - J_{\nu+1}(t). \quad (2.8)$$

For $x > 0$, from (2.4)

$$(I_{0-}^q (t-x)^{1-q}) (x) = 1 - \frac{x^{\mu+1} {}_1F_2 \left[\frac{\mu+1}{2}, \left[\frac{\mu+3}{2}, \mu+1 \right], \frac{-x^2}{4} \right]}{2^\mu \Gamma(\mu+2)}. \quad (2.9)$$

$$(D_{0-}^q ((t-x)^{1-q})) (x) = - \left[J_{\mu+1}(t) - \frac{\mu J_\mu(t)}{t} \right]. \quad (2.10)$$

Property 2.2 Let $\text{Re}(\nu) \leq -1$, then

$$(I_{0+}^p ((x-t)^{1-p})) (x) = \int_0^x J_\nu(t) dt. \quad (2.11)$$

Property 2.3 For $(\text{Re}(\mu) \leq 0$ and $x > 0$ or $x < 0$) and $x \leq 0$



$$\left(I_{0-}^q (t-x)^{1-q}\right)(x) = \int_x^{\infty} J_{\mu}(t) dt. \quad (2.12)$$

Property 2.4 If $0 < a < b < \infty$ and $\operatorname{Re}(\mu) + \operatorname{Re}(p) > -1$, then from (1.5)

$$\left(I_{0+}^p \left((x-t)^{1-p} t^p\right)\right)(x) = \frac{x^{\nu+p+1} {}_1F_2 \left[\frac{\nu+p+1}{2}, \left[\nu+1, \frac{\nu+p+3}{2} \right], \frac{-x^2}{4} \right]}{2^{\nu} (\nu+p+1) \Gamma(p) \Gamma(\nu+1)}. \quad (2.13)$$

$$\left(D_{0+}^p \left((x-t)^{1-p} t^p\right)\right)(x) = \frac{t^p (\nu+p) J_{\nu}(t)}{t} - t^p J_{\nu+1}(t). \quad (2.14)$$

For $x > 0$, from (1.6)

$$\left(I_{0-}^q (t-x)^{1-q} x^q\right)(x) = x^q - \frac{x^{q+\mu+1} {}_1F_2 \left[\frac{\mu+1}{2}, \left[\frac{\mu+3}{2}, \mu+1 \right], \frac{-x^2}{4} \right]}{2^{\mu} \Gamma(q) \Gamma(\mu+2)}. \quad (2.15)$$

$$\left(D_{0-} \left((t-x)^{1-q} x^q\right)\right)(x) = -x^p \left[J_{\mu+1}(t) - \frac{\mu J_{\mu}(t)}{t} \right]. \quad (2.16)$$

Property 2.5 Let $\operatorname{Re}(\nu) + \operatorname{Re}(p) \leq -1$, then

$$\left(I_{0+}^p \left((x-t)^{1-p} t^p\right)\right)(x) = \frac{1}{\Gamma(p)} \int_0^x t^p J_{\nu}(t) dt. \quad (2.17)$$

Property 2.6 For $(\operatorname{Re}(\nu) \leq 0$ and $x > 0$ or $x < 0$) and $x \leq 0$

$$\left(I_{0-}^q (t-x)^{1-q} x^q\right)(x) = \frac{1}{\Gamma(q)} \int_x^{\infty} x^q J_{\mu}(t) dt. \quad (2.18)$$

Property 2.7 If $\operatorname{Re}(p) > 0, \operatorname{Re}(q) > 0, 0 < a < b < \infty$; then

$$\begin{aligned} D_{0+}^q \left(D_{0+}^p \left((x-t)^{1-p} t^p \right) \right) (x) &= D_{0+}^p \left(D_{0+}^q \left((x-t)^{-q} t^q \right) \right) (x) \\ &= t^{p+q} J_{\nu+1}(t) J_{\mu+1}(t) - t^{p+q-1} (\mu+2p+q-1) J_{\mu}(t) J_{\nu+1}(t) \\ &\quad - t^{p+q-1} (\nu+p) J_{\nu}(t) J_{\mu+1}(t) \\ &\quad + t^{p+q-2} J_{\mu}(t) J_{\nu}(t) (\mu\nu - p - \nu + \mu p + 2\nu p + \nu q + pq + \nu^2 + p^2 - t^2). \end{aligned} \quad (2.19)$$

