A Classical Geometric Relationship That Reveals The Golden Link in Nature

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Abstract

This paper introduces the perfect complementary relationship between the 3-4-5 Pythagorean triangle and the 1:2:$\sqrt{5}$ right-angled triangle. The classical geometric intimacy between these two right triangles not only provides for the ultimate geometric substantiation of Golden Ratio, but it also reveals the fundamental Pi: Phi correlation (π: φ), with an extreme level of precision, and which is firmly based upon the classical geometric principles.

Keywords: Golden Ratio, Fibonacci Sequence, Pi, Phi, Pythagorean Triangle, Golden Triangle, Divine Proportion

Introduction

One of the greatest geometers of all times; Johannes Kepler had mentioned "The Two Great Treasures of Geometry," namely, "The Theorem of Pythagoras" and "The Golden Ratio", the former he compared to 'a measure of gold', and the later he named 'a precious jewel'! And, both of them were brought together in Kepler's triangle.

However, Kepler's golden postulate is now observed to be truly corroborated by the classical geometric relationship between 3-4-5 Pythagorean triple and 1:2:$\sqrt{5}$ triangle, which is found to be the very origin of the Golden Ratio in geometry.

The right angled triangle 1:2:$\sqrt{5}$, with its catheti in ratio 1:2, is described in this paper, to possess several unique geometric features, which make it a perfect complementary triangle for the 3-4-5 Pythagorean triple.

The astonishing properties of this 1:2:$\sqrt{5}$ triangle also prove it to be the very origin of the Golden Ratio (φ) in geometry. Moreover, this triangle provides a very important link between $\boldsymbol{\pi}$ and $\boldsymbol{\varphi}$, for example; as an "extremely precise" and stunning relationship, the ratio of the area of 1:2:$\sqrt{5}$ triangle to the area of its Incircle is exactly $\frac{\varphi^4}{\pi}$; where $\boldsymbol{\varphi}$ is the Golden Ratio, and this is **the ultimate geometric substantiation of** φ, and this is **the extremely precise Pi: Phi correlation**, premised upon classical geometric ratios, revealed by this unique 1:2:$\sqrt{5}$ triangle.

This paper introduces the intimate geometric relationship between the 1:2:$\sqrt{\mathbf{5}}$ triangle and the 3-4-5 triangle, which is the first of the primitive Pythagorean triples. These two right angled triangles, with all their corresponding sides and angles collaborating with each other, conjointly provide for the ultimate geometric substantiation of φ, in a phenomenal way described in this paper. Also, the confluence of these two triangles disclose the precise geometric interrelation between Pi(π) and the Golden Ratio (Phi;φ). The classical correspondence between 1:2:$\sqrt{5}$ triangle and the first primitive Pythagorean triple provides the most precise method for the mathematical determination of the Pi:Phi correlation, firmly premised upon classical geometric principles.

Hence, the aim of this paper is to introduce the peculiar relationship between two very special right-angled triangles, which in turn reveals the august link between the two most important constants in the geometry of nature; the $\boldsymbol{\pi}$ and the $\boldsymbol{\varphi}$.

The paper presents the author's investigations and findings.

The 1:2:$\sqrt{5}$ triangle

The right-angled triangle, with its catheti in ratio 1:2, is observed to possess several peculiar geometric properties. Noticeably, three sides of this 1:2:$\sqrt{5}$ triangle provide the fractional expression for Golden Ratio**:** $\frac{1+\sqrt{5}}{2}$**;** which is the solution to the quadratic equation $x^2 - x - 1 = 0$ for the Golden Ratio (φ).

A perfect classical geometric relationship is observed between this 1:2:$\sqrt{5}$ triangle and the 3-4-5 Pythagorean triple, and it is elaborated in this paper, on basis of following precise points;

1) Formation of the two triangles
2) Classical intimacy between the angles of two triangles
3) Side Lengths Synergism; and the Golden Ratio embedded in
4) The Fibonacci-Pythagorean Triples
5) π and φ : The General Correlation
6) π and φ : The Special Correlation

Formation of the Two Triangles

As a well-known geometric method of producing 3-4-5 triangle, a square can be dissected by three line segments, as shown here, in figure 1. Remarkably, such dissection of a square invariably produces multiple 1:2:$\sqrt{5}$ triangles alongside the Pythagorean triple.

In square ABCD, points P and R are the midpoints of side AD and DC, respectively. Connecting point P to the vertices B and C, and point R to the vertex B, produces the triangle PBQ in centre; which is a 3-4-5 Pythagorean triangle. And remarkably, all other triangles formed in the figure, namely, ΔCQR, ΔCQB, ΔAPB, ΔCDP and ΔCBR; all are 1:2:$\sqrt{5}$ triangles of various sizes.

