# Some Result on Contact Pseudo-Slant Submanifolds of a Sasakian Manifold

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**Abstract**

In this paper, we study the geometry of the contact pseudo-slant submanifolds of a Sasakian manifold. We derive the integrability conditions of distributions in the definition of a contact pseudo-slant submanifold. The notions contact pseudo-slant product is defined, and the necessary and sufficient conditions for a submanifold to contact pseudo-slant product is given. Also, a non-trivial example is used to demonstrate that the method presented in this paper is effective.

**Keywords:** Sasakian Manifold, Slant Submanifold, Contact Pseudo-Slant Submanifold, Contact Pseudo-Slant Product.

## 1. Introduction

The differential geometry of slant submanifolds has shown an increasing development since B. Y. Chen defined slant submanifolds in complex manifolds as a natural generalization of both the invariant and anti-invariant submanifolds[5]. After then, Papaghuic initiated the notion of semi-slant submanifolds as a generalization of slant submanifolds and CR-submanifolds[12]. Furthermore, Carriazo defined pseudo-slant submanifold with the name anti-slant submanifolds as a special class of bi-slant submanifolds [2, 3, 4]. Also, pseudo-slant submanifolds have been studied by Khan et. al. in [10]. Later, U. C. De et. al. studied and characterized pseudo-slant submanifolds of trans-Sasakian Manifolds [6]. Recently, M. Atceken and S. Dirik also have investigated contact pseudo-slant submanifolds in Cosymplectic, Kenmotsu, and Sasakian space forms and gave some results on mixed-geodesic, totally geodesic and the induced tensor fields to be parallel [7, 8, 9].

In this paper, we study geometry of the contact pseudo-slant submanifolds of a Sasakian manifold. In Section 2, we review basic formulas and definitions for a Sasakian manifold and their submanifolds. In Section 3, we derive the integrability conditions of distributions in the definition of a contact pseudo-slant submanifold. The notions contact pseudo-slant product is defined, and the necessary and sufficient conditions for a submanifold to be contact pseudo-slant product is given. Also, a non-trivial example is used to demonstrate that the method presented in this paper is effective.

## 2. Preliminaries

Given an odd-dimensional Riemannian manifold $(\bar{M}, g)$, let $\varphi$ be a $(1,1)$-type tensor field $\xi$ is a unit vector field and $\eta$ is a 1-form on $\bar{M}$. If we have

$$\varphi^2 X = -X + \eta(X)\xi, \ \ g(X,\xi) = \eta(X) \tag{2.1}$$

and

$$g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y) \tag{2.2}$$

for any vector fields $X, Y$ on $\bar{M}$, then $\bar{M}$ is said to have an almost contact metric structure $(\varphi, \xi, \eta, g)$ and it is called an almost contact metric manifold.

Let $\Phi$ denote the fundamental form 2-form in $\bar{M}$, given by $\Phi(X,Y)=g(X,\varphi Y)$, for any vector fields $X,Y$ on $\bar{M}$. If $\Phi=d\eta$, then $\bar{M}$ is said to be a contact metric manifold. Furthermore, the contact metric structure is called a K-contact structure if $\xi$ is a Killing vector field, that is, $\bar{\nabla}_X\xi=-\varphi X$, for any vector field $X$ on $\bar{M}$, where $\bar{\nabla}$ denotes the Levi-Civita connection on $\bar{M}$.

The structure $(\varphi,\xi,\eta,g)$ is said to be normal if $[\varphi,\varphi]+2d\eta\otimes\xi=0$, where $[\varphi,\varphi]$ is the Nijenhuis torsion of $\varphi$. A normal contact metric manifold is called a Sasakian manifold. So every Sasakian manifold is a K-contact manifold. It is well-know that an almost contact metric manifold is a Sasakian if and only if

$$(\bar{\nabla}_X\varphi)Y=g(X,Y)\xi-\eta(Y)X, \tag{2.3}$$

for any vector fields $X,Y$ on $\bar{M}$.