$$\begin{aligned} D_{0-}^q \left(D_{0-}^p \left((x-t)^{1-p} t^p \right) \right) (x) &= D_{0-}^p \left(D_{0-}^q \left((t-x)^{-q} t^q \right) \right) (x) \\ &= x^{p+q} \left[\frac{J_{\mu+1}(t) J_{\nu+1}(t) - \frac{\nu J_{\nu}(t) J_{\mu+1}(t) + (\mu-1) J_{\nu+1}(t) J_{\mu}(t)}{t}}{t^2} \right. \\ &\quad \left. - \frac{(\nu - \nu^2 - \nu\mu + t^2) J_{\mu}(t) J_{\nu}(t)}{t^2} \right]. \end{aligned} \quad (2.20)$$

If $\operatorname{Re}(\mu) + \operatorname{Re}(p) > -1$, and $x > 0$ or $x < 0$ then



$$\begin{aligned}
 & I_{0+}^q \left(I_{0+}^p \left((x-t)^{1-p} t^p \right) \right) (x) \\
 &= \frac{1}{\Gamma(p)\Gamma(q)} \int_0^x t^p J_\nu(t) \left(\int_0^x t^q J_\mu(t) dt \right) dt + \\
 & \frac{x^{\mu+\nu+p+q+2} {}_1F_2 \left[\frac{\nu+q+1}{2}, \left[\nu+1, \frac{\nu+q+3}{2} \right], \frac{-x^2}{4} \right] {}_1F_2 \left[\frac{\mu+p+1}{2}, \left[\mu+1, \frac{\mu+p+3}{2} \right], \frac{-x^2}{4} \right]}{2^{\mu+\nu} (\mu+p+1)(\nu+q+1)\Gamma(p)\Gamma(q)\Gamma(\mu+1)\Gamma(\nu+1)}.
 \end{aligned} \tag{2.21}$$

Property 2.8 If $\text{Re}(\mu) + \text{Re}(p) \leq -1$, and $x > 0$ or $x < 0$ then

$$\begin{aligned}
 & I_{0+}^q \left(I_{0+}^p \left((x-t)^{1-p} t^p \right) \right) (x) \\
 &= \frac{1}{\Gamma(p)\Gamma(q)} \int_0^x t^p J_\nu(t) \left(\int_0^x t^q J_\mu(t) dt \right) dt \\
 & + \int_0^x t^p J_\nu(t) x^{\mu+q+1} \frac{{}_1F_2 \left[\frac{\mu+q+1}{2}, \left[\mu+1, \frac{\mu+q+3}{2} \right], \frac{-x^2}{4} \right]}{2^\mu (\mu+q+1)\Gamma(\mu+1)\Gamma(p)\Gamma(q)} dt.
 \end{aligned} \tag{2.22}$$

Property 2.9 If $x = 0$; then

$$I_{0+}^q \left(I_{0+}^p \left((x-t)^{1-p} t^p \right) \right) (x) = 0 \tag{2.23}$$

Property 2.10. For $x > 0$;

$$\begin{aligned}
 & I_{0-}^q \left(I_{0-}^p \left((t-x)^{1-p} t^p \right) \right) (x) \\
 &= \frac{x^{p+q} \left[\left\{ 2^\mu \Gamma(\mu+2) - x^{\mu+1} {}_1F_2 \left[\frac{\mu+1}{2}, \left[\mu+1, \frac{\mu+3}{2} \right], \frac{-x^2}{4} \right] \right\} \right. \\
 & \left. - \left\{ 2^\nu \Gamma(\nu+2) - x^{\nu+1} {}_1F_2 \left[\frac{\nu+1}{2}, \left[\nu+1, \frac{\nu+3}{2} \right], \frac{-x^2}{4} \right] \right\} \right]}{2^{\mu+\nu} \Gamma(\mu+1)\Gamma(\nu+2)\Gamma(p)\Gamma(q)}.
 \end{aligned} \tag{2.24}$$

