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The Geometry of Y^n Space and Its Application to Theory of General Relativity, Derivation of Field Equations

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ABSTRACT

In this paper we study the geometry of Y^n space and applications of this space to general theory of relativity. In Y^n space we obtained analog Ricci - Jacobi identity; We study the hypersutface Y^{n-1} in Y^n space; the geodesic lines equation have been researched; we introduced analog of Darboux theory in case of Y^n space, so it was shown the tensor $\pi_{\alpha\beta}$ can be presented as the sum of two tensors symmetrical and antisymmetrical with property $\pi_{[\alpha\beta]} = \frac{1}{2} g_{ij} S^i_{pq} \xi^p_\beta \xi^q_\alpha \nu^j$. We discussed some partial cases gravitational and electromagnetic interaction, and their connection to geometry structure; we considered stronger electromagnetic field in Y^n space. We derived the general field equations (electromagnetic and gravitational) from the variation principle.

Indexing terms/Keywords

metric, torsion, curvature, connection, Ricci – Jacobi identity, geodesic equation, tangent bundle, covariant derivative, tensor densities, field equation, gravitation, hypersurfaces, electromagnetic field.

Academic Discipline And Sub-Disciplines

Geometry; theory of relativity;

SUBJECT CLASSIFICATION

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TYPE (METHOD/APPROACH)

Geometry methods

INTRODUCTION

1. For describing objective reality we have some 4-dimentional key scientific theories: classical mechanics, general relativity, quantum physics, Maxwell's electromagnetic theory [8-10], Yang–Mills theory and the Standard Model. All this theory have many applications and were made many experiments that prove it correctness except general relativity, of cause we have some phenomenon that convenient explain by this theory, like gravitational lensing, black holes and gravitational wave, but all these phenomenon can be study by another methods, such if we have two objects orbiting a common center of mass (or any other pulsating massive object) and we assume finite interaction speed then we obtain correct low for gravitational wave by classical mechanics. According to Albert Einstein [9] idea general relativity had to be the theory that united electromagnetic and gravitational interactions, but at present days classical theory of general relativity don't include Maxwell's theory as a natural part (we can't count electromagnetic theory in Riemann space as electromagnetic-gravitational theory, because if we assume space without mass we can't obtain Maxwell's theory in the absence of mass), so in reality theory of general relativity is only classical mechanics theory with finite interaction speed, with electromagnetic amendment.

There were many attempts to build higher dimensional theory, that could describe electromagnetic and gravitational interactions consubstantialy, but the problem is open. We believe that the problem can be solved in four dimensional continuum so we don't discuss higher dimensional theory and we only will give resume Einstein–Cartan [4-9, 17, 18] theory which didn't solve the problem.

2. Preliminary consideration and the Einstein–Cartan theory. The Einstein–Cartan theory [4-10, 17, 18] is a theory of gravitation similar to general relativity, but with presumption that the affine connection has vanishing antisymmetric part (torsion tensor), so that the torsion can be coupled to the intrinsic angular momentum (spin) of matter, much in the same way in which the curvature is coupled to the energy and momentum of matter. The theory was first proposed by Elie Cartan in 1922 [4, 5] and developed in the following years then Tom Kibble afresh it in the 1960s, and 1976. Next, in 1982 Penrose has shown that torsion appears when spinors are allowed to be recalled by a complex conformal factor. Then in 1995, the theory has been generalized by F.W. Hehl.

The space-time in Einstein–Cartan theory is four dimensional metric-affine space with a connection that is metric, originally Albert Einstein [8, 10] studied the space that was compound of Riemann and affine spaces (it's so call Einstein theory of gravitation with teleparallelism, where he considered two connections one without and with torsion and he postulated that torsion is associated with electromagnetism like metric with gravity).

According to Andrzej Trautman [18]: "The Einstein-Cartan theory is a viable theory of gravitation that differs very slightly from the Einstein theory; the effects of spin and torsion can be significant only at densities of matter that are very high, but nevertheless much smaller than the Planck density at which quantum gravitational effects are believed to dominate", so modern Einstein–Cartan don't try to unite electromagnetic and gravitational theory, but rather make some amendments to gravitational theory. The field equations of Einstein–Cartan theory come from exactly the same approach as in general relativity, except that a general asymmetric affine connection is assumed rather than the symmetric Levi-Civita connection (i.e., space-time is assumed to have torsion in addition to curvature).

Essential problems Einstein–Cartan theory emerged originally in early A. Einstein works devoted gravitation theory with teleparallelism and concern with space-time structure Einstein–Cartan space. The Einstein–Cartan space has compound structure of Riemann [13] and affine four-dimensional space-time, but A. Einstein assumed that in this space exists "the local n-bein consists of n orthogonal unit vectors with components h_i^a with respect to any Gaussian coordinate system" or "distant parallelism" with $g_{ik} = h_{ia}h_{ka}$, (the space with such geometrical structure is differ from Y^n - spaces) and one of the variants of gravitational theory in space with teleparallelism [2, 9].

The Einstein–Cartan theory is different from theory of teleparallelism but related, then was attempt to improve this theory in "the new teleparallel theory of gravity" with space-time that has a quadruplet of parallel vector fields as the fundamental structure and these parallel vector fields generated the metric tensor (A. Einstein worked on this idea also).

The crucial idea, this theory, was the introduction of a tetrad field, i.e., a set $\{y_1,\dots,y_4\}$ of four vector fields defined on all of set M such that for every $p \in M$, the set $\{y_1(p),\dots,y_4(p)\}$ is a basis of T_pM, where T_pM, denotes the fiber over p of the tangent vector bundle TM. Hence, the four-dimensional space-time manifold M, must be a parallelizable manifold. The tetrad field was introduced to allow the distant comparison of the direction of tangent vectors at different points of the manifold, hence the name distant parallelism. But this attempt was not successful. We believe that problem arose due the space structure, so we introduced space Y^n with different geometrical structure.

The natural approach to obtaining field equations are derived them by varying Einstein action with respect to the metric and torsion independently. Principle of least action is one of natural way to obtain field equations and it unified the gravitational theory with Maxwell theory, but it is not the only one.

The variation principle of least action can be formulated in the form: $\delta\left(W_m + W_g\right) = \mathbf{0}$, where W_m and W_g - action respectively for matter and field values, and we are varying metric and torsion. The scalar density can be taken as $\left(R_{ik} + S_{im}^n S_{kn}^m\right) g^{ik}\sqrt{-g}$, and we postulate that all the variations of the integral $\int\left(R_{ik} + S_{im}^n S_{kn}^m\right) g^{ik}\sqrt{-g}dV$ are zero (analog of this scalar density was introdused in A. Einstein workes). As result we obtain the field equations, this is the general schem for obtaining field equations from principle of least action, result depends on what function we take like Lagrangian and what variables we count independent in many cases its obvious.

In modern Einstein – Cartan theory, they consider Lagrangian in more complicated form, for example $\delta L = L_a \wedge \delta\varphi^a + \frac{1}{2}\tau^{ij}\delta g_{ij} + \delta\theta^i \wedge t_i - \frac{1}{2}\delta\omega_i^j \wedge s_j^i + EX$, but complication of scalar density in variation principle of least action don't bring any novelty in theory and don't solve the problem of obtaining field equations that discrabed gravitational and electromagnetic fields from one point of view.

On the other hand, acoding to A. Einstein "there should be a consensus about the consubstantiality of the gravitational and electromagnetic field" and more essential is to obtain the theory that discribes electromagnetic and gravitational field togather and from one point of view.

Now, to make the our discussion more tangible, we will gave concise description of the geometrical structure of Y^n space, develop the geometry of hypersurfaces Y^{n-1} [22, 23].

The remainder of this paper is organized as follows. In Section 1 consider the geometrical structure of Y^{n-1} space and consist of four subsection dedicated to: general geometry, geodesic, theory of hypersurfaces, identities in accordance; Section 2 discusses the field equations and its application to gravitational theory, consist of two subsection empirical approach and deriving the field equations from the variation principle in accordance; conclusions.

1. The geometry of Y^n space

1.1. Structure of Y^n - space. Let be n - dimensional continuum equipped with a field twice covariant symmetric tensor which is non-degenerate $g_{ik}(M)$, where $Det|g_{ik}| \neq \mathbf{0}$ and $g_{ik} = g_{ki}$, this metric tensor is chosen arbitrarily, but in addition to conditions laid above we demand that manifold was sufficiently smooth.

The connection $\Gamma^i_{jk}(M)$ is a geometric object on a manifold and is subjected to the law of the transformation from one coordinate system x^i to another $x^{i'}$ by the formula:

$$\Gamma^{i'}_{j'k'} = \Gamma^i_{jk} \frac{\partial x^{i'}}{\partial x^i} \frac{\partial x^j}{\partial x^{j'}} \frac{\partial x^k}{\partial x^{k'}} + \frac{\partial^2 x^i}{\partial x^{j'} \partial x^{k'}} \frac{\partial x^{i'}}{\partial x^i}, \qquad (1.1)$$

where the functions Γ^i_{jk} are sufficiently smooth.