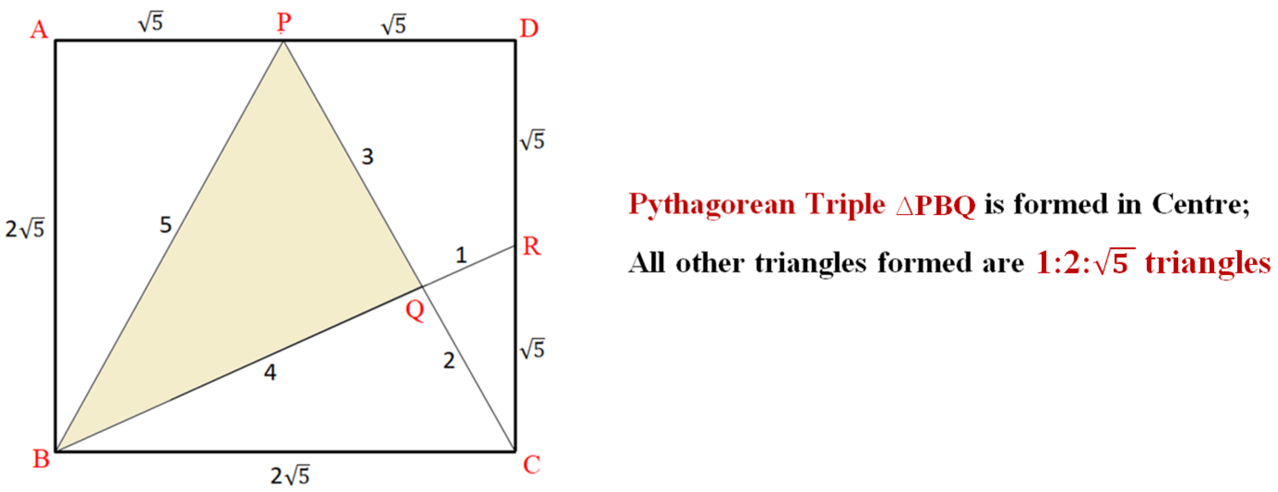


Figure 1: Dissecting the Square; the triangle PBQ produced in centre is the 3-4-5 Pythagorean triple, and all other triangles produced are 1:2:$\sqrt{5}$ triangles of various sizes.

The abovementioned method of the dissection of a square can divide any square into a 3-4-5 Pythagorean triple and multiple 1:2:$\sqrt{5}$ triangles. And the square, so dissected, possess several unique geometric properties, as mentioned below.

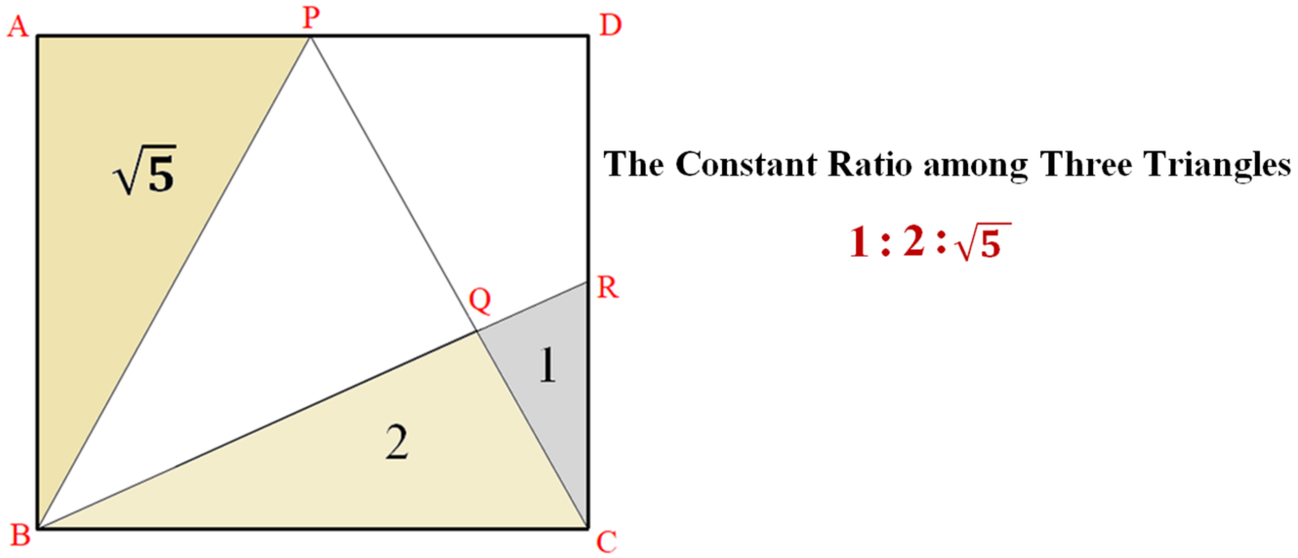


Figure 2: The constant ratio among the Sides, Perimeters, Inradii and Circumradii of three triangles

The square, so dissected, consists of following three types of geometric figures;

1. The 3-4-5 Pythagorean triangle ΔPQB
2. The 1:2:$\sqrt{5}$ triangles, namely, ΔCQR, ΔCQB, and ΔAPB.

Remarkably, these three 1:2:$\sqrt{5}$ triangles are in a definite proportion with each other. They have their corresponding sides, perimeters, as well as the radii of their incircles and circumcircles; all in the **ratio 1:2:$\sqrt{5}$**.