Property 2.11 And for $\text{Re}(\mu) > 0$ and $x = 0$

$$I_{0-}^q \left(I_{0-}^p \left((t-x)^{1-p} t^p \right) \right) (x) = 0 \tag{2.25}$$

3. BESSEL-TYPE OF FRACTIONAL ORDER FUNCTION ON FINITE OR INFINITE INTERVAL OF THE REAL-AXIS

Let $[a, b]$ be a finite or infinite interval of the real-axis \square . Considering the left-sided and right-sided integrals of fractional order $\text{Re}(p) > 0; \text{Re}(q) > 0$.

$$(I_{a+}^p f)(x) = \frac{1}{\Gamma(p)} \int_a^x \left\{ (x-t)^{p-1} t^p J_\nu(t) \right\} f(t) dt. \quad (a < x < b) \tag{3.1}$$



$$(I_{b-}^q f)(x) = \frac{1}{\Gamma(q)} \int_x^b \{(t-x)^{q-1} x^q J_\mu(t)\} f(t) dt. \quad (a < x < b) \quad (3.2)$$

respectively. When $a = 0$ and $b = 1$, equation (3.1) and (3.2) are given by

$$(I_{0+}^p f)(x) = \frac{1}{\Gamma(p)} \int_0^x \{(x-t)^{p-1} t^p J_\nu(t)\} f(t) dt. \quad (x > 0) \quad (3.3)$$

$$(I_{0-}^q f)(x) = \frac{1}{\Gamma(q)} \int_x^1 \{(t-x)^{q-1} t^q J_\mu(t)\} f(t) dt. \quad (x > 0) \quad (3.4)$$

The function which is considered in the study in this paper is $f = (x-a)^p J_\nu(x-a)$, where p is initially arbitrary and $J_\nu(x-a)$ is Bessel function of order ν , with variable 'x' and constant 'a'. It is observed that p must exceed -1 for differintegration to have the properties of the operator as in [V].

Property 3.1 If $[a, b]$ and $\text{Re}(\nu) > -1$, then from (3.3)

$$(I_{0+}^0((x-t))) (x) = \frac{x^{\nu+1} {}_1F_2\left[\frac{\nu+1}{2}, \left[\nu+1, \frac{\nu+3}{2}\right], \frac{-x^2}{4}\right]}{2^\nu \Gamma(\nu+2)}. \quad (3.5)$$

$$(D_{0+}^0((x-t))) (x) = \frac{x^{\nu+1} (\nu+1) {}_1F_2\left[\frac{\nu+1}{2}, \left[\nu+1, \frac{\nu+3}{2}\right], \frac{-x^2}{4}\right]}{2^\nu \Gamma(\nu+2)} - \frac{x^{\nu+2} {}_1F_2\left[\frac{\nu+3}{2}, \left[\nu+2, \frac{\nu+5}{2}\right], \frac{-x^2}{4}\right]}{2^{\nu+2} (\nu+3) \Gamma(\nu+2)} \quad (3.6)$$

Property 3.2 For $x > 0$, equation (3.4) gives

$$(I_{0-}^0(t-x))(x) = \int_x^1 J_\mu(t) dt. \quad (3.7)$$

$$(D_{0-}^0((t-x)))(x) = -J_\mu(x). \quad (3.8)$$

Property 3.3 Let $\text{Re}(\nu) \leq -1$, then

$$(I_{0+}^p((x-t)^{1-p}))(x) = \int_0^x J_\nu(t) dt. \quad (3.9)$$

Property 3.4 For $(\text{Re}(\mu) \leq 0$ and $x > 0$ or $x < 0$) and $x \leq 0$

$$(I_{0-}^q(t-x)^{1-q})(x) = \int_x^1 J_\mu(t) dt. \quad (3.10)$$

Property 3.5 If $0 < a < b < \infty$ and $\text{Re}(\mu) + \text{Re}(p) > -1$, then from (3.3)



$$\left(I_{0+}^p \left((x-t)^{1-p} t^p \right)\right)(x) = \frac{x^{\nu+p+1} {}_1F_2 \left[\frac{\nu+p+1}{2}, \left[\nu+1, \frac{\nu+p+3}{2} \right], \frac{-x^2}{4} \right]}{2^\nu (\nu+p+1) \Gamma(p) \Gamma(\nu+1)}. \quad (3.11)$$