Always below we would not require the symmetry of connection. And so if the metric g_{ik} is well defined, then a geometric object Γ^i_{jk} subject to certain requirements, but still there is some freedom in the choice of connection of the space, more precisely, we need to define a torsion tensor:

$$S^i_{jk} \equiv \Gamma^i_{jk} - \Gamma^i_{kj}, \qquad (1.2)$$

then the geometric object Γ^i_{jk} that is generated the connection is determined uniquely. The object Γ^p_{kl}, which generate space connection, is completely determined by two tensors g_{ik} and S^m_{ik}. Therefore the connection Γ^p_{kl} is the sum of a geometric object P^p_{kl} which is composed of derivatives of the metric tensor g_{ik} and tensor L^p_{kl} is compiled of g_{ik} and the tensor S^m_{kl}, by formula $\Gamma^p_{kl} = \mathrm{P}^p_{kl} + L^p_{kl}$.

The main assumption is that a scalar product of two any vectors in parallel transportation along an arbitrary path does not change.

Next we introduce the notation and from the last formula we see that

$$\mathrm{P}^p_{kl} = \frac{1}{2} g^{pi} \left(g_{ik,l} + g_{li,k} - g_{kl,i} \right) \qquad (1.3)$$

is geometric object

$$L^p_{kl} \equiv \frac{1}{2} S^p_{kl} + \frac{1}{2} g^{pi} \left(g_{km} S^m_{li} + g_{lm} S^m_{ki} \right) \qquad (1.4)$$

is tensor.

Remark . It is not difficult to prove the relation:

$$\Gamma^p_{pl} = \frac{1}{2} g_{ip,l} g^{ip} = \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial x^l}, \text{ where } g = \det|g_{ik}|.$$

Then, we consider the difference of first general derivatives:

$$u_{i;l} - u_{l;i} = u_{i,l} - u_{l;i} - S^k_{il} u_k.$$

Similarly, we obtained the difference:

$$u_{i;l;k} - u_{i;k;l} = R^p_{kli} u_p + S^q_{kl} u_{i;q} \qquad (1.5)$$

where R^p_{kli} is curvature tensor. Similarly, we have

$$u^i_{;l;k} - u^i_{;k;l} = -R^i_{klp} u^p + S^q_{kl} u^i_{;q} \qquad (1.6)$$

here we notated:

$$\mathrm{P}_{ikl}^{p} \equiv \mathrm{P}_{li,k}^{p} - \mathrm{P}_{lk,i}^{p} + \mathrm{P}_{qk}^{p}\mathrm{P}_{li}^{q} + \mathrm{P}_{qi}^{p}\mathrm{P}_{lk}^{q} \quad (1.7)$$

is a tensor like the Riemann curvature tensor, composed of the metric tensor and its derivatives.

$$Z_{ikl}^{p} \equiv L_{qk}^{p}L_{li}^{q} - L_{qi}^{p}L_{lk}^{q} \text{ is tensor} \quad (1.8)$$

$$\mathrm{T}_{ikl}^{p} \equiv L_{li,k}^{p} - L_{lk,i}^{p} + \mathrm{P}_{qk}^{p}L_{li}^{q} + \mathrm{P}_{li}^{q}L_{qk}^{p} - \mathrm{P}_{qi}^{p}L_{lk}^{q} - \mathrm{P}_{lk}^{q}L_{qi}^{p} \text{ is tensor.} \quad (1.9)$$

Then we get:

$$\mathrm{T}_{ikl}^{p} = L_{li;k}^{p} - L_{lk;i}^{p} - L_{li}^{q}\Gamma_{qk}^{p} + L_{qi}^{p}\Gamma_{lk}^{q} + L_{lq}^{p}\Gamma_{ik}^{q} + L_{lk}^{q}\Gamma_{qi}^{p} - L_{qk}^{p}\Gamma_{li}^{q} - L_{lq}^{p}\Gamma_{ki}^{q} +$$

$$+\mathrm{P}_{qk}^{p}L_{li}^{q} + \mathrm{P}_{li}^{q}L_{qk}^{q} - \mathrm{P}_{qi}^{p}L_{lk}^{q} - \mathrm{P}_{lk}^{q}L_{qi}^{p} = L_{li;k}^{p} - L_{lk;i}^{p} - L_{li}^{q}L_{qk}^{p} + L_{qi}^{p}L_{lk}^{q} + L_{lk}^{q}L_{qi}^{p} - L_{qk}^{p}L_{li}^{q} + L_{lq}^{p}S_{ik}^{q},$$

since, the absolute derivatives have tensor character it is tensor.

If we denote:

$$\mathrm{M}_{ikl}^{p} \equiv \mathrm{T}_{ikl}^{p} + Z_{ikl}^{p}, \quad (1.10)$$

then we get

$$\mathrm{M}_{ikl}^{p} = L_{li;k}^{p} - L_{lk;i}^{p} + L_{lq}^{p}S_{ik}^{q} + L_{qi}^{p}L_{lk}^{q} - L_{qk}^{p}L_{li}^{q}, \quad (1.11)$$

or $R_{ikl}^{p} = \mathrm{P}_{ikl}^{p} + \mathrm{M}_{ikl}^{p}$.

Remark. Since, torsion tensor is antisymmetric, we have identities: $S_{jp}^{i}S_{ki}^{p} = S_{pk}^{i}S_{ij}^{p}$ and $S_{ip}^{i}S_{jk}^{p} = \mathbf{0};$ then we obtain the equation: $S_{jp}^{i}S_{ki}^{p} + S_{kp}^{i}S_{ij}^{p} + S_{ip}^{i}S_{jk}^{p} = \mathbf{0}$.

1.2. The geodesics in Y^n space. Definition (geodesic). *A geodesic by definition is a curve whose tangent vectors remain parallel when they are transported along this curve.*

Theorem 1. *So that not isotropic line in Y^n space was geodesic it is necessary and sufficient that a variation of the arc of the line δs was equaled to*

$$\delta s = \int_{t_1}^{t_2} g_{ij}S_{pk}^{j}\frac{dx^i}{dt}dx^p\delta x^k.$$

Theorem 2. *For that true Riemannian space with a connection P_{kl}^{p} has shared geodesic lines with Y^n space with connection Γ_{ij}^{k} with torsion tensor S_{ij}^{k}, it is necessary and sufficient that the connections to be differed by tensor:*

$$\Gamma_{ij}^{k} - P_{ij}^{k} = \frac{\mathbf{1}}{\mathbf{2}}\frac{\mathbf{1}}{n+\mathbf{1}}\left(\delta_i^k S_{jl}^{l} + \delta_j^k S_{il}^{l}\right).$$

Remark. In work of A. Einstein "Unified field theory based on riemannian metrics and distant parallelism" definition of geodesic is different, there geodesic defined as the shortest in sense of riemannian metrics.

1.3. The theory of hypersurfaces Y^{n-1} in Y^n. We assume that hypersurfaces Y^{n-1} with coordinates is embedded in Y^n space with coordinates. The hypersurface can be defined by a system of equations:

$$x^i = x^i\left(y^{\mathbf{1}},...,y^{n-\mathbf{1}}\right),$$

where the rank of the matrix $\left[\dfrac{\partial x^i}{\partial y^\alpha}\right]$ is equal to $n-\mathbf{1}$. The metric tensor of $Y^{n-\mathbf{1}}$ is calculated by the formula:

$$a_{\alpha\beta} = g_{ij}\frac{\partial x^i}{\partial y^\alpha}\frac{\partial x^j}{\partial y^\beta}, \qquad (3.1)$$

and the torsion tensor of Y^{n-1} by definition:

$$T^{\gamma}_{\alpha\beta} = a^{\gamma\eta} g_{pq} S^{p}_{ij}\frac{\partial x^i}{\partial y^\alpha}\frac{\partial x^j}{\partial y^\beta}\frac{\partial x^q}{\partial y^\eta}. \qquad (3.2)$$

By this two tensors $a_{\alpha\beta}$ and $T^{\gamma}_{\alpha\beta}$ one could fully defined the geometry of the Y^{n-1} space by itself, without embedded space Y^n.