3. And, the irregular quadrilateral PQRD.

The internal angles of this quadrilateral, beside two opposite right angles, as shown in Fig.3, are angle DPQ = 63.435^0 and angle DRQ = 116.565^0, which are remarkably same as **"the angles of the Golden Rhombus"**.

Also, it is conspicuous in Figure 3, that the quadrilateral ABQP also has the same internal angles and same shape as □ PQRD. This quadrilateral ABQP consists of the 3-4-5 Pythagorean triangle PQB and an equivalent sized 1:2:$\sqrt{5}$ triangle PAB, having common hypotenuse PB.

And surprisingly, the perimeters of ΔCQR, □PQRD and □ABQP are invariably in the proportion **1: φ : φ ^{2},** where **φ** is the Golden Ratio.

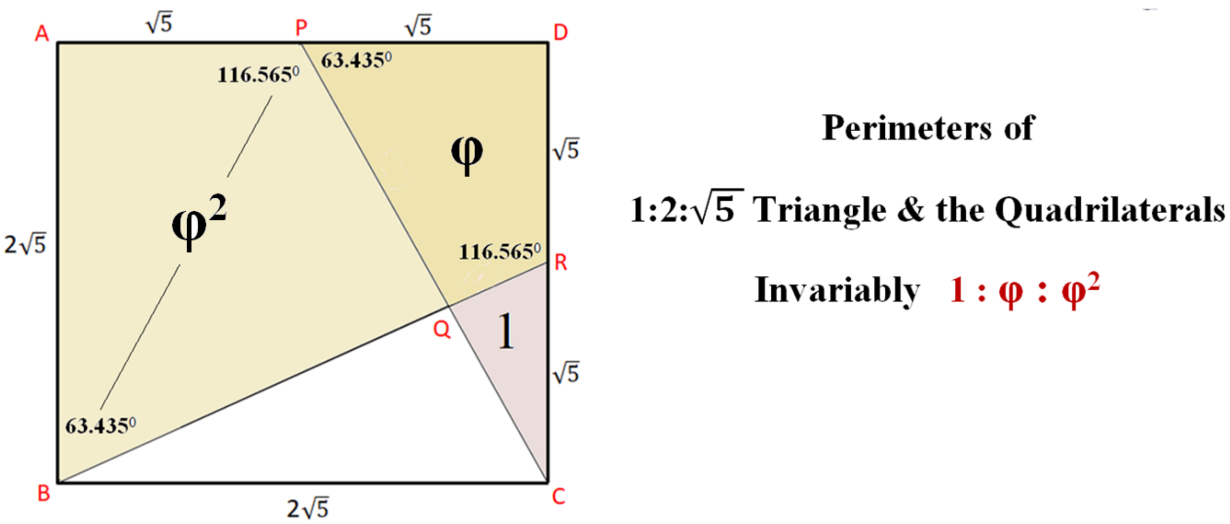


Figure 3: The1:2:$\sqrt{5}$ triangle & the two Quadrilaterals: the Constant Perimeters' Ratio

All these 1:2:$\sqrt{5}$ triangles, the irregular quadrilaterals, and the square ABCD as a whole are full of the Golden Ratios, embedded in their very structure, which will be elaborated subsequently.

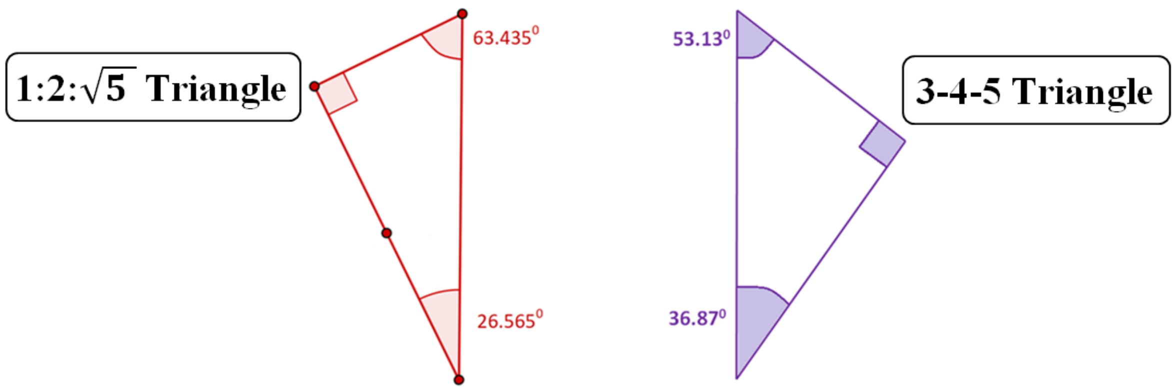
The 1:2:$\sqrt{5}$ Triangle and the First Pythagorean Triple: A Classical Geometric Intimacy