$$\left(D_{0+}^p \left((x-t)^{1-p} t^p \right)\right)(x) = \frac{x^{\nu+p} (\nu+1) {}_1F_2 \left[\frac{\nu+p+1}{2}, \left[\nu+1, \frac{\nu+p+3}{2} \right], \frac{-x^2}{4} \right]}{2^\nu \Gamma(\nu+2) \Gamma(p)} \quad (3.12)$$

$$- \frac{x^{\nu+p+2} {}_1F_2 \left[\frac{\nu+p+3}{2}, \left[\nu+2, \frac{\nu+p+5}{2} \right], \frac{-x^2}{4} \right]}{2^{\nu+1} (\nu+p+3) \Gamma(\nu+2) \Gamma(1-p)}$$

Property 3.6 For $x > 0$, equation (3.4) becomes

$$\left(I_{0-}^q \left((t-x)^{1-q} x^q \right)\right)(x) = \frac{1}{\Gamma(q)} \int_x^1 t^q J_\mu(t) dt. \quad (3.13)$$

$$\left(D_{0-}^q \left((t-x)^{1-q} x^q \right)\right)(x) = - \frac{x^q J_\mu(x)}{\Gamma(1-q)}. \quad (3.14)$$

Property 3.7 Let $\text{Re}(\nu) + \text{Re}(p) \leq -1$, then

$$\left(I_{0+}^p \left((x-t)^{1-p} t^p \right)\right)(x) = \frac{1}{\Gamma(p)} \int_0^x t^p J_\nu(t) dt. \quad (3.15)$$

Property 3.8 For $\text{Re}(\nu)+1 \leq \text{Im}(p)$;

$$\left(I_{0+}^{ip} \left((x-t)^{1-ip} t^{ip} \right)\right)(x) = \frac{1}{\Gamma(ip)} \int_0^x t^{ip} J_\nu(t) dt. \quad (3.16)$$

$$\left(D_{0+}^{ip} \left((t-x)^{1-ip} t^{ip} \right)\right)(x) = - \frac{x^{ip} J_\nu(x)}{\Gamma(1-ip)}.$$

Property 3.9 Let $\text{Re}(\nu)+1 > \text{Im}(p)$; then

$$\left(I_{0+}^{ip} \left((x-t)^{1-ip} t^{ip} \right)\right)(x) = \frac{x^{\nu+ip+1} {}_1F_2 \left[\frac{\nu+ip+1}{2}, \left[\nu+1, \frac{\nu+ip+3}{2} \right], \frac{-x^2}{4} \right]}{2^\nu (\nu+ip+1) \Gamma(ip) \Gamma(\nu+1)}. \quad (3.17)$$

$$\left(D_{0+}^{ip} \left((x-t)^{1-ip} t^{ip} \right)\right)(x) = \frac{x^{\nu+ip} (\nu+1) {}_1F_2 \left[\frac{\nu+ip+1}{2}, \left[\nu+1, \frac{\nu+ip+3}{2} \right], \frac{-x^2}{4} \right]}{2^\nu \Gamma(\nu+2) \Gamma(ip)} \quad (3.18)$$

$$- \frac{x^{\nu+ip+2} {}_1F_2 \left[\frac{\nu+ip+3}{2}, \left[\nu+2, \frac{\nu+ip+5}{2} \right], \frac{-x^2}{4} \right]}{2^{\nu+1} (\nu+ip+3) \Gamma(\nu+2) \Gamma(1-ip)}$$

$$\left(I_{0-}^q \left((t-x)^{1-q} x^q \right)\right)(x) = \frac{1}{\Gamma(qi)} \int_x^1 t^{qi} J_\mu(t) dt. \quad (3.19)$$



$$\left(D_{0-}^{iq} \left((t-x)^{1-iq} t^{iq} \right)\right)(x) = \frac{J_{\mu}(x)}{\Gamma(1-qi)}. \quad (3.20)$$

4. GRAPHICAL REPRESENTATIONS OF DERIVATIVES OF BESSEL-TYPE OF FRACTIONAL ORDER FUNCTION ON FINITE OR INFINITE INTERVAL

In [VI], the authors have published Fractional Integrals and Derivatives, Theory and Applications. In [VII], authors have developed theory and applications of fractional differential equations. The graphical representation studied in this paper are from the developments of fractional integrals and derivatives of Bessel-type and may be used in developing the study in applications of fractional differential equations. The wider applications are found in the Mathematical Physics which also helps researchers for the future scope and developments from this study.