The connection of Y^{n-1} can be calculated by:

$$G^{\alpha}_{\beta\gamma} = \frac{1}{2}\left(a^{\alpha\eta}\left(a_{\beta\eta,\gamma} + a_{\gamma\eta,\beta} + a_{\beta\gamma,\eta} + a_{\beta\mu}T^{\mu}_{\gamma\eta} + a_{\gamma\mu}T^{\mu}_{\beta\eta}\right) + T^{\alpha}_{\gamma\beta}\right). \qquad (3.3)$$

Let $\xi^i_\alpha = \dfrac{\partial x^i}{\partial y^\alpha}$ and at each point of Y^{n-1} we build the rapper consisting of the vectors $\xi^i_1,\ldots,\xi^i_{n-1},\nu^i$, where $\xi^i_1,\ldots,\xi^i_{n-1}$ linearly independent tangent vectors and ν^i normal vector, defined since the metric exists. Since the normal and any tangent vector is orthogonal we have $g_{ik}\nu^i\xi^k_\alpha = 0$, and write decomposition:

$$\xi^i_{\beta;\alpha} = \pi_{\alpha\beta}\nu^i. \qquad (3.4)$$

Here $\pi_{\alpha\beta}$ is tensor, second fundamental tensor of hypersurfaces Y^{n-1}. Due to, the existence of metric, we have obtained by differentiating $g_{ij}\nu^i\xi^j_\alpha = 0$ by γ:

$$\pi_{\gamma\alpha} = -g_{ij}\nu^i_{;\gamma}\xi^j_\alpha. \qquad (3.5)$$

Similarly, by differentiating $g_{ij}\nu^i\nu^j = 1$ by γ, we obtain:

$$\nu^i_{;\gamma} = -a^{\eta\mu}\pi_{\mu\gamma}\xi^i_\eta = -\pi^{\eta}_{\gamma}\xi^i_\eta. \qquad (3.6)$$

Further, we obtain:

$$\xi^i_{\beta;\chi;\lambda} - \xi^i_{\beta;\lambda;\chi} = -R^i_{klp}\xi^k_\lambda\xi^l_\chi\xi^p_\beta + R^{\sigma}_{\lambda\chi\beta}\xi^i_\sigma + T^{\sigma}_{\lambda\chi}\xi^i_{\beta;\sigma} =$$
$$= \left(\pi_{\chi\beta;\lambda} - \pi_{\lambda\beta;\chi}\right)\nu^i - \left(\pi_{\chi\beta}\pi_{\eta\lambda}a^{\eta\sigma} - \pi_{\lambda\beta}\pi_{\eta\chi}a^{\eta\sigma}\right)\xi^i_\sigma. \qquad (3.7)$$

Equation (3.7) is multiplying by $g_{ij}\xi^j_\alpha$, we have:

$$R_{\alpha\lambda\chi\beta} = R_{iklp}\xi^k_\lambda\xi^l_\chi\xi^p_\beta\xi^i_\alpha - \left(\pi_{\chi\beta}\pi_{\alpha\lambda} - \pi_{\lambda\beta}\pi_{\alpha\chi}\right). \qquad (3.8)$$

Similarly, we derive a formula:

$$\nu^i_{;\chi;\lambda} - \nu^i_{;\lambda;\chi} = -R^i_{klp}\xi^k_\lambda\xi^l_\chi\nu^p + T^{\sigma}_{\lambda\chi}\nu^i_{;\sigma} = \left(\pi_{\eta\lambda;\chi}a^{\eta\sigma} - \pi_{\eta\chi;\lambda}a^{\eta\sigma}\right)\xi^i_\sigma. \qquad (3.9)$$

We contract (3.7) with $g_{ij}\nu^j$, then:

$$-R_{iklp}\xi^k_\lambda\xi^l_\chi\xi^p_\beta\nu^i + T^{\sigma}_{\lambda\chi}\pi_{\sigma\beta} = \pi_{\chi\beta;\lambda} - \pi_{\lambda\beta;\chi}.$$

Formula (3.9) is multiplying by $g_{ij}\xi^j_\alpha$, we concluded that:

$$-R_{iklp}\xi^k_\lambda\xi^l_\chi\nu^p\xi^i_\alpha - T^\sigma_{\lambda\chi}\pi_{\alpha\sigma} = \pi_{\alpha\lambda;\chi} - \pi_{\alpha\chi;\lambda}$$

we can represent this formula like

$$-R_{iklp}\xi^i_\alpha\xi^k_\lambda\xi^l_\chi\nu^p + T^\sigma_{\chi\lambda}\pi_{\alpha\sigma} = \pi_{\alpha\lambda;\chi} - \pi_{\alpha\chi;\lambda}.$$

Thus, we have the two types of formulas. Formula (3.8) does not contain the torsion tensor explicitly, but it is counted in the tensor $\pi_{\alpha\beta}$. In the formula (3.9) the torsion tensor of the hypersurface Y^{n-1} present explicitly and in the form of coefficients of $\pi_{\alpha\beta}$, and appears in the calculation of the covariant derivative.

We denote $\vartheta_{\alpha\beta}$ symmetrical tensor $g_{ij}\nu^i_{;\alpha}\nu^j_{;\beta}$ and we have

$$\vartheta_{\alpha\beta} = g_{ij}\nu^i_{;\alpha}\nu^j_{;\beta} = g_{ij}a^{\eta\mu}\pi_{\mu\alpha}\xi^i_\eta a^{\chi\delta}\pi_{\delta\beta}\xi^j_\chi = g_{ij}\pi^\eta_\alpha\pi^\chi_\beta\xi^i_\eta\xi^j_\chi = a_{\eta\chi}\pi^\eta_\alpha\pi^\chi_\beta = a^{\eta\chi}\pi_{\eta\alpha}\pi_{\chi\beta},$$

or

$$\vartheta_{\alpha\beta} = g_{ij}\nu^i_{;\alpha}\nu^j_{;\beta} = g_{ij}\left(\nu^i_{,\alpha} + \Gamma^i_{lk}\nu^k\xi^l_\alpha\right)\left(\nu^j_{,\beta} + \Gamma^j_{pq}\nu^q\xi^p_\beta\right) =$$
$$= g_{ij}\nu^i_{,\alpha}\nu^j_{,\beta} + g_{ij}\nu^j_{,\beta}\Gamma^i_{lk}\nu^k\xi^l_\alpha + g_{ij}\nu^i_{,\alpha}\Gamma^j_{pq}\nu^q\xi^p_\beta + g_{ij}\Gamma^i_{lk}\Gamma^j_{pq}\nu^k\xi^l_\alpha\nu^q\xi^p_\beta$$

so we see, that asymmetrical part vanished.

We denote $M = \frac{1}{2}a^{\alpha\beta}\pi_{\alpha\beta}$, we have

$$\vartheta_{\alpha\beta} = a_{\eta\chi}\pi^\eta_\alpha\pi^\chi_\beta = a^{\eta\chi}\pi_{\eta\alpha}\pi_{\chi\beta}, \qquad \mathbf{2}Ma^{\eta\chi} = a^{\alpha\beta}\pi_{\alpha\beta}a^{\eta\chi}$$

then

$$a^{\eta\chi}\left(\pi_{\eta\alpha}\pi_{\chi\beta} - a^{\alpha\beta}\pi_{\alpha\beta}\pi_{\eta\chi}\right) = -\frac{\pi}{a}a_{\alpha\beta},$$

$$a^{\eta\chi}\left(\pi_{\eta\alpha}\pi_{\chi\beta} - a^{\alpha\beta}\pi_{\alpha\beta}\pi_{\eta\chi}\right) + \frac{\pi}{a}a_{\alpha\beta} = \mathbf{0},$$

$$\vartheta_{\alpha\beta} - \mathbf{2}M\pi_{\alpha\beta} + \frac{\pi}{a}a_{\alpha\beta} = \mathbf{0}$$

and we obtain

$$\vartheta_{\alpha\beta} = \mathbf{2}M\pi_{\alpha\beta} - \frac{\pi}{a}a_{\alpha\beta}.$$

We calculate

$$\vartheta_{\alpha\beta;\omega} - \vartheta_{\omega\beta;\alpha} = a^{\eta\chi}\pi_{\eta\alpha;\omega}\pi_{\chi\beta} + a^{\eta\chi}\pi_{\eta\alpha}\pi_{\chi\beta;\omega} - a^{\eta\chi}\pi_{\eta\omega;\alpha}\pi_{\chi\beta} - a^{\eta\chi}\pi_{\eta\omega}\pi_{\chi\beta;\alpha} =$$
$$= a^{\eta\chi}\pi_{\chi\beta}\left(\pi_{\eta\alpha;\omega} - \pi_{\eta\omega;\alpha}\right) + a^{\eta\chi}\pi_{\eta\alpha}\pi_{\chi\beta;\omega} - a^{\eta\chi}\pi_{\eta\omega}\pi_{\chi\beta;\alpha}$$

and

$$\vartheta_{\alpha\beta;\omega} - \vartheta_{\omega\beta;\alpha} = a^{\eta\chi}\pi_{\chi\beta}\left(-R_{iklp}\xi^i_\eta\xi^k_\alpha\xi^l_\omega\nu^p + T^\sigma_{\omega\alpha}\pi_{\eta\sigma}\right) + a^{\eta\chi}\pi_{\eta\alpha}\pi_{\chi\beta;\omega} - a^{\eta\chi}\pi_{\eta\omega}\pi_{\chi\beta;\alpha} =$$
$$= -R_{iklp}\xi^i_\eta\xi^k_\alpha\xi^l_\omega\nu^p a^{\eta\chi}\pi_{\chi\beta} + T^\sigma_{\omega\alpha}\pi_{\eta\sigma}a^{\eta\chi}\pi_{\chi\beta} + a^{\eta\chi}\pi_{\eta\alpha}\pi_{\chi\beta;\omega} - a^{\eta\chi}\pi_{\eta\omega}\pi_{\chi\beta;\alpha}.$$

Tensor $\vartheta_{\alpha\beta}$ can be associated with square of angle between normal and adjacent normal $\vartheta_{\alpha\beta}dy^\alpha dy^\beta = d\varphi^2$. So, let in space Y^n with coordinates $x^1,...,x^n$ given the system of non degenerate equations $x^i = x^i\left(y^1,...,y^{n-1}\right)$ so is determined the hypersurface Y^{n-1} and the metric and torsion of Y^{n-1} and since the connection of Y^{n-1}. We can considered the hypersurface like Y^{n-1} space and so we obtain all internal (intrinsic) geometry structure of Y^{n-1}, but formulas $x^i = x^i\left(y^1,...,y^{n-1}\right)$ define more, then internal (intrinsic) geometry structure of Y^{n-1}, they define external geometry of Y^{n-1} (imbedding) as well. External geometry or "how the hypersurface Y^{n-1} is imbedded" define by one of tensors $\pi_{\alpha\beta}$ or $\vartheta_{\alpha\beta}$ which determinate position of hypersurface in Y^n space. As example, internal (intrinsic) geometry Y^{n-1} we considered geodesic in Y^{n-1}.