The 1:2:$\sqrt{5}$ triangle is not just a geometric structure formed alongside the 3-4-5 triangle in the abovementioned dissected square, but it is observed that there exists a precise complementary relationship between these two right-angled triangles. The word 'complementary' here, should not be confused with the 'complementary angles,' which add up to 90 degrees. Here, the 'complementary relationship' between 1:2:$\sqrt{5}$ triangle and Pythagorean triple signifies the classical synergism between the two triangles that culminates in unconventional geometric outcomes.

All corresponding angles and sides of these two triangles, synergize with each other to reflect the Golden Ratio, as follows.

The angles of the 1:2:$\sqrt{5}$ triangle and the 3-4-5 triangle are closely associated with each other, and they add up to very peculiar values.

$$63.435^0 + 53.13^0 = 116.565^0 = 2 \arctan \varphi$$



$$26.565^0 + 36.87^0 = 63.435^0 = 2 \arctan \frac{1}{\varphi}$$

Figure 4: The corresponding angles of 1:2:$\sqrt{5}$ triangle and Pythagorean triangle add up to reflect the precise Golden Ratio

As shown in the figure, the corresponding angles of the two triangles add up, to reflect **the Golden Ratio**;

$$\frac{63.435^0 + 53.13^0}{2} = \textbf{arctan } \boldsymbol{\varphi}$$

and also, $\frac{26.565^0 + 36.87^0}{2} = \textbf{arctan } \frac{1}{\varphi}$

Noticeably, the corresponding angles of the two triangles add up to the **angles of Golden Rhombus**, whose diagonals are perfectly in Golden Ratio;

$$\mathbf{26.565^0 + 36.87^0 = 63.435^0}$$

and also, $\mathbf{53.13^0 + 63.435^0 = 116.565^0}$.

Such peculiar values of the angles of 1:2:$\sqrt{5}$ triangle and the 3-4-5 triangle, and their interconnectedness are bound to produce the outstanding results, as follows. Figure 5 shows the 3-4-5 triangle ADC and equivalent sized 1:2:$\sqrt{5}$ triangle ABC, with their common hypotenuse AC, and surprisingly the figure reveals the Golden Ratio, precisely embedded in this blend of the two triangles.

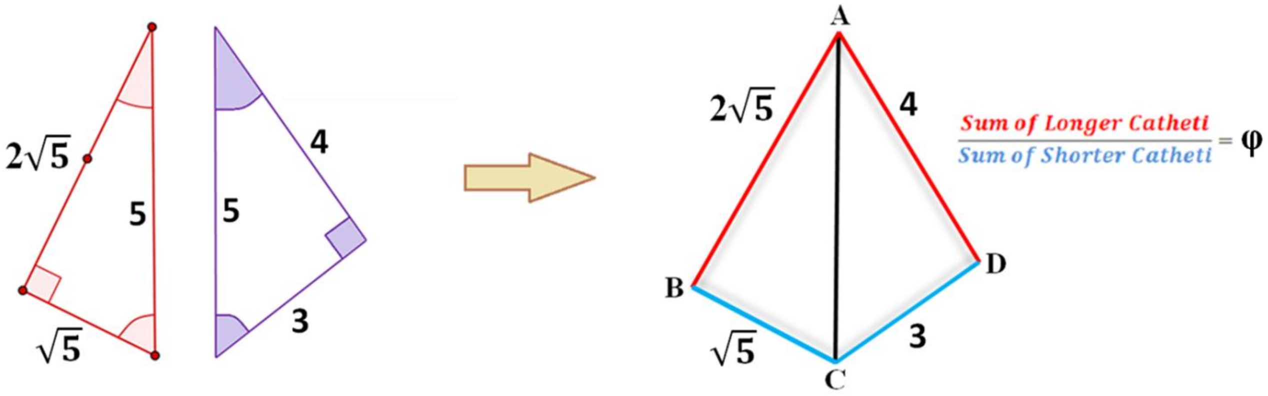


Figure 5: The Golden Ratio in Confluence of 1:2:$\sqrt{5}$ triangle and 3-4-5 triangle.