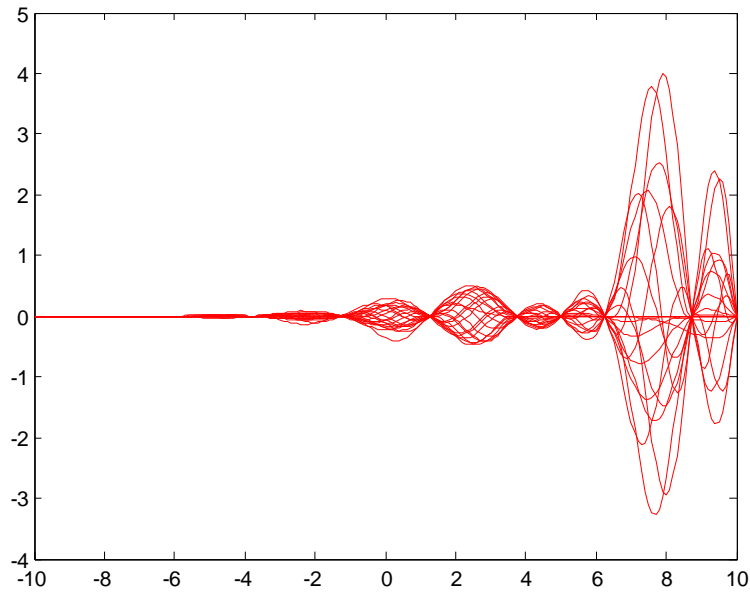


Figure 1 plot of (x,y); $y = (\text{besselj}(\nu, x - a) \cdot (x - a)^p) / \gamma(1 - p)$; $\nu, p < 201$; $x = [-10, 10]$; for $\text{Re}(\nu) + \text{Re}(p) > -1$; $a < 25$.

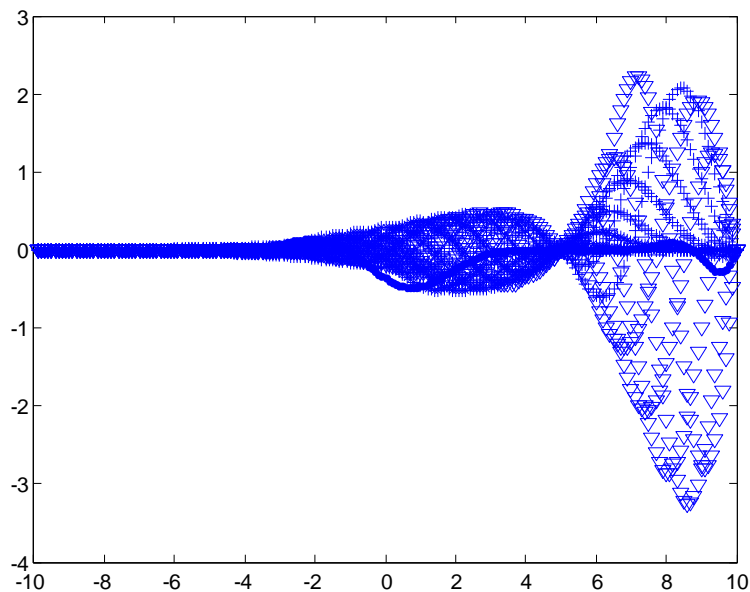


Figure 2 plot (x,y); $y = -((\text{besselj}(\mu, a - x) \cdot (a - x)^q) / \gamma(1 - q))$; $\nu, p < 201$; $x = [-10, 10]$; $a < 20$.