Geodesic on Y^{n-1}. According to definition geodesic on Y^{n-1} determined by formula

$$\frac{d^2y^\alpha}{ds^2} = -G^\alpha_{\beta\gamma}\frac{dy^\beta}{ds}\frac{dy^\gamma}{ds}.$$

Let a curve: $y^\alpha = y^\alpha(\tau), \quad \tau \in [\tau_1;\tau_2]$. We calculate the variation of length of geodesic δS of the curve S:

$$\delta\left(a_{\alpha\beta}\frac{dy^\alpha}{d\tau}\frac{dy^\beta}{d\tau}\right) = a_{\alpha\beta}\tilde{D}\frac{dy^\alpha}{d\tau}\frac{dy^\beta}{d\tau} + a_{\alpha\beta}\frac{dy^\alpha}{d\tau}\tilde{D}\frac{dy^\beta}{d\tau} = 2a_{\alpha\beta}\frac{dy^\alpha}{d\tau}\tilde{D}\frac{dy^\beta}{d\tau}$$

$$\tilde{D}\frac{dy^\alpha}{d\tau} = \delta\frac{dy^\alpha}{d\tau} + G^\alpha_{\beta\gamma}\frac{dy^\beta}{d\tau}\delta y^\gamma$$

$$D\frac{\delta y^\alpha}{d\tau} = \frac{d}{d\tau}\delta y^\alpha + G^\alpha_{\gamma\beta}\frac{dy^\gamma}{d\tau}\delta y^\beta$$

$$\tilde{D}\frac{\delta y^\alpha}{d\tau} = D\frac{dy^\alpha}{d\tau} + T^\alpha_{\beta\gamma}\frac{dy^\beta}{d\tau}\delta y^\gamma$$

where denotes $\tilde{D}$ the absolute differential at the parameter curves of the family at a constant value τ, and D is absolute differential displacement $d\tau$ curve at a constant parameter of the family, then

$$\delta\left(g_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}\right) = 2g_{ij}\frac{dx^i}{dt}\left(D\frac{\delta x^j}{dt} + S^j_{pk}\frac{dx^p}{dt}\delta x^k\right),$$

$$\delta s = \int_{\tau_1}^{\tau_2} a_{\alpha\beta}\frac{dy^\alpha}{ds}D\delta y^\beta + \int_{\tau_1}^{\tau_2} a_{\alpha\beta}T^\beta_{\gamma\lambda}\frac{dy^\alpha}{d\tau}dy^\gamma\delta y^\lambda =$$

$$\int_{\tau_1}^{\tau_2} D\left(a_{\alpha\beta}\frac{dy^\alpha}{ds}\delta y^\beta\right) - \int_{\tau_1}^{\tau_2} a_{\alpha\beta}D\frac{dy^\alpha}{ds}\delta y^\beta + \int_{\tau_1}^{\tau_2} a_{\alpha\beta}T^\beta_{\gamma\lambda}\frac{dy^\alpha}{d\tau}dy^\gamma\delta y^\lambda$$

since the ends of the variable curve are fixed

$$\delta s=\int_{\tau_1}^{\tau_2}\left(a_{\alpha\beta}T_{\gamma\lambda}^{\beta}\frac{dy^{\alpha}}{d\tau}dy^{\gamma}\delta y^{\lambda}-a_{\alpha\beta}D\frac{dy^{\alpha}}{ds}\delta y^{\beta}\right),$$

suppose considered curve has a fixed length (analytically $\delta s=0$), then we obtain:

$$\delta s=\int_{\tau_1}^{\tau_2}\left(a_{\alpha\beta}T_{\gamma\lambda}^{\beta}\frac{dy^{\alpha}}{d\tau}dy^{\gamma}\delta y^{\lambda}-a_{\alpha\beta}D\frac{dy^{\alpha}}{ds}\delta y^{\beta}\right)=0.$$

By the fundamental lemma of calculus of variations, it follows:

$$a_{\alpha\lambda}T_{\gamma\beta}^{\lambda}\frac{dy^{\alpha}}{d\tau}dy^{\gamma}-a_{\alpha\beta}D\frac{dy^{\alpha}}{ds}=0.$$

The variation of the length of the geodesic is:

$$\delta s=\int_{t_1}^{t_2}a_{\alpha\lambda}T_{\gamma\beta}^{\lambda}\frac{dy^{\alpha}}{d\tau}dy^{\gamma}.$$

We remark, that the geodesics on Y^{n-1} which are determined by connection $G_{\beta\gamma}^{\alpha}$ don't depending on terms that contain tensor $T_{\beta\gamma}^{\alpha}$.

Now, we can construct a semi-geodesics coordinate system in any point of Y^{n-1}, but we can't integrate it.

Application of tensor $\pi_{\alpha\beta}$. *Now we will repeat ours reasoning scheme of construction of hypersurface and attempt more completely to understand the structure of imbedding space* Y^n. *In space* Y^n *with coordinates* $x^1,...,x^n$ *we have the system of nondegenerate equations* $x^i=x^i\left(y^1,...,y^{n-1}\right)$ *which is determined the hypersurface* Y^{n-1}, *then we calculate the metric and torsion of* Y^{n-1} *by formulas (3.1) and (3.2), and connection by (3.3). Then we studied some tensors* $\xi_1^i,..,\xi_{n-1}^i,\nu^i$ *and obtained tensor* $\pi_{\alpha\beta}$, *which is similar to the second tensor of Riemannian hypersurface but not symmetrical* $\pi_{\alpha\beta}-\pi_{\beta\alpha}=g_{ij}S_{pq}^{i}\xi_{\beta}^{p}\xi_{\alpha}^{q}\nu^{j}$.

From theory of surface in R^3, we know, that covariant derivative of second tensor of any enough smooth surface is symmetrical tensor, on another hand, as we can see from $-R_{iklp}\xi_{\lambda}^{k}\xi_{\chi}^{l}\xi_{\beta}^{p}\nu^{i}+T_{\lambda\chi}^{\sigma}\pi_{\sigma\beta}=\pi_{\chi\beta;\lambda}-\pi_{\lambda\beta;\chi}$ tensor $\pi_{\lambda\beta;\chi}$ is not symmetrical.

Formula $\pi_{\alpha\beta}-\pi_{\beta\alpha}=g_{ij}S_{pq}^{i}\xi_{\beta}^{p}\xi_{\alpha}^{q}\nu^{j}$ shows that external properties of geometry (imbedding) of hypersurface can be associated with tensor $\pi_{\alpha\beta}$ and torsion S_{pq}^{i} of imbedding space Y^n is influenced not only tensor $T_{\beta\gamma}^{\alpha}$ but also $\pi_{\alpha\beta}$ and $\vartheta_{\alpha\beta}$.

We associate with Y^{n-1} some coordinate system in Y^n, which denote by $u^1,...,u^{n-1},u^n$ by the rule

$$u^1=y^1,...,u^{n-1}=y^{n-1},u^n=z,$$

with new metric $\tilde{g}_{ik}$ defined by $\tilde{g}_{\alpha\beta}=a_{\alpha\beta}$, $\tilde{g}_{n\alpha}=0$, $\tilde{g}_{nn}=1$. Where z is a geodesic line directed along ν^i - the normal to hypersurface.