In quadrilateral ABCD,

$$\frac{\textbf{The Sum Of the Longer Catheti of the Two Triangles}}{\textbf{Sum of their Shorter Catheti}} = \frac{\mathbf{AB+AD}}{\mathbf{BC+CD}} = \boldsymbol{\varphi}$$

Noticeably, the above quadrilateral ABCD formed by the merger of 1:2:$\sqrt{5}$ triangle and Pythagorean triple, is exactly the same as □ABQP in Figures 1, 2, and 3.

And also, the Golden Ratio is not just restricted to above blend of the two triangles, but it is all embedded in all geometric formations in the dissected square: in all the 1:2:$\sqrt{5}$ triangles, and in the combinations of triangles, and also in the irregular quadrilaterals.

As shown in Figure 6, the two Quadrilaterals □ABQP and □DRQP possess Golden Ratio,

as: $\frac{\mathbf{BA+BQ}}{\mathbf{PA+PQ}} = \frac{\mathbf{PQ+PD}}{\mathbf{QR+DR}} = \boldsymbol{\varphi}$

Also, the five 1:2:$\sqrt{5}$ triangles, namely, ΔCQR, ΔCQB, ΔAPB, ΔCDP, and ΔCBR retain Golden Ratio in their side lengths, as**:** $\frac{\text{Hypotenuse + Shorter Cathetus}}{\text{Longer Cathetus}} = \varphi$

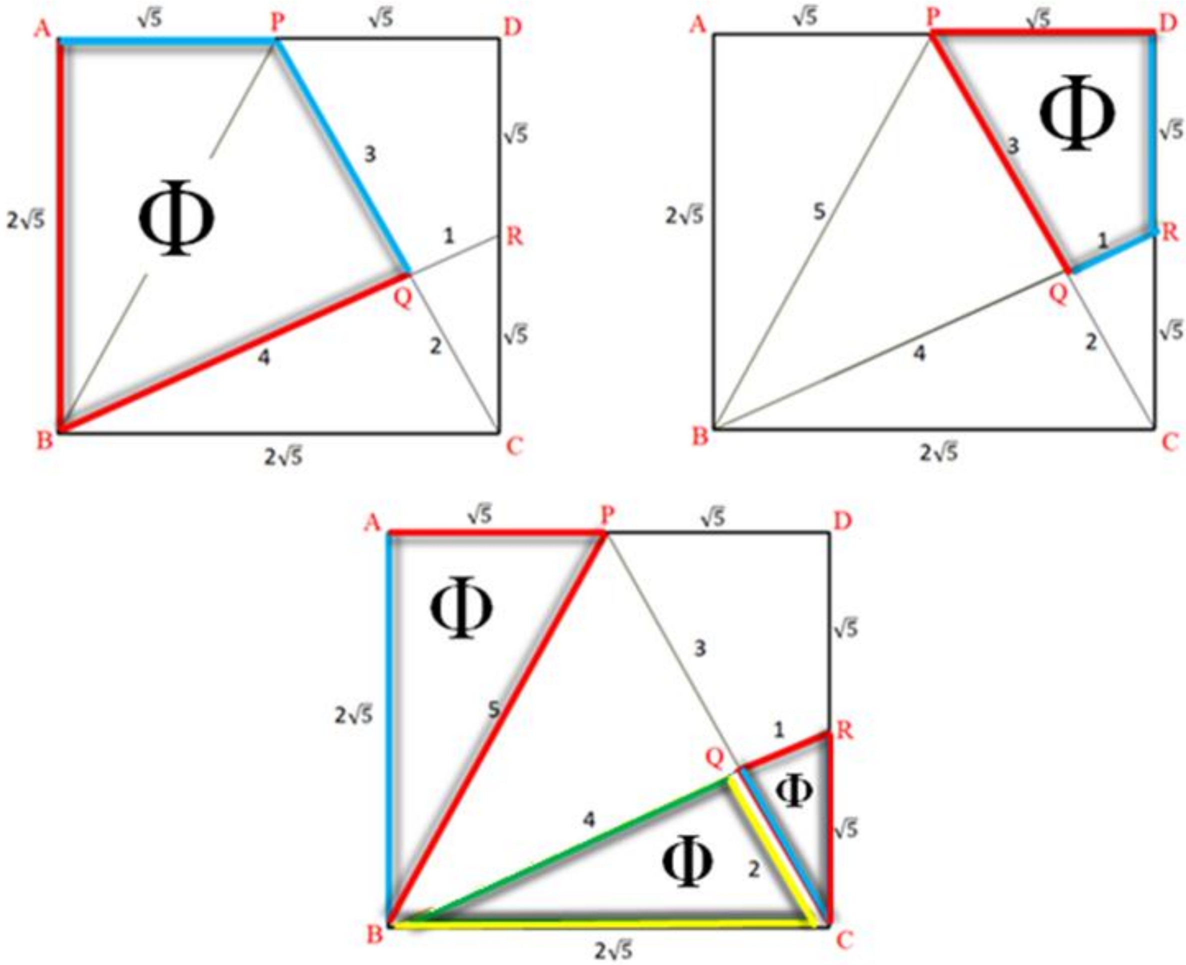


Figure 6: Golden Ratio embedded in the formations.