Let $\bar{M}(c)$ be a Sasakian space form with constant $\varphi$-holomorphic sectional curvature $c$. Then the curvature tensor $\bar{R}$ of $\bar{M}(c)$ is given by

$$\bar{R}(X,Y)Z=(\frac{c+3}{4})\{g(Y,Z)X-g(X,Z)Y\}+(\frac{c-1}{4})\{\eta(X)\eta(Z)Y-\eta(Y)\eta(Z)X$$

$$+g(X,Z)\eta(Y)\xi-g(Y,Z)\eta(X)\xi+g(X,\varphi Z)\varphi Y-g(Y,\varphi Z)\varphi X+2g(X,\varphi Y)\varphi Z\} \tag{2.4}$$

for any vector fields $X,Y,Z$ on $\bar{M}(c)$.

Now, let $M$ be a submanifold of an almost contact metric manifold $\bar{M}$, we denote the induced connections on $M$ and the normal bundle $T^{\perp}M$ by $\nabla$ and $\nabla^{\perp}$, respectively, then the Gauss and Weingarten formulas are given by

$$\bar{\nabla}_X Y=\nabla_X Y+h(X,Y) \tag{2.5}$$

and

$$\bar{\nabla}_X V=-A_V X+\nabla^{\perp}_X V, \tag{2.6}$$

for any $X,Y\in\Gamma(TM)$, $V\in\Gamma(T^{\perp}M)$, where $h$ is the second fundamental form and $A_V$ is the Weingarten map associated with $V$ as

$$g(A_V X,Y)=g(h(X,Y),V). \tag{2.7}$$

We denote the Riemannian curvature tensor of $M$ by $R$, then the Gauss equation imply

$$\bar{R}(X,Y)Z=R(X,Y)Z-A_{h(Y,Z)}X+A_{h(X,Z)}Y+(\nabla_X h)(Y,Z)-(\nabla_Y h)(X,Z), \tag{2.8}$$

for any $X,Y,Z\in\Gamma(TM)$.

The covariant derivative $\nabla h$ of $h$ is defined by

$$(\nabla_X h)(Y,Z) = \nabla_X^{\perp} h(Z,Y) - h(\nabla_X Y, Z) - h(\nabla_X Z, Y), \tag{2.9}$$

and the covariant derivative $\nabla A$ of $A$ is also defined by

$$(\nabla_X A)_V Y = \nabla_X (A_V Y) - A_{\nabla_X^{\perp} V} Y - A_V \nabla_X Y, \tag{2.10}$$

for any $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$.

Taking the normal component of (2.8), we reach at equation of Codazzi

$$(\bar{R}(X,Y)Z)^{\perp} = (\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z). \tag{2.11}$$

If $(\bar{R}(X,Y)Z)^{\perp} = 0$, then submanifold is said to be curvature-invariant.

The Ricci equation is given by

$$g(R^{\perp}(X,Y)V,U) = g(\bar{R}(X,Y)U,V) - g([A_V, A_U]X,Y), \tag{2.12}$$

for any $X, Y \in \Gamma(TM)$ and $U, V \in \Gamma(T^{\perp}M)$. If $R^{\perp} = 0$, then the normal connection of the $M$ is called flat.

Furthermore, for any $X \in \Gamma(TM)$, we can write

$$\varphi X = TX + NX, \tag{2.13}$$

where $TX$ and $NX$ denote the tangential and normal components of $\varphi X$, respectively. Similarly, for $V \in \Gamma(T^{\perp}M)$, $\varphi V$ also can be written

$$\varphi V = BV + CV, \tag{2.14}$$

where $BV$ and $CV$ denote, respectively, the tangential and normal components of $\varphi V$.

Taking into account (2.4) and (2.12), we have

$$g(\bar{R}^{\perp}(X,Y)V,U) = g([A_V, A_U]X,Y) + (\frac{c-1}{4})\{g(X,\varphi V)g(U,\varphi Y)$$
$$- g(Y,\varphi V)g(\varphi X,U) + 2g(X,\varphi Y)g(\varphi V,U)\}, \tag{2.15}$$

for any $X, Y \in \Gamma(TM)$ and $V, U \in \Gamma(T^{\perp}M)$.