Since the rank of the matrix $\left[\frac{\partial x^i}{\partial y^{\alpha}}\right]$ equal $n-1$ suppose that $rank\left[\frac{\partial x^{\gamma}}{\partial y^{\alpha}}\right]>0$ then exist the solution of system of equations

$$x^1=x^1\left(y^1,...,y^{n-1}\right),$$

$$\ldots\ldots,$$

$$x^{n-1} = x^{n-1}\left(y^1,\ldots,y^{n-1}\right),$$

which we denote by

$$u^1 = y^1 = y^1(x^1,\ldots,x^{n-1}),$$

$$\ldots\ldots,$$

$$u^{n-1} = y^{n-1} = y^{n-1}(x^1,\ldots,x^{n-1}),$$

and

$$u^n = z = z(x^1,\ldots,x^{n-1},x^n)$$

herewith cometric tensor $\tilde{g}^{ik}$ equals

$$\tilde{g}^{ik} = \xi^i_\alpha \xi^k_\beta a^{\alpha\beta} + \nu^i \nu^k.$$

Remark. Whereas all ours researches have local character we will mark some remark about Taylor series. Let $\Delta\xi^i$ is infinitesimal vector on the hypersurface Y^{n-1}, we can represent it as an infinite sum of terms - Taylor series and contract this infinitesimal vector with $g_{ij}\nu^j$.

$$g_{ij}\nu^j \Delta\xi^i = \frac{1}{2} g_{ij}\nu^j \xi^i_{\alpha;\beta} Du^\alpha Du^\beta + \frac{1}{6} g_{ij}\nu^j \xi^i_{\alpha;\beta;\gamma} Du^\alpha Du^\beta Du^\gamma + \ldots.$$

since $\pi_{\alpha\beta} = g_{ij}\xi^i_{\beta;\alpha}\nu^j$ and $\pi_{\alpha\beta;\gamma} = g_{ij}\nu^j \xi^i_{\beta;\alpha;\gamma}$ can be written as

$$g_{ij}\nu^j \Delta\xi^i = \frac{1}{2}\pi_{\alpha\beta} Du^\alpha Du^\beta + \frac{1}{6}\pi_{\alpha\beta;\gamma} Du^\alpha Du^\beta Du^\gamma + \ldots.$$

where $Du^\alpha = du^\alpha + G^\alpha_{\beta\gamma} u^\beta du^\gamma$ and G define by (3.3).

Definition. If $\varphi^{\alpha\beta}\omega_{\alpha\beta\gamma\ldots} = 0$ for all $\gamma\ldots$, then we will call tensor $\varphi_{\alpha\beta}$ apolar with tensor $\omega_{\alpha\beta\gamma\ldots}$.

We present tensor $\pi_{\alpha\beta}$ in the form of sum two tensors symmetrical $\pi_{(\alpha\beta)} = \frac{1}{2} g_{ij}\nu^j(\xi^i_{\beta;\alpha} + \xi^i_{\alpha;\beta})$ and antisymmetrical $\pi_{[\alpha\beta]} = \frac{1}{2} g_{ij}\nu^j(\xi^i_{\beta;\alpha} - \xi^i_{\alpha;\beta}) = \ = \frac{1}{2} g_{ij}\nu^j(\xi^i_{\beta,\alpha} - \xi^i_{\alpha,\beta})$ with property $2\pi_{[\alpha\beta]} = g_{ij} S^i_{pq}\xi^p_\beta \xi^q_\alpha \nu^j$.

We denote $a^{\alpha\beta}\pi_{\alpha\beta;\gamma} = (n-1)F_\gamma$ and $\pi^{(\alpha\beta)}\pi_{(\alpha\beta);\gamma} = H_\gamma$, here tensor $\pi^{(\alpha\beta)}$ is constructed from minors of tensor $\pi_{(\alpha\beta)}$ multiplied by C. It is easy to see $a^{\alpha\beta}\pi_{\alpha\beta;\gamma} = (n-1)F_\gamma = a^{\alpha\beta}\pi_{(\alpha\beta);\gamma}$, but connection $G^\gamma_{\alpha\beta}$ isn't symmetrical, so $\pi_{\alpha\beta;\gamma}$ isn't symmetrical at γ.

By applying equality $a_{\alpha\beta}a^{\alpha\beta} = n-1$ we have two equality

$$a^{\alpha\beta}(\pi_{\alpha\beta;\gamma} - a_{\alpha\beta}F_\gamma) = 0,$$

$$\pi^{(\alpha\beta)}(\pi_{(\alpha\beta);\gamma} - \frac{1}{C(n-1)}\pi_{(\alpha\beta)}H_\gamma) = 0.$$

Therefore we obtained two tensor $\pi_{\alpha\beta;\gamma} - a_{\alpha\beta}F_\gamma$ and $\pi_{(\alpha\beta);\gamma} - \frac{1}{C(n-1)}\pi_{(\alpha\beta)}H_\gamma$ which are apolar with $a_{\alpha\beta}$ and $\pi_{(\alpha\beta)}$ correspondingly.

Tensor $\pi_{(\alpha\beta);\gamma} - \frac{1}{C(n-1)}\pi_{(\alpha\beta)}H_\gamma$, which is apolar with tensor $\pi_{(\alpha\beta)}$ (we find constant from apolarity condition), can be symmetrized (in case of space R^n that tensor called Darboux's tensor) and written in the form

$$\theta_{\alpha\beta\gamma} = \pi_{(\alpha\beta);\gamma} - \frac{1}{C(n+1)}(\pi_{(\alpha\beta)}H_\gamma + \pi_{(\beta\gamma)}H_\alpha + \pi_{(\gamma\alpha)}H_\beta)$$

it thrice covariant symmetric tensor of the third order, defined on the hypersurface.

By using the tensor $\theta_{\alpha\beta\gamma}$, we can write the third degree equation

$$\theta_{\alpha\beta\gamma}dy^\alpha dy^\beta dy^\gamma = \pi_{(\alpha\beta);\gamma}dy^\alpha dy^\beta dy^\gamma - \frac{1}{C(n+1)}(\pi_{(\alpha\beta)}H_\gamma + \pi_{(\beta\gamma)}H_\alpha + \pi_{(\gamma\alpha)}H_\beta)dy^\alpha dy^\beta dy^\gamma = 0,$$

$$\pi_{(\alpha\beta);\gamma}dy^\alpha dy^\beta dy^\gamma = \frac{1}{C(n+1)}(\pi_{(\alpha\beta)}H_\gamma + \pi_{(\beta\gamma)}H_\alpha + \pi_{(\gamma\alpha)}H_\beta)dy^\alpha dy^\beta dy^\gamma,$$

it is easy to see that here symmetry isn't essential, we can rewrite

$$\pi_{\alpha\beta;\gamma}dy^\alpha dy^\beta dy^\gamma = \frac{3}{C(n+1)}\pi_{\alpha\beta}dy^\alpha dy^\beta H_\gamma dy^\gamma.$$

1.4. Identities. These identities are analogical to A. Einstein identities which were obtained in his theory of teleparallelism, so:

$$H^{ji} = g^{kp}g^{js}S^i_{sk;p} - g^{kp}g^{js}S^i_{qp}S^q_{sk},$$

$$F^{jp} = g^{kp}g^{js}S^i_{sk;i}.$$

Next we assume that $F_{pq} = S^i_{pq;i}$ and we calculate:

$$H^{ji}_{;i} - F^{ji}_{;i} - g^{kp}g^{js}S^q_{sk}F_{pq} = g^{kp}g^{js}R^t_{ips}S^i_{tk} + g^{kp}g^{js}R^t_{ipk}S^i_{st} - g^{kp}g^{js}R^i_{ipt}S^t_{sk}. \quad (4.1)$$

Then, we are denoting $S^p_{ip} = \varphi_i$ and since $S^p_{ij}S^q_{pq} = \mathbf{0}$, we are obtaining:

$$S^p_{ij;p} = \varphi_{i,j} - \varphi_{j,i}.$$

If $S^p_{ij;p} = \mathbf{0}$, then $\varphi_{i,j} - \varphi_{j,i} = \mathbf{0}$ and hence S^p_{ip} can be expressed in terms of the partial derivative of the scalar $S^p_{ip} = \varphi_i = (\ln\psi)_{,i}$. System (4.1) can be rewritten:

$$H^{ji}_{;i} = g^{kp}g^{js}R^t_{ips}S^i_{tk} + g^{kp}g^{js}R^t_{ipk}S^i_{st} - g^{kp}g^{js}R^i_{ipt}S^t_{sk}, \quad F^{ij} = \mathbf{0}.$$

We can write the tensor $C^{ijk} = g^{pj}g^{qk}S^i_{pq} + g^{pk}g^{qi}S^j_{pq} + g^{pi}g^{qj}S^k_{pq}$ and from this follows at once, that this tensor is antisymmetric in any pair of indices.

We consider

$$C^{ikj}_{;i} = -C^{ijk}_{;i} = -C^{ijk}_{,i} - \frac{\mathbf{1}}{\mathbf{2}}S^j_{pq}C^{kpq} + \frac{\mathbf{1}}{\mathbf{2}}S^k_{pq}C^{jpq} - \Gamma^q_{pq}C^{pkj},$$

and

$$H^{jk}-H^{kj}-F^{jk}=-C^{ijk}_{,i}-\left(\ln\left(\psi\sqrt{-g}\right)\right)_{,i}C^{ikj}.$$

We multiple by $\psi\sqrt{-g}$, have

$$\psi\sqrt{-g}\left(H^{jk}-H^{kj}-F^{jk}\right)=-\psi\sqrt{-g}\left(C^{ijk}_{,i}+\left(\ln\left(\psi\sqrt{-g}\right)\right)_{,i}C^{ikj}\right),$$

$$\psi\sqrt{-g}\left(H^{jk}-H^{kj}-F^{jk}\right)=-\left(\psi\sqrt{-g}C^{ijk}\right)_{,i}.$$

We differentiated the last equality, in view of the antisymmetry of the tensors and we obtained the equality: $\left(\psi\sqrt{-g}\left(H^{jk}-H^{kj}-F^{jk}\right)\right)_{,k}=\mathbf{0}$.