The Fibonacci Pythagorean Triples:

The classical relationship between 1:2:$\sqrt{5}$ triangles and 3-4-5 triple is also upheld by the well-known methods of generating Pythagorean triples from Fibonacci and Fibonacci-like sequences. And noticeably, the Pythagorean triples formed using the Fibonacci or Fibonacci-like sequence, tend to approach the 1:2:$\sqrt{5}$ triangles. The Pythagorean triples so derived from Fibonacci series, with the alternate Fibonacci numbers, starting from 5, as the hypotenuse, approach the 1:2:$\sqrt{5}$ triangles proportions, as the series, advances.

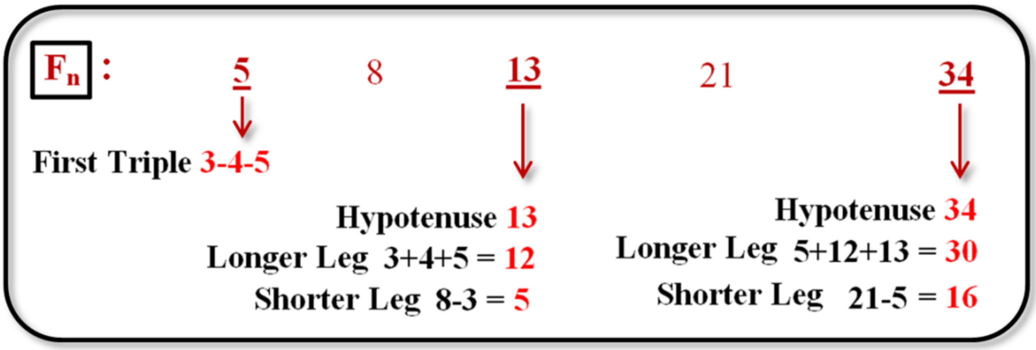
Following method can generate infinite number of the Pythagorean triples from Fibonacci sequence.

The Alternate Fibonacci Number as Hypotenuse, for example, say **13**

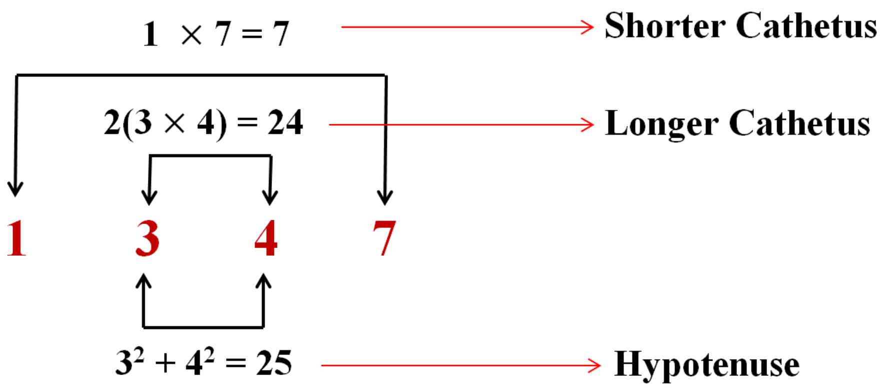
Sum of all three sides of previous triple gives Longer Cathetus: 3+4+5 = **12**

The missed Fibonacci number before 13 minus the Shorter Leg of the previous triple gives the Shorter Cathetus: 8-3 = **5,** and hence, the triple formed is **5-12-13**.

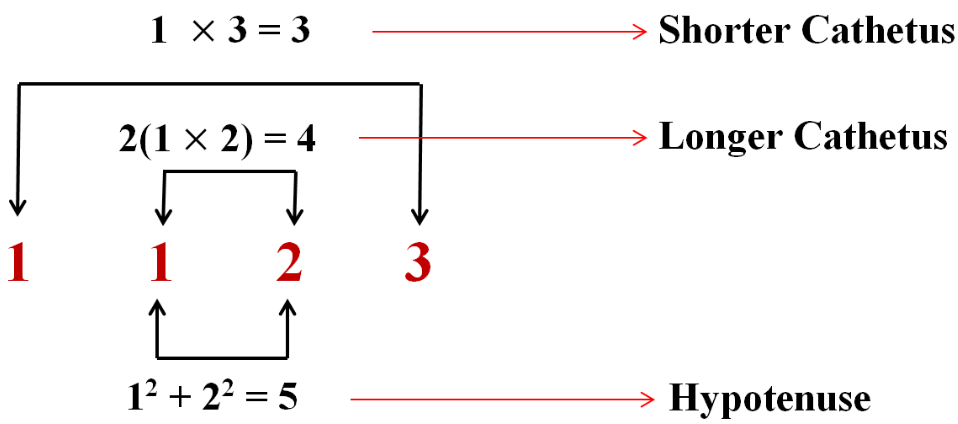
The successive Fibonacci-Pythagorean Triples, with 3-4-**5** as the first of them, and every second Fibonacci number as hypotenuse, form the series of Pythagorean triples.