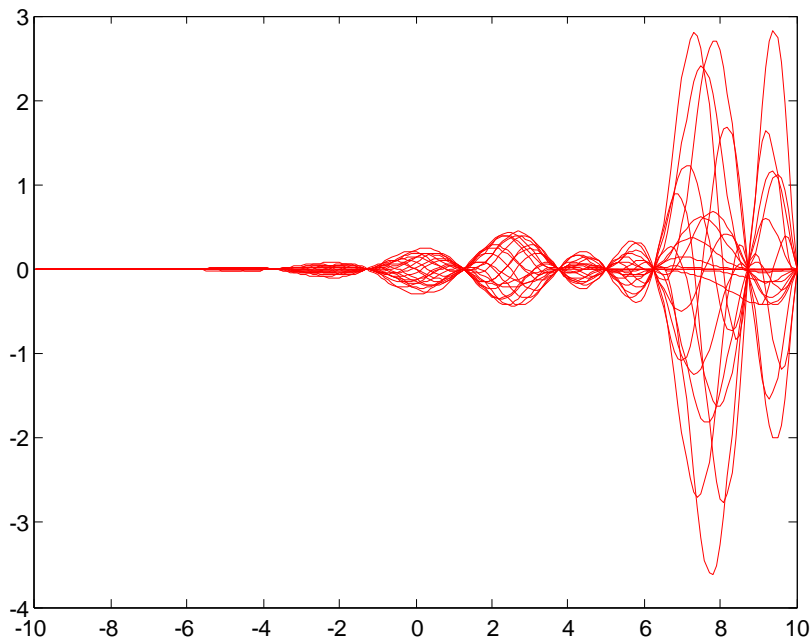


Figure 3 plot (x,y) nu,p<201;x=[-10,10]; for Re(nu)+ Re(p)>-1; a<25

In Figure 3 the derivative is obtained from $(1/\gamma(1-p)) \cdot \text{diff}(\int((x-t)^{(p-1)}) \cdot ((x-t)^{-(p+1)}) \cdot ((t-a)^{(p)}) \cdot \text{besselj}(\nu, t-a), 't', a, x), 'x', 2)$. $y = -((\text{besselj}(\nu+1, x-a) + (\nu \cdot \text{besselj}(\nu, x-a)) / (a-x)) \cdot (x-a)^p - p \cdot \text{besselj}(\nu, x-a) \cdot (x-a)^{(p-1)}) / \gamma(1-p)$;

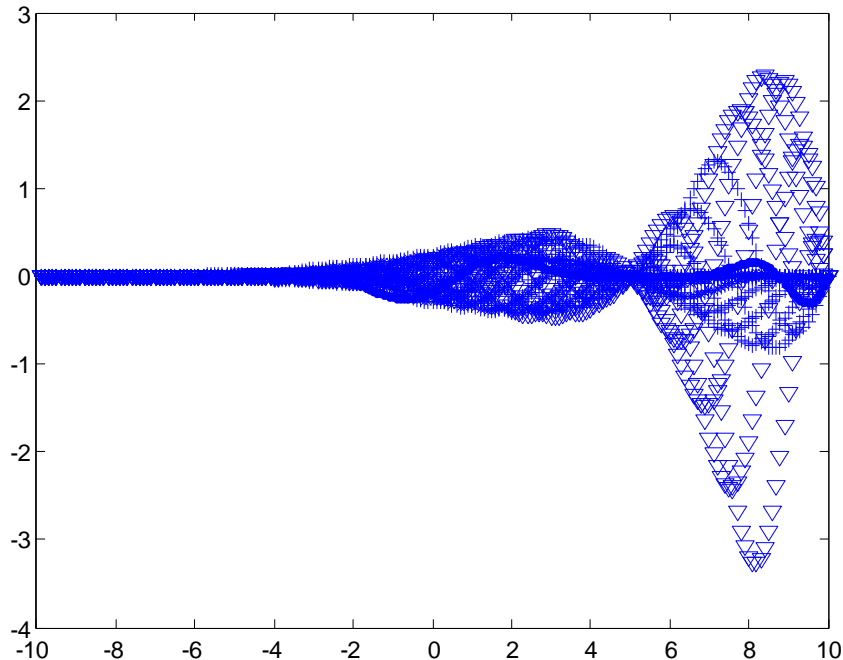


Figure 4 plot (x,y) where nu,p<201;x=[-10,10];a<20.