2. The field equations

2.1. Empirical approach. The geometrical theory above was developed only with geometrical and logical origins without any additional assumption or physical hypotheses, below we will make such assumptions.

We won't use principle of least action for deriving field equations based on Y^n space and we don't try to develop "Unified field theory" here (we make it in next section), but we analyze possible applications geometrical theory and make some physical hypotheses. First of we discuss some partial cases.

Newtonian gravitational theory bases on latent assumption that geometry physical world is flat and can be describe by Galilean metric, so if we consider degenerated Y^n space i.e. space where tensors $S^i_{jk}=\mathbf{0}$ and $R^p_{ikl}=\mathbf{0}$ then in this space we can develop all Newton-Hilbert-Maxwell theory.

We remind that according to Albert Einstein proposal: *the free falling gravitating massive bodies follow geodesic line.* If we postulate this proposal we can obtain some results of Newton theory as a consequence. We have another important assumption of Albert Einstein that the geodesic equation of motion can be derived from the field equations for empty space.

In Einstein-Hilbert theory, the metric tensor can be thought of as a generalization of the Newtonian gravitational potential. If we consider Y^n space where tensor $S^i_{jk}=\mathbf{0}$ and tensor R^p_{ikl} can be nonzero then we can obtain Einstein-Hilbert-Maxwell theory, for example Schwarzschild solution and Einstein- Maxwell electromagnetic field equations in form $F^{ik}{}_{;k}=J^i$ and $F_{ij,k}+F_{ki,j}+F_{jk,i}=0$ where exists a 4-potential A_i such that $F_{ij}=A_{i;k}-A_{k;i}$. So, the field equations yield equations, that correspond to the Newton gravitation theory and to Maxwell's electromagnetic field theory. So in cases when Y^n space degenerate we obtain well-known field theory.

Now, we consider pure electromagnetic field in Y^n space presume we stand far enough from mass but there is strong enough electromagnetic field, or we can think that electromagnetic field much stronger that gravitational field and we can neglect gravitational component. We start from identity (4.1) in the form

$$H^{ji}_{;i}-F^{ji}_{;i}-g^{kp}g^{js}S^q_{sk}F_{pq}=0,$$

the most simple possible field equations will be conditions for the tensor S^p_{ij} . We obtain the field equations:

$$H^{ik}=0,$$

$$F^{ik}=0.$$

We denote $F_{ik}=F_{i;k}-F_{k;i}$ and $F_i=\theta-\dfrac{\partial\log\psi}{\partial x^i}=0$, then we have we obtained the equality:

$$\left(\psi\sqrt{-g}\left(H^{jk}-H^{kj}-F^{jk}\right)\right)_{,k}=\mathbf{0}$$

which is derived from presumption of $F_i = \theta_i - \dfrac{\partial \log \psi}{\partial x^i} = 0$ and $F_{ik} = F_{i;k} - F_{k;i}$, here we assumed $S_{ip}^{p} = \varphi_i = \left(\ln \psi\right)_{,i}$, $\varphi_i = \theta_i$.

The form of Maxwell's equations is $F_{ij,k} + F_{ki,j} + F_{jk,i} = 0$ is similar to equation

$$C^{ijk} = g^{pj} g^{qk} S_{pq}^{i} + g^{pk} g^{qi} S_{pq}^{j} + g^{pi} g^{qj} S_{pq}^{k} = \mathbf{0}$$

so it could be postulate as a new electromagnetic field equation, but it is partial case of $\left(\psi\sqrt{-g}\left(H^{jk} - H^{kj} - F^{jk}\right)\right)_{,k} = \mathbf{0}$.

So, if there isn't a gravitational field, that means the g_{ik} is Minkowski metric and there is an electromagnetic field, that means the nonzero S_{jk}^{i}. The torsion can be finding from field equations (presumption that the space has Minkowski metric is too strong since field equations have to be solve together and the metric tensor is included in all field equations, but we simplify situation).

From pure mathematical point of view we can consider the surface S. At point A on S construct a tangent plane P. We choose an arbitrary infinitesimal square ABCD in the plane P with vertex A. From point A on the surface S will draw the geodesic in the direction of AB. We pass along it the distance corresponding parameter equal to the length of AB, get to point B'. Similarly, from A on S draw geodesic towards AD, get into D'. We perform a parallel transportation of vector AD to point B' along the geodesic AB' and draw of geodesic B' along the transportation of this vector, we reach the point C'. Similarly, the vector AB will be move parallel along the AD' and along the transported vector from D' draw geodesic get to C''. If torsion is zero, then C '= C'', and geodesic square up to small higher-order will be closed, otherwise not. In our case, due to the presence of the metric, the length of the gap can be calculated. Let this gap denote by Ψ^k, then $\Psi^k = S_{ij}^{k} A^i B^j \tau^{\mathbf{2}}$, where the parallelogram $A^i\tau$ and $B^j\tau$ shrinks to a point at $\tau \to \mathbf{0}$. In this case, we can write the square of the length: $\left|\Psi\right|^{\mathbf{2}} = g_{pq} S_{ij}^{p} A^i B^j S_{kl}^{q} A^k B^l \tau^{\mathbf{4}}$. These considerations are true only up to the second order relative to the length of square side. If we want more strict result we must consider the component of curvature tensor. Next, this example is true only when length of square side tends to zero i.e. remains very small in other words in general it is a local property.

We are going to discuss physical interpretation of this example. The physical properties of the space-time are defined by the presence of matter (electromagnetic fields and mass) in this space and from the viewpoint of mathematics are described by the geometrical structure of space (torsion and metric tensors). The empty space (without matter) is corresponded the geometric structure of Euclidean space (torsion tensor and curvature tensor are identically equal to zero). Similarly gravitation (mass and without electromagnetic fields) is corresponded the geometric structure of Riemannian space (torsion tensor is identically equal to zero). And similarly the electromagnetism (electromagnetic fields without mass) in corresponded the geometric structure of space with connection (curvature tensor is identically equal to zero). In the last two cases the result is conditional (not strict) because the matter division by the mass and field is conditional.

Metric and torsion tensors are calculated from the differential equations of the field. Hence torsion as the curvature arises from the physical features of the distribution of matter in space-time. Roughly, the same way as the masses leads to curvature space-time, electromagnetism leads to appearance of torsion. But on the other hand from the mathematical point of view if we assume that the space-time embedded in Euclidean space of higher dimension then the appearance of torsion can be explained by violation of the smoothness embedding. Therefore, we can conditionally determine the torsion and curvature by violation of smoothness regardless of the dimension and embedment.

The question then arises: Where we can observe this phenomenon? Though the answer is simple in any system, where electromagnetic field strong enough to change structure of space and bring non zero torsion, but the problem is that in such system appear phenomena bounded with energy and momentum and as consequevce the curvature of space, so for research torsion of space-time, we have to reduce the factors that leading to curvature of space-time and evaluate the torsion of space-time. Although, the square of the length of space-time $\left|\Psi\right|^{\mathbf{2}} = g_{pq} S_{ij}^{p} A^i B^j S_{kl}^{q} A^k B^l \tau^{\mathbf{4}}$ limits to zero when $\tau \to \mathbf{0}$ the effects in three-dimensional world can be very essential, it depends on electromagnetic field.

2.2. The field equations from the variation principle. We will derive the field equations from the variation principle of least action, by varying the function S_{jk}^{i} and g_{ik} independently.