A similar method can generate Pythagorean triples from 'Fibonacci like sequence,' where each term is the sum of previous two terms in the series: $X_n = X_{n-1} + X_{n-2}$. From any four consecutive terms of such Fibonacci like series, say 1, 3, 4, and 7, the triple 7-24-25 in this case, can be generated as shown below.



Similarly, the first primitive triple 3-4-5 can be generated from first four Fibonacci numbers, by the same method:



Unlike rest of the contents of this paper, the abovementioned methods for generating Fibonacci-Pythagorean triples are not part of the Author's findings. These both are well-known methods of producing Pythagorean triples. However, these methods are elaborated here, because the triples' series produced by both of these methods, starting with 3-4-5 as the first triple, invariably approaches the 1:2:$\sqrt{5}$ triangles proportions, and hence endorse the classical geometric relationship between 1:2:$\sqrt{5}$ triangles and 3-4-5 triangle revealed in this paper.

In conformity with the 1:2:$\sqrt{5}$ triangles and Pythagorean triangle intimacy, the series of such triples, starting with the first primitive triple 3-4-5, inevitably approaches 1:2:$\sqrt{5}$ triangle ratios. As such series advances, like **5**-4-3, **13**-12-5, **34**-30-16, **89**-80-39, **233**-208-105, **610**-546-272 and so on, the triples so formed approach 1:2:$\sqrt{5}$ triangle proportions, and attain the precise side length ratio **1:2:$\sqrt{5}$**, exactly in the same manner as the ratio between consecutive Fibonacci numbers, $\frac{\text{Fn}}{\text{Fn}-1}$ approaches the Golden Ratio, $\lim_{n\to\infty}\frac{\text{Fn}}{\text{Fn}-1} \cong \boldsymbol{\varphi}$. And hence, it clearly endorses: while **φ** is the Golden Ratio in nature, the **1:2:$\sqrt{5}$ triangle** is truly the Golden Trigon in geometry, in every sense of the term.

Incorporating the Circles: The advent of Pi(π)

The 1:2:$\sqrt{5}$ triangle is a very special triangle with Golden Ratio embedded in its side lengths, and it is also the perfect complementary triangle for the 3-4-5 Pythagorean triangle. Moreover, this unique triangle also provides the ultimate geometric link between nature's two most important constants, namely, the Pi(**π**) and the Phi(**φ**). If the incircles and the circumcircles are incorporated into various geometric formations described so far, the 1:2:$\sqrt{5}$ triangle reveals the Golden link between Pi and Phi, in conformity with precise geometric measurements. Such **π**:**φ** correlations, revealed by 1:2:$\sqrt{5}$ triangle can be considered under two subsections, namely, the General Correlation and the extremely precise Special Correlation.

Circumcircle and π: φ Genaral Relationship

A very peculiar relationship between **π** and **φ** is revealed by **the circumcircle of 1:2:$\sqrt{5}$ triangle -Pythagorean triangle conflux.** The 3-4-5 Pythagorean triangle, when merged with 1:2:$\sqrt{5}$ triangle, precisely reveals the Pi:Phi correlation, thus further ratifying the geometric intimacy between the two triangles.

As shown in the figure below, the usual dissected square □ABCD is circumscribed in the circle whose radius is half the diagonal of the square, and it gives the very interesting relations;