By using (2.4) and (2.8), the Riemannian curvature tensor $R$ of an immersed submanifold $M$ of a Sasakian space form $\bar{M}(c)$ is given by

$$R(X,Y)Z = (\frac{c+3}{4})\{g(Y,Z)X - g(X,Z)Y\} + (\frac{c-1}{4})\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X$$

$$+\eta(Y)g(X,Z)\xi-\eta(X)g(Y,Z)\xi+g(X,\varphi Z)\varphi Y-g(Y,\varphi Z)\varphi X+2g(X,\varphi Y)\varphi Z\}$$

$$+A_{h(Y,Z)}X-A_{h(X,Z)}Y+(\nabla_Y h)(X,Z)-(\nabla_X h)(Y,Z). \quad (2.16)$$

The normal part of (2.16), we have

$$(\nabla_X h)(Y,Z)-(\nabla_Y h)(X,Z)=(\frac{c-1}{4})\{g(X,TZ)NY-g(Y,TZ)NX+2g(X,TY)NZ\}$$

By using (2.1), (2.13), (2.14) and taking into account of $\xi$ being tangent to $M$, we get

$$T^2+BN=-I+\eta\oplus\xi,\ \ NT+CN=0, \quad (2.18)$$

and

$$TB+BC=0,\ \ NB+C^2=-I. \quad (2.19)$$

Furthermore, the covariant derivatives of the tensor field $T$, $N$, $B$ and $C$ are, respectively, defined by

$$(\nabla_X T)Y=\nabla_X TY-T\nabla_X Y \quad (2.20)$$

$$(\nabla_X N)Y=\nabla_X^{\perp} NY-N\nabla_X Y \quad (2.21)$$

$$(\nabla_X B)V=\nabla_X BV-B\nabla_X^{\perp} V \quad (2.22)$$

and

$$(\nabla_X C)V=\nabla_X^{\perp} CV-C\nabla_X^{\perp} V \quad (2.23)$$

for any $X,Y\in\Gamma(TM)$ and $V\in\Gamma(T^{\perp}M)$.

By using (2.4), (2.5), (2.6) and (2.13), we can easily to see that

$$(\nabla_X T)Y=A_{NY}X+Bh(X,Y)+g(X,Y)\xi-\eta(Y)X, \quad (2.24)$$

$$(\nabla_X N)Y=-h(X,TY)+Ch(X,Y), \quad (2.25)$$

$$(\nabla_X B)V=A_{CV}X-TA_V X, \quad (2.26)$$

$$(\nabla_X C)V=-NA_V X-h(X,BV) \quad (2.27)$$

and for any $X,Y\in\Gamma(TM)$ and $V\in\Gamma(T^{\perp}M)$.

By using (2.25) and (2.26), we can easily to see that

$$g((\nabla_X N)Y,V)=-g(h(X,TY),V)+g(Ch(X,Y),V)$$

$$= -g(A_V X, TY) - g(h(X,Y), CV)$$

$$= g(TA_V X, Y) - g(A_{CV} X, Y)$$

$$= g(TA_V X - A_{CV} X, Y)$$

$$= -g(A_{CV} X - TA_V X, Y) = -g((\nabla_X B)V, Y)$$

thus we have

$$g((\nabla_X N)Y, V) = -g((\nabla_X B)V, Y). \tag{2.28}$$

Since $\xi$ is tangent to $M$, making use of (2.5), (2.6), (2.7) and (2.13), we infer that

$$\nabla_X \xi = -TX,\ h(X,\xi) = -NX,\ A_V \xi = BV, \tag{2.29}$$

for all $V \in \Gamma(T^{\perp}M)$ and $X \in \Gamma(TM)$.

**Definition 2.1** *Let M be a submanifold of an almost contact metric manifold $(\bar{M}, \varphi, \xi, \eta, g)$. Then M is said to be a contact slant submanifold if the angle $\theta(X)$ between $\varphi X$ and $T_M(p)$ is constant at any point $p \in M$ for any $X$ linearly independent of $\xi$. Thus the invariant and anti-invariant submanifolds are special class of slant submanifolds with slant angles $\theta = 0$ and $\theta = \frac{\pi}{2}$, respectively. If the slant angle $\theta$ is neither zero nor $\frac{\pi}{2}$, then slant submanifold is said to be proper contact slant submanifold. The slant submanifolds of an almost contact metric manifold, the following theorem is well known.[11]*

**Theorem 2.2** *Let $M$ be a submanifold of an almost contact metric manifold $\bar{M}$ such that $\xi \in \Gamma(TM)$. $M$ is a contact slant submanifold if and only if there exists a constant $\lambda \in (0,1)$ such that*

$$T^2 = \lambda(-I + \eta \otimes \xi) \tag{2.30}$$

Furthermore, if $\theta$ is slant angle of $M$, then it satisfies $\lambda = \cos^2\theta$ [3].