We form the scalar density as $\left(R_{ik} + S_{im}^{n} S_{kn}^{m}\right) g^{ik} \sqrt{-g}$ and postulate that all the variations of the integral:

$$\int\left(R_{ik}+S_{im}^{n}S_{kn}^{m}\right)g^{ik}\sqrt{-g}dV$$

with respect to S_{jk}^{i} and g_{ik} as the independent variables are zero (at the boundaries they do not vary)

$$\delta\int(R_{ik}g^{ik}\sqrt{-g}+S_{im}^{n}S_{kn}^{m}g^{ik}\sqrt{-g})dV=0.$$

Now, we obtain some preliminaries results. For variation the second term we have formula

$$\delta(S_{im}^{n}S_{kn}^{m})=(S_{kl}^{q}\delta_{i}^{p}+S_{il}^{q}\delta_{k}^{p})\delta(S_{pq}^{l}),$$

and

$$\delta\int S_{im}^{n}S_{kn}^{m}g^{ik}\sqrt{-g}dV=\int(S_{kl}^{q}\delta_{i}^{p}+S_{il}^{q}\delta_{k}^{p})\delta(S_{pq}^{l})g^{ik}\sqrt{-g}dV.$$

Recalling that $\Gamma_{ik}^{j}=P_{ik}^{j}+L_{ik}^{j}$ where P_{ik}^{j} are function only of g_{ik} and tensor L_{ik}^{j} are function of S_{jk}^{i} and g_{ik} we have $\delta(\Gamma_{ik}^{j})=\delta(L_{ik}^{j})$. Then, it easy to obtain

$$\delta(L_{ik}^{j})=\frac{1}{2}(\delta_{i}^{p}\delta_{k}^{q}\delta_{l}^{j}+g^{jq}g_{il}\delta_{k}^{p}+g^{jq}g_{kl}\delta_{i}^{p})\delta(S_{pq}^{l}).$$

We can rewrite

$$R_{ik}=S_{in,k}^{n}+\frac{1}{2}(g_{nm,i}g^{nm})_{,k}-(P_{ik}^{n}+L_{ik}^{n})_{,n}+P_{mk}^{n}P_{in}^{m}+P_{mk}^{n}L_{in}^{m}+$$

$$+L_{mk}^{n}P_{in}^{m}+L_{mk}^{n}L_{in}^{m}-P_{mn}^{n}P_{ik}^{m}-P_{mn}^{n}L_{ik}^{m}-L_{mn}^{n}P_{ik}^{m}-L_{mn}^{n}L_{ik}^{m}.$$

and

$$\delta\int R_{ik}g^{ik}\sqrt{-g}dV=\int(-(g^{ik}\sqrt{-g})_{,k}\delta(S_{in}^{n})+(g^{ik}\sqrt{-g})_{,n}\delta(L_{ik}^{n})+$$

$$+[P_{mk}^{n}\delta(L_{in}^{m})+P_{in}^{m}\delta(L_{mk}^{n})+L_{in}^{m}\delta(L_{mk}^{n})+L_{mk}^{n}\delta(L_{in}^{m})-$$

$$-P_{mn}^{n}\delta(L_{ik}^{m})-P_{ik}^{m}\delta(L_{mn}^{n})-L_{mn}^{n}\delta(L_{ik}^{m})-L_{ik}^{m}\delta(L_{mn}^{n})]g^{ik}\sqrt{-g})dV.$$

Then we calculate

$$\delta\int\left(R_{ik}+S_{im}^{n}S_{kn}^{m}\right)g^{ik}\sqrt{-g}dV=\frac{1}{2}\int\Big(-2(g^{pk}\sqrt{-g})_{,k}\delta_{l}^{q}+(g^{pq}\sqrt{-g})_{,l}+(g^{ip}\sqrt{-g})_{,n}g^{nq}g_{il}+$$

$$+(g^{pk}\sqrt{-g})_{,n}g^{nq}g_{kl}+g^{ik}\sqrt{-g}[P_{lk}^{q}\delta_{i}^{p}+P_{mk}^{p}g^{mq}g_{il}+P_{mk}^{n}g^{mq}g_{nl}\delta_{i}^{p}+$$

$$+P_{il}^{p}\delta_{k}^{q}+P_{in}^{m}g^{nq}g_{ml}\delta_{k}^{p}+P_{in}^{p}g^{nq}g_{kl}+L_{lk}^{q}\delta_{i}^{p}+L_{mk}^{p}g^{mq}g_{il}+L_{mk}^{n}g^{mq}g_{nl}\delta_{i}^{p}-$$

$$-P_{ln}^{n}\delta_{i}^{p}\delta_{k}^{q}-P_{mn}^{n}g^{mq}g_{il}\delta_{k}^{p}-P_{mn}^{n}g^{mq}g_{kl}\delta_{i}^{p}-$$

$$-P_{ik}^{p}\delta_{l}^{q}-P_{ik}^{m}g^{pq}g_{ml}-P_{ik}^{p}\delta_{l}^{q}-L_{ln}^{n}\delta_{i}^{p}\delta_{k}^{q}-L_{mn}^{n}g^{mq}g_{il}\delta_{k}^{p}-$$

$$-L_{mn}^{n}g^{mq}g_{kl}\delta_{i}^{p}-L_{ik}^{p}\delta_{l}^{q}-L_{ik}^{m}g^{pq}g_{ml}-L_{ik}^{p}\delta_{l}^{q}+(S_{kl}^{q}\delta_{i}^{p}+S_{il}^{q}\delta_{k}^{p})]\Big)(\delta S_{pq}^{l})dV=0.$$

By using the principle of variational calculus, we have the field equations

$$-2(g^{pk}\sqrt{-g})_{,k}\delta_{l}^{q}+(g^{pq}\sqrt{-g})_{,l}+(g^{ip}\sqrt{-g})_{,n}g^{nq}g_{il}+$$

$$+(g^{pk}\sqrt{-g})_{,n}g^{nq}g_{kl}+g^{ik}\sqrt{-g}[P_{lk}^{q}\delta_{i}^{p}+P_{mk}^{p}g^{mq}g_{il}+P_{mk}^{n}g^{mq}g_{nl}\delta_{i}^{p}+$$

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$$+P^{p}_{il}\delta^{q}_{k}+P^{m}_{in}g^{nq}g_{ml}\delta^{p}_{k}+P^{p}_{in}g^{nq}g_{kl}+L^{q}_{lk}\delta^{p}_{i}+L^{p}_{mk}g^{mq}g_{il}+L^{n}_{mk}g^{mq}g_{nl}\delta^{p}_{i}-$$

$$-P^{n}_{ln}\delta^{p}_{i}\delta^{q}_{k}-P^{n}_{mn}g^{mq}g_{il}\delta^{p}_{k}-P^{n}_{mn}g^{mq}g_{kl}\delta^{p}_{i}-$$

$$-P^{p}_{ik}\delta^{q}_{l}-P^{m}_{ik}g^{pq}g_{ml}-P^{p}_{ik}\delta^{q}_{l}-L^{n}_{ln}\delta^{p}_{i}\delta^{q}_{k}-L^{n}_{mn}g^{mq}g_{il}\delta^{p}_{k}-$$

$$-L^{n}_{mn}g^{mq}g_{kl}\delta^{p}_{i}-L^{p}_{ik}\delta^{q}_{l}-L^{m}_{ik}g^{pq}g_{ml}-L^{p}_{ik}\delta^{q}_{l}+(S^{q}_{kl}\delta^{p}_{i}+S^{q}_{il}\delta^{p}_{k})]=0.$$

and we can rewrite

$$-2(g^{pk}\sqrt{-g})_{,k}\delta^{q}_{l}+(g^{pq}\sqrt{-g})_{,l}+(g^{ip}\sqrt{-g})_{,n}g^{nq}g_{il}+(g^{pk}\sqrt{-g})_{,n}g^{nq}g_{kl}+$$

$$+\sqrt{-g}[P^{q}_{lm}g^{pm}+3P^{p}_{lm}g^{mq}+2P^{n}_{mk}g^{mq}g_{nl}g^{pk}-P^{n}_{ln}g^{pq}-2P^{n}_{mn}g^{mq}\delta^{p}_{l}-$$

$$-2P^{p}_{ik}\delta^{q}_{l}g^{ik}-P^{m}_{ik}g^{pq}g_{ml}g^{ik}+L^{q}_{lk}g^{kp}+L^{p}_{ml}g^{mq}+L^{n}_{mk}g^{mq}g_{nl}g^{pk}-L^{n}_{ln}g^{pq}-$$

$$-2L^{n}_{mn}\delta^{p}_{l}g^{mq}-2L^{p}_{nm}\delta^{q}_{l}g^{nm}-L^{m}_{ik}g^{pq}g_{ml}g^{ik}+2S^{q}_{nl}g^{pn}]=0.$$

So we obtained the first system of field equations by variation of the torsion tensor.

Remark. We could transform it by using formulas $g_{,l}=\dfrac{\partial g}{\partial x^{l}}=gg^{ik}g_{ik,l}=-gg_{ik}g^{ik}_{,l}$ and $(g^{ik}\sqrt{-g})_{,l}=(g^{ik}_{,l}-\dfrac{1}{2}g^{ik}g^{pq}g_{pq,l})\sqrt{-g}$, then thay would be free from $\sqrt{-g}$.

We will derive the field equations from the variation principle of least action, but now by varying the function g_{ik}.

We form the scalar density as $\left(R_{ik}+S^{n}_{im}S^{m}_{kn}\right)g^{ik}\sqrt{-g}$ and postulate that all the variations of the integral by varying the function g_{ik} are equal zero.