The derivative of $\text{Dbminus2} = (1/\gamma(1-q)) \cdot \text{diff}(\int((t-x)^{(q-1)}) \cdot ((t-x)^{-(q+1)}) \cdot ((a-t)^{(q)}) \cdot \text{besselj}(\mu, a-t), 't', x, 1), 'x', 2)$ where $y = -((\text{besselj}(\mu + 1, a - x) - (\mu \cdot \text{besselj}(\mu, a - x)) / (a - x)) \cdot (a - x)^q - q \cdot \text{besselj}(\mu, a - x) \cdot (a - x)^{(q-1)}) / \gamma(1 - q)$ in figure 4.

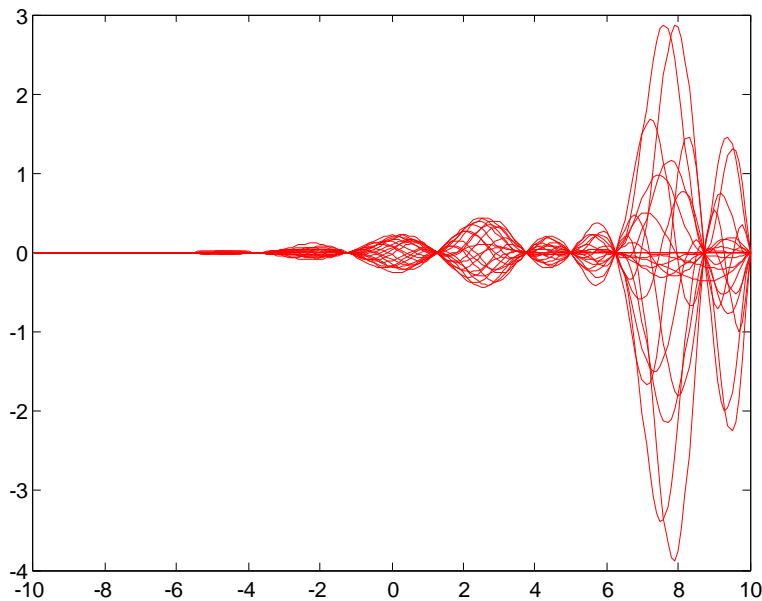


Figure 5 plot (x,y), $\nu, p < 201; x = [-10, 10]$; for $\text{Re}(\nu) + \text{Re}(p) > -1; a < 25$.

Figure 5 is obtained from the calculation of $D_{\text{aplus3}} = \frac{1}{\Gamma(1-p)} \cdot \text{diff}(\int ((x-t)^{(p-1)}) \cdot ((x-t)^{-(p+1)}) \cdot ((t-a)^p) \cdot \text{besselj}(\nu, t-a), 't', a, x), 'x', 3)$ considering $\nu, p < 201; x = [-10, 10]$; for $\text{Re}(\nu) + \text{Re}(p) > -1; a < 25$.

The value of $y = -((x-a)^p \cdot (\text{besselj}(\nu, x-a) - (\nu \cdot \text{besselj}(\nu+1, x-a) + (\nu \cdot \text{besselj}(\nu, x-a)) / (a-x)) / (a-x) + (\nu \cdot \text{besselj}(\nu, x-a)) / (a-x)^2 + (\text{besselj}(\nu+1, x-a) \cdot (\nu+1)) / (a-x) + 2 \cdot p \cdot (\text{besselj}(\nu+1, x-a) + (\nu \cdot \text{besselj}(\nu, x-a)) / (a-x)) \cdot (x-a)^{(p-1)} - p \cdot \text{besselj}(\nu, x-a) \cdot (x-a)^{(p-2)} \cdot (p-1)) / \Gamma(1-p)$;