As a consequence of the above Theorem and (2.18), we have the following relations;

$$g(TX, TY) = \cos^2\theta\{g(X,Y) - \eta(X)\eta(Y)\}, \tag{2.31}$$

$$g(NX, NY) = \sin^2\theta\{g(X,Y) - \eta(X)\eta(Y)\}, \tag{2.32}$$

$$BN = \sin^2\theta(-I + \eta \oplus \xi). \tag{2.33}$$

For a slant submanifold $M$ of an almost contact metric manifold $\bar{M}$, the normal bundle $T^{\perp}M$ of $M$ is decomposable as

$$T^{\perp}M = N(TM) \oplus \mu,$$

where $\mu$ is the invariant normal subbundle with respect to $\varphi$.

## 3. Some Result on Contact Pseudo-Slant Submanifolds of a Sasakian Manifold

In this section, we study contact pseudo-slant submanifolds in a Sasakian manifold and we give some characterization results.

**Definition 3.1** *Let $M$ be a submanifold of a Sasakian manifold $\bar{M}(\varphi,\xi,\eta,g)$. We say that $M$ is a contact pseudo-slant submanifold if there exists a pair of orthogonal distributions $D^{\theta}$ and $D^{\perp}$ on $M$ such that*

**i.)** the distribution $D^{\perp}$ is anti-invariant, i.e., $\varphi(D^{\perp}) \subseteq T^{\perp}M$,

**ii.)** the distribution $D^{\theta}$ is slant with slant angle $\theta$,

**iii.)** the tangent space $TM$ admits the orthogonal direct decomposition

$TM = D^{\perp} \oplus D^{\theta},\ \xi \in \Gamma(TM)$ [10].

If we denote the dimensions of $D^{\perp}$ and $D^{\theta}$ by $n$ and $m$, respectively, the we have

the following possible cases;

**i.)** if $n=0$, then $M$ is a slant submanifold,

**ii.)** if $m=0$, then $M$ is an anti-invariant submanifold,

**iii.)** if $nm \neq 0$, $\theta = 0$, then $M$ is a contact CR-submanifold.

For a pseudo-slant submanifold $M$ of a Sasakian manifold $\bar{M}$, the normal bundle $T^{\perp}M$ of a pseudo-slant submanifold $M$ is decomposable as

$$T^{\perp}M = \varphi(D^{\perp}) \oplus N(D^{\theta}) \oplus \mu,\ \ \varphi(D^{\perp}) \perp N(D^{\theta}). \tag{3.1}$$

**Theorem 3.2** *Let $M$ be a contact pseudo-slant submanifold of a Sasakian manifold $\bar{M}$. Then anti-invariant distribution $D^{\perp}$ is always integrable.*

**Proof** For any $Y, Z \in \Gamma(D^{\perp})$, we have

$$\bar{\nabla}_Z \varphi Y = \varphi \bar{\nabla}_Z Y + g(Z,Y)\xi$$

$$-A_{NY}Z + \nabla^{\perp}_Z NY = T\nabla_Z Y + N\nabla_Z Y + Bh(Z,Y) + Ch(Z,Y) + g(Z,Y)\xi,$$