By standard calculations, we have:

$$\delta\int\left(R_{ik}+S^{j}_{il}S^{l}_{kj}\right)g^{ik}\sqrt{-g}dV=\int\Big(R_{ik}\sqrt{-g}\delta g^{ik}+R_{ik}g^{ik}\delta\sqrt{-g}+$$

$$+g^{ik}\sqrt{-g}\delta R_{ik}+S^{n}_{im}S^{m}_{kn}\sqrt{-g}\delta g^{ik}+S^{n}_{im}S^{m}_{kn}g^{ik}\delta\sqrt{-g}\Big)dV,$$

and

$$R_{ik}g^{ik}\delta\sqrt{-g}=-\frac{1}{2}R_{pq}g^{pq}g_{ik}\sqrt{-g}\delta g^{ik}.$$

Similarly, we obtain:

$$S^{n}_{im}S^{m}_{kn}g^{ik}\delta\sqrt{-g}=-\frac{1}{2}S^{n}_{pm}S^{m}_{qn}g^{pq}g_{ik}\sqrt{-g}\delta g^{ik}.$$

Now we compute $g^{ik}\sqrt{-g}\delta R_{ik}$ directly by using the definition, thus obtain two types of summands, the first have the standard form $g^{ik}\left(\delta\Gamma^{l}_{ki}\right)_{,l}-g^{ik}\left(\delta\Gamma^{l}_{kl}\right)_{,i}=\left(g^{ik}\delta\Gamma^{l}_{ki}-g^{il}\delta\Gamma^{p}_{lp}\right)_{,l}$ and by Stokes' theorem turns into zeros. The term of the second type exists due to the absence of symmetry connection and then we express the connection coefficients via the metric and torsion, after a calculation, we obtain:

$$g^{ik}\delta R_{ik}=g^{ik}\delta\left(\Gamma^{p}_{qk}\Gamma^{q}_{ip}-\Gamma^{p}_{qp}\Gamma^{q}_{ik}\right)=$$

$$=\frac{1}{4}g^{ik}\delta(2g^{pn}g_{is}S^{m}_{kn}S^{s}_{pm}+g^{pn}g_{km}g^{qt}g_{is}S^{m}_{qn}S^{s}_{pt}-2g^{pn}g_{is}S^{m}_{pm}S^{s}_{kn}-2g^{pn}g_{ks}S^{m}_{pm}S^{s}_{in}=$$

$$=\frac{1}{4}(4S_{im}^{m}S_{kn}^{n}+2S_{nk}^{m}S_{im}^{n}+2g_{is}g^{pn}S_{nk}^{m}S_{pm}^{s}+2g_{ms}g^{pn}S_{pk}^{m}S_{in}^{s}+$$

$$+2g_{km}g_{is}g^{pn}g^{qt}S_{qn}^{m}S_{tp}^{s}+2g_{is}g^{pn}S_{pm}^{m}S_{kn}^{s}+2g_{ks}g^{pn}S_{pm}^{m}S_{in}^{s})\delta g^{ik}.$$

Thus, we have:

$$\delta\int\left(R_{ik}+S_{im}^{n}S_{kn}^{m}\right)g^{ik}\sqrt{-g}dV=\int\left(R_{ik}-\frac{1}{2}g_{ik}R+\right.$$

$$+\frac{1}{2}(2S_{im}^{m}S_{kn}^{n}+S_{nk}^{m}S_{im}^{n}+g_{is}g^{pn}S_{nk}^{m}S_{pm}^{s}+g_{ms}g^{pn}S_{pk}^{m}S_{in}^{s}+$$

$$+g_{km}g_{is}g^{pn}g^{qt}S_{qn}^{m}S_{tp}^{s}+g_{is}g^{pn}S_{pm}^{m}S_{kn}^{s}+g_{ks}g^{pn}S_{pm}^{m}S_{in}^{s})+$$

$$\left.+S_{im}^{n}S_{kn}^{m}-\frac{1}{2}S_{pm}^{n}S_{qn}^{m}g^{pq}g_{ik}\right)\sqrt{-g}\delta g^{ik}dV=0.$$

Then we obtain the conclusions:

$$R_{ik}-\frac{1}{2}g_{ik}R+\frac{1}{2}(2S_{im}^{m}S_{kn}^{n}+S_{kn}^{m}S_{im}^{n}+g_{is}g^{pn}S_{nk}^{m}S_{pm}^{s}+g_{ms}g^{pn}S_{pk}^{m}S_{in}^{s}+$$

$$+g_{km}g_{is}g^{pn}g^{qt}S_{qn}^{m}S_{tp}^{s}+g_{is}g^{pn}S_{pm}^{m}S_{kn}^{s}+g_{ks}g^{pn}S_{pm}^{m}S_{in}^{s})-\frac{1}{2}S_{pm}^{n}S_{qn}^{m}g^{pq}g_{ik}=0.$$

We have obtained the system of field equations where g_{ik} and S_{ik}^{j} are unknown functions, these equations must be solved together. They determine the metric tensor and torsion tensor of space-time for a given arrangement of energy and matter in the space-time.

It is a set of non-linear partial differential equations with regard to g_{ik} and S_{ik}^{j}. The solutions of these E.Q. are the components of the metric and torsion tensors. These metric and torsion together describe the structure of the space-time including the inertial motion of objects and electromagnetic fields in the space-time.

Conclusions

In this paper we compared Einstein – Cartan theory with electro-gravitational theory base on Y^n space (precisely on $Y^{\mathbf{4}}$) and developed electro-gravitational theory base on $Y^{\mathbf{4}}$. For convenience we gave concise resume Einstein–Cartan theory and described the geometrical structure of Y^n space, and development of geometry hypersurfaces in Y^n. We have studied some special cases of the theory of field equations in $Y^{\mathbf{4}}$. We derived from the variation principle the general field equations (electromagnetic and gravitational) base on $Y^{\mathbf{4}}$ space.

For further develop this theory needs more experiments for couple physical phenomena with mathematical predictions.

REFERENCES

I. Agricola I. and Friedrich T. A note on flat metric connections with antisymmetric torsion. Differential Geometry and its Applications, vol. 2, pp. 480–487. – 2010.

II. Bonneau G. Compact Einstein-Weyl four-dimensional manifolds. Classical and Quantum Gravity, vol. 16, pp. 1057–1068. – 1999.

III. Bredies Kristian. Symmetric tensor fields of bounded deformation. Zbl 06226689 Ann. Mat. Pura Appl. vol. 192 (4), N. 5, pp. 815-851. – 2013.

IV. Élie Cartan. "Sur une généralisation de la notion de courbure de Riemann et les espaces à torsion." C. R. Acad. Sci. (Paris) 174, pp. 593–595. –1922.

V. Élie Cartan. "Sur les variétés à connexion affine et la théorie de la relativité généralisée." Part I: Ann. Éc. Norm. 40, 325–412 (1923) and ibid. 41, 1–25 (1924); Part II: ibid. 42, pp. 17–88. – 1925.

VI. Cartan E. and Schouten J. On Riemannian geometries admitting an absolute parallelism. Nederlandse Akademie van Wetenschappen. Proceedings. Series A, vol. 29, pp. 933–946. – 1926.

VII. Cartan E. and Schouten J. On the geometry of the group manifold of simple and semisimple groups. Nederlandse Akademie van Wetenschappen. Series A, vol. 29, pp. 803–815. – 1926.

VIII. Einstein A. The Meaning of Relativity. Princeton Univ. Press. Princeton. – 1921.

IX. Einstein A. Relativity: The Special and General Theory, New York: H. Holt and Company. – 1920.

X. Einstein A. Theorie unitaire de champ physique. Ann. Inst. H. Poincare, №1 pp. 1-24. – 1930.

XI. Hammond, R.T. Torsion gravity. Rep. Prog. Phys. 65: pp. 599–649. – 2002.

XII. Hehl F., McCrea J., Mielke E., Ne'eman Y. Metric-affine gauge theory of gravity: field equations, Noether identities, world spinors, and breaking of dilation invariance. Phys. Reports 258: P. 1–171. – 1995.

XIII. Jost J. Riemannian Geometry and Geometric Analysis. Springer-Verlag, Berlin. – 2005.

XIV. Penrose, R Spinors and torsion in general relativity. Found. of Phys. 13: pp. 325–339. – 1983.

XV. Pedersen H. and Swann A. Riemannian submersions, four-manifolds and Einstein-Weyl geometry. Proceedings of the London Mathematical Society, vol. 66, pp. 381–399. – 1993.

XVI. Sean Dineen. Multivariate calculus and geometry. 3rd ed. Springer Undergraduate Mathematics Series. Berlin, Springer, 259 p., 2014.

XVII. Trautman, A. Spin and torsion may avert gravitational singularities, Nature (Phys. Sci.) P. 242, 7. – 1973.

XVIII. Trautman, A. Einstein-Cartan Theory. – 2006.

XIX. Vargas José G. Differential geometry for physicists and mathematicians. Moving frames and differential forms: From Euclid past Riemann. – 2014.

XX. Sean Dineen. Multivariate calculus and geometry. 3rd ed. Springer Undergraduate Mathematics Series. Berlin, Springer, 259 p. – 2014.

XXI. Vargas Josy G. Differential geometry for physicists and mathematicians. Moving frames and differential forms: From Euclid past Riemann. – 2014.

XXII. Yaremenko M.I. Derivation of Field Equations in Space with the Geometric Structure Generated by Metric and Torsion / M.I.Yaremenko // Journal of Gravity, Volume 2014 13P. – 2014.

XXIII. Yaremenko N.I. The geometric structure of space agreed generated by metric and torsion, Proceedings of the internation geometry center. Volume 6, N 4, pp. 22-34. – 2013.

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