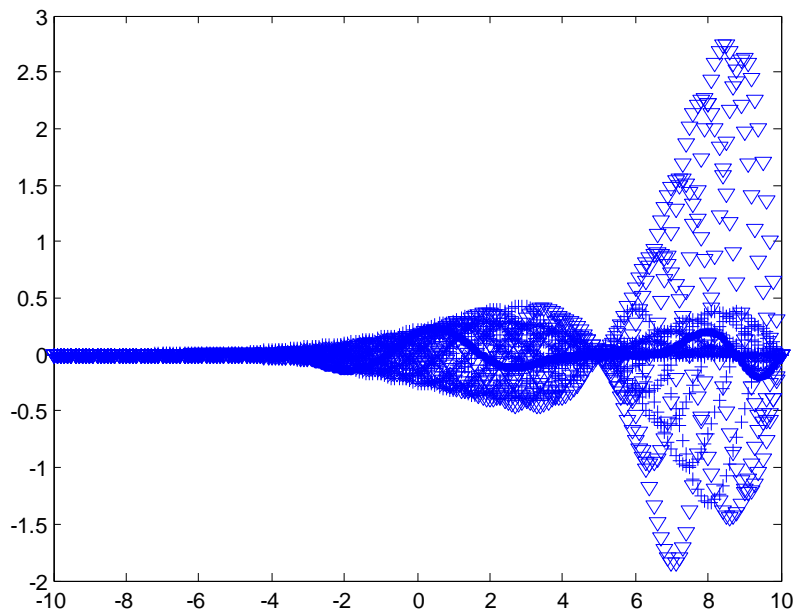


Figure 6 plot (x,y) where $\nu, p < 201; x = [-10, 10]$; $a < 20$.

In Figure 6 $D_{\text{bminus3}} = \frac{1}{\Gamma(1-q)} \cdot \text{diff}(\int ((t-x)^{(q-1)}) \cdot ((t-x)^{-q+1}) \cdot ((a-t)^q) \cdot \text{besselj}(\mu, a-t), 't', x, 1), 'x', 3)$; where

$y = (((a-x)^q \cdot (\text{besselj}(\mu, a-x) + (\mu \cdot (\text{besselj}(\mu+1, a-x) - (\mu \cdot \text{besselj}(\mu, a-x)) / (a-x))) / (a-x) + (\mu \cdot \text{besselj}(\mu, a-x)) / (a-x)^2 - (\text{besselj}(\mu+1, a-x) \cdot (\mu+1)) / (a-x) + 2 \cdot q \cdot (\text{besselj}(\mu+1, a-x) - (\mu \cdot \text{besselj}(\mu, a-x)) / (a-x)) \cdot (a-x)^{(q-1)} - q \cdot \text{besselj}(\mu, a-x) \cdot (a-x)^{(q-2)} \cdot (q-1)) / \Gamma(1-q)$.



5. CONCLUSION

The properties of fractional integrals of Bessel-type Fractional Integrals from left-sided and right-sided integrals of fractional order are established on finite and infinite interval of the real-line, half axis and real axis. The fractional derivatives of Bessel-type of fractional order on finite of the real-line are studied by graphical representation. Results are comprising of results from MATLAB R2011b.

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Author' biography with Photo



Dr. V. R. Lakshmi Gorty has overall twenty years service. She is PhD form University of Pune. Her research interests are: Integral transforms and operational calculus (AMS classification: 44, 42), Wavelet Transforms (33C05), Functional Analysis Generalized functions (distributions): (AMS-classification: 46 F12), Mixed Radix System, Transforms to Kekre's function, Discrete transforms, Interdisciplinary Research, Fractional Calculus and generalizations (AMS classification: 26, 26A33), Fractional Transforms and generalizations. She has so far published more than 62 papers in national, international journals and conferences. She has attended 34 seminars and workshops. She has been a resource person for MATLAB in state and national level and visited many colleges with this Performa. She is recipient of Bharat Jyoti Award, Certificate of Excellence is awarded by India International Friendship Society on 15 April 2015 at New Delhi. She has been awarded Outstanding Scientist Award in December 2015, by VIFRA 2015 at Chennai. The Award intends to publicly recognize an individual's Efforts and Success. The award is given based on the Expert Committee report and APEX Committee recommendations. She is member of many professional bodies like IAIAM, ISTE, INS and IMS. She is also Congress Member of IAENG. She was the co-chair of ICTSM (International Conference) 2011 held at SVKM's NMIMS University, MPSTME. She worked as an editor of Springer series, where the selected papers of peer reviewed were published. She worked coordinator of the International conference ICATE 2013, sponsored by IEEE X-plore digital library. She is editor of Indian journal of science, engineering & technology management, approved with an ISSN 0975-525 X, Techno Path (National Journal) since its first issue (vol. I), January 2009.