which implies that

$$-A_{NY}Z = T\nabla_Z Y + Bh(Y,Z) + g(Z,Y)\xi.$$

Thus we have

$$T[Y,Z] = A_{NZ}Y - A_{NY}Z. \tag{3.2}$$

Since the ambient manifold $\bar{M}$ is Sasakian, we have

$$g(A_{NZ}Y - A_{NY}Z, U) = g(h(Y,U), NZ) - g(h(Z,U), NY)$$

$$= g(h(Y,U), NZ) - g(\bar{\nabla}_U Z, NY)$$

$$= g(h(Y,U), NZ) + g(\bar{\nabla}_U \varphi Y, Z)$$

$$= g(h(Y,U), NZ) + g((\bar{\nabla}_U \varphi)Y + \varphi\bar{\nabla}_U Z, Z)$$

$$= g(h(Y,U), NZ) + g(\varphi\bar{\nabla}_U Z, Z)$$

$$+ g(g(U,Y)\xi - \eta(Y)U, Z)$$

$$= g(h(Y,U), NZ) - g(\bar{\nabla}_U Y, \varphi Z)$$

$$= g(h(Y,U), NZ) - g(h(U,Y), NZ) = 0,$$

for any $U \in \Gamma(TM)$, that is,

$$A_{NZ}Y = A_{NY}Z. \tag{3.3}$$

From (3.2) and (3.3), we conclude that $T[Y,Z] = 0$ i.e., $[Y,Z] \in \Gamma(D^{\perp})$. The proof is completes.

**Theorem 3.3** *Let $M$ be contact pseudo-slant submanifold of a Sasakian manifold $\bar{M}$. Then the slant distribution $D^{\theta}$ is integrable if and only if*

$$g(A_{CNY}Z + TA_{NZ}Y, X) = g(A_{CNX}Z + TA_{NZ}X, Y),$$

for any $X, Y \in \Gamma(D^{\theta})$ and $Z \in \Gamma(D^{\perp})$.

**Proof** By using (2.2) and (2.3), we have

$$g([X,Y], Z) = g(\nabla_X Y, Z) - g(\nabla_Y X, Z)$$

$$= g(\bar{\nabla}_Y Z, X) - g(\bar{\nabla}_X Z, Y)$$

$$= g(\varphi\bar{\nabla}_Y Z, \varphi X) - g(\varphi\bar{\nabla}_X Z, \varphi Y)$$

$$= g(\bar{\nabla}_Y \varphi Z - (\bar{\nabla}_Y \varphi)Z, \varphi X) - g(\bar{\nabla}_X \varphi Z - (\bar{\nabla}_X \varphi)Z, \varphi Y)$$

$$= g(\bar{\nabla}_Y \varphi Z, \varphi X) - g(g(Y.Z)\xi + \eta(Z)Y, \varphi X)$$

$$- g(\bar{\nabla}_X \varphi Z, \varphi Y) + g(g(X.Z)\xi + \eta(Z)X, \varphi Y).$$

Thus, from the equations numbered (2.5), (2.6), (2.7), (2.13) and (2.33)

$$g([X,Y],Z) = g(\bar{\nabla}_Y \varphi Z, TX) + g(\bar{\nabla}_Y \varphi Z, NX) - g(\bar{\nabla}_X \varphi Z, TY) - g(\bar{\nabla}_X \varphi Z, NY)$$

$$= -g(A_{\varphi Z}TX, Y) + g(A_{\varphi Z}TY, X) + g(\bar{\nabla}_Z X, \varphi NY) - g(\bar{\nabla}_Y Z, \varphi NX)$$

$$= g(A_{\varphi Z}TY, X) - g(A_{\varphi Z}TX, Y) + g(\nabla_X Z, BNY) - g(\nabla_Y Z, BNX)$$

$$+ g(\bar{\nabla}_X Z, CNY) - g(\bar{\nabla}_Y Z, CNX)$$

$$= g(A_{\varphi Z}TY + TA_{\varphi Z}Y, X) - \sin^2\theta g(\nabla_X Z, Y)$$

$$+ \sin^2\theta g(\nabla_Y Z, X) + g(h(X,Z), CNY) - g(h(Y,Z), CNX)$$

$$= g(TA_{\varphi Z}Y + A_{\varphi Z}TY, X) + \sin^2\theta g([X,Y],Z) + g(A_{CNY}X - A_{CNX}Y, Z),$$

for any $X, Y \in \Gamma(D^\theta)$ and $Z \in \Gamma(D^\perp)$. Consequently, we reach at

$$\cos^2\theta g([X,Y],Z) = g(TA_{\varphi Z}Y + A_{\varphi Z}TY, X) + g(A_{CNY}X - A_{CNX}Y, Z),$$

which proves our assertion.

**Definition 3.4** *Let $M$ be a contact pseudo-slant submanifold of a Sasakian manifold $\bar{M}$. $M$ is said to be contact pseudo-slant product if the distributions $D^\perp$ and $D^\theta$ are totally geodesic in $M$.*

**Theorem 3.5** *Let $M$ be a proper contact pseudo-slant submanifold of a Sasakian manifold $\bar{M}$. If the tensor field $N$ is parallel, then $M$ is a contact pseudo-slant product.*

**Proof** $N$ is parallel if and only if $B$ is parallel, from (2.28) and (2.26), we have

$$A_{CV}X - TA_V X = 0 \tag{3.4}$$

taking $V = NZ$ in equation (3.4) and from (2.18), we get

$$TA_{NZ}X = 0, \quad X \in \Gamma(TM), \quad Z \in \Gamma(D^\perp).$$

This implies that $A_{\varphi Z}X \in \Gamma(D^\perp)$ and $Bh(X,Z) = 0$. The proof is completes.

**Theorem 3.6** *Let $M$ be a contact pseudo-slant submanifold of a Sasakian space form $\bar{M}(c)$. $M$ is either anti-invariant submanifold or $\bar{M}$ is flat if $N$ is parallel.*

**Proof** Since $N$ is parallel, we can easily to see that

$$h(TX,Y) = Ch(X,Y) = h(X,TY),$$

Thus we have

$$g(h(TX,Y),V) = g(h(X,TY),V) = g(A_V TX, Y) = g(A_V X, TY) = -g(TA_V X, Y)$$

which is equivalent to

$$TA_V X + A_V TX = 0, \qquad (3.5)$$

for any $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$. From (3.5), we have

$$g(A_V TX, BU) = g(A_V X, TBU) = -g(A_V X, BCU) = -g(NA_V X, CU) = 0, \qquad (3.6)$$

for any vector fields $U, V$ normal to $M$. Taking the covariant derivative of (3.6), for $Y \in \Gamma(TM)$, we obtain

$$g(\overline{\nabla}_Y A_V TX, BU) + g(A_V TX, \overline{\nabla}_Y BU) = 0.$$

This means that

$$0 = g((\nabla_Y A)_V TX + A_{\nabla_Y^{\perp} V} TX + A_V \nabla_Y TX, BU) \ + g(A_V TX, (\nabla_Y B)U + B\nabla_Y^{\perp} U).$$

Taking into account (2.28) and (3.6), we reach at

$$g((\nabla_Y A)_V TX + A_V\{(\nabla_Y T)X + T\nabla_Y X\}, BU) = 0,$$

from which

$$g((\nabla_Y A)_V TX, BU) + g(A_V\{A_{NX} Y + Bh(X,Y) + g(X,Y)\xi - \eta(X)Y\}, BU) = 0,$$

or,

$$g((\nabla_Y A)_V TX, BU) + g(A_V BU, A_{NX} Y) + g(A_V BU, Bh(X,Y)) = 0.$$

This implies that

$$g((\nabla_{TY} h)(TX, BU), V) = g((\nabla_{TY} A)_V TX, BU) \ = -g(A_V BU, A_{NX} TY) - g(A_V BU, Bh(TY, X)).$$

Thus we conclude that

$$\begin{aligned} g((\nabla_{TX} h)(TY, BU) - (\nabla_{TY} h)(TX, BU), V) &= g(A_V BU, A_{NX} TY) \\ &\quad - g(A_V BU, A_{NY} TX) \\ &= g(A_V BU, A_{NX} TY - TA_{NY} X) \\ &= g(A_V A_{NX} TY, BU) \\ &\quad - g(A_V TA_{NY} X, BU) \\ &= g(A_V TA_{NX} Y, BU) = 0. \end{aligned} \qquad (3.7)$$

On the other hand, form the Codazzi equation, we have

$$g((\nabla_{TX}h)(TY,BU)-(\nabla_{TY}h)(TX,BU),V)=(\frac{c-1}{4})\{g(T^2Y,BV)g(NX,U)$$
$$-g(T^2X,BV)g(NY,U)$$
$$+2g(TX,T^2Y)g(NBV,U)\}$$
$$=-\cos^2\theta(\frac{c-1}{4})\{g(Y,BV)g(NX,U)$$
$$-g(X,BV)g(NY,U) \qquad (3.8)$$
$$+2g(TX,Y)g(NBV,U)\}.$$

In (3.8), taking $X,Y\in\Gamma(D^\theta)$ and $U=V=NZ\in\Gamma(T^\perp M)$ for $Z\in\Gamma(D^\perp)$, and corresponding (3.7) and (3.8), we get

$$\cos^2\theta(\frac{c-1}{2})g(TX,Y)g(Z,Z)=0.$$

This proves our assertion.

**Example 3.9** Let $M$ be a submanifold of $\mathrm{R}^9$ defined by the following equation

$$\chi(u,v,w,t,z)=(3u\sin\alpha,-v\cos\alpha,-2u\sin\alpha,v\cos\alpha,-w\cos t,\cos t,w\sin t,-\sin t,z).$$

We can easily to see that the tangent bundle of $M$ is spanned by the tangent vectors

$$e_1=3\sin\alpha\frac{\partial}{\partial x_1}-2\sin\alpha\frac{\partial}{\partial x_2},e_2=-\cos\alpha\frac{\partial}{\partial y_1}+\cos\alpha\frac{\partial}{\partial y_2},e_5=\xi=\frac{\partial}{\partial z}.$$

$$e_3=-\cos t\frac{\partial}{\partial x_3}+\sin t\frac{\partial}{\partial x_4},e_4=w\sin t\frac{\partial}{\partial x_3}-\sin t\frac{\partial}{\partial y_3}+w\cos t\frac{\partial}{\partial x_4}-\cos t\frac{\partial}{\partial y_4}.$$

For the almost contact metric structure $\varphi$ of $\mathrm{R}^9$, whose coordinate systems $(x_1,\ y_1,\ x_2,\ y_2,\ x_3,\ y_3,\ x_4,\ y_4,\ z)$, choosing

$$\varphi(\frac{\partial}{\partial x_i})=\frac{\partial}{\partial y_i},\varphi(\frac{\partial}{\partial y_j})=-\frac{\partial}{\partial x_j},1\le i,j\le 4$$

then we have

$$\varphi e_1=3\sin\alpha\frac{\partial}{\partial y_1}-2\sin\alpha\frac{\partial}{\partial y_2},\varphi e_2=\cos\alpha\frac{\partial}{\partial x_1}-\cos\alpha\frac{\partial}{\partial x_2},\varphi e_3=-\cos t\frac{\partial}{\partial y_3}+\sin t\frac{\partial}{\partial y_4},$$

and

$$\varphi e_4 = w\sin t\frac{\partial}{\partial y_3} + \sin t\frac{\partial}{\partial x_3} + w\cos t\frac{\partial}{\partial y_4} + \cos t\frac{\partial}{\partial x_4}.$$

By direct calculations, we can infer $D_\theta = span\{e_1, e_2\}$ is a slant distribution with slant angle $\cos\theta = \frac{g(e_1, \varphi e_2)}{\|e_1\|\|\varphi e_2\|} = \frac{5\sin\theta.\cos\theta}{\sqrt{13\sin^2\theta}\sqrt{2\cos^2\theta}} = \frac{5\sqrt{26}}{26}$ $\theta = arc\cos(\frac{5\sqrt{26}}{26})$. Since $g(\varphi e_3, e_i) = 0, \quad i = 1,2,4,5$ and $(\varphi e_4, e_j) = 0, \quad j = 1,2,3,5$ are orthogonal to $M$, $D^\perp = span\{e_3, e_4, e_5\}$ is an anti-invariant distribution. Thus $M$ is a $5$-dimensional proper contact pseudo-slant submanifold of $R^9$ with it's usual almost contact metric structure.

**Conclusions and Remarks**

In this paper, we study geometry of the contact pseudo-slant submanifolds of a Sasakian manifold. We derive the integrability conditions of distributions in the definition of a contact pseudo-slant submanifold. The notions contact pseudo-slant product is defined and the necessary and sufficient conditions for a submanifold to be contact pseudo-slant product is given. Also, an non-trivial example is used to demonstrate that the method presented in this paper is effective . It is well known that open problems are so interesting in this area, especially for almost contact structures.

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