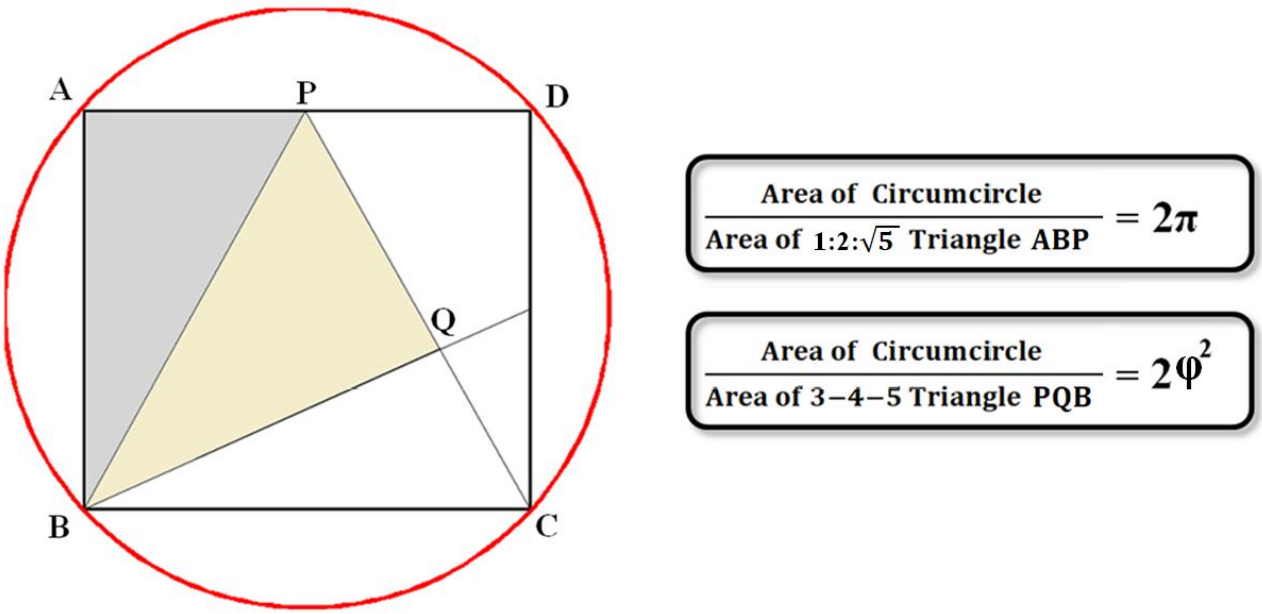


Figure 7: Circumcircle of the dissected square

From above figure, it has been observed that,

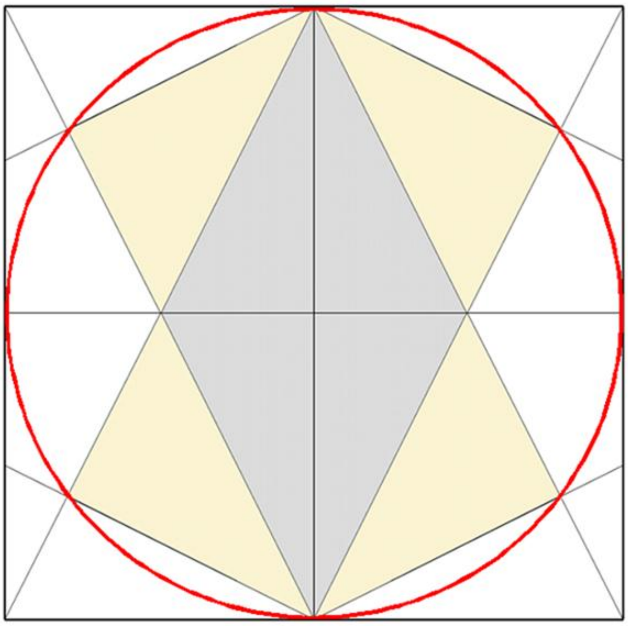
$$\frac{\textbf{Area of Circumcircle}}{\textbf{Area of 1:2:}\sqrt{5}\textbf{ triangle ABP}} = \mathbf{2\pi}$$

and, $$\frac{\textbf{Area of Circumcircle}}{\textbf{Area of Pythagorean Triangle PQB}} = \mathbf{2\varphi^2}$$

Hence, $\frac{\textbf{Area of } 1{:}2{:}\sqrt{5}\textbf{ triangle}}{\textbf{Area of Pythagorean Triangle}} = \frac{\varphi^2}{\pi} = \frac{5}{6}$

which gives the Pi:Phi correlation as **5π = 6φ²**, which is precise up to four decimal places.

Interestingly, if such **Four** squares are taken into account, and incircle is incorporated into the larger square formed, this also gives the exactly identical results.



$$\frac{\textbf{Area of Incircle}}{\textbf{Area of Four } 1{:}2{:}\sqrt{5}\textbf{ Triangles}} = \pi$$

$$\frac{\textbf{Area of Incircle}}{\textbf{Area of Four } 3-4-5\textbf{ Triangles}} = \varphi^2$$

Figure 8: Incircle of the Symmetric arrangement of four squares

Here, $\frac{\textbf{Area of Circle}}{\textbf{Area of Four } 1{:}2{:}\sqrt{5}\textbf{ triangles}} = \boldsymbol{\pi}$

and, $\frac{\textbf{Area of Circle}}{\textbf{Area of Four Pythagorean Triangles}} = \boldsymbol{\varphi^2}$

Now, it is well known that,

$$\frac{\text{Area of Square}}{\text{Area of Incircle}} = \frac{\text{Perimeter of Square}}{\text{Circumference of Incircle}} = \frac{4}{\pi} \approx \sqrt{\varphi}$$

However, the methods mentioned above, using 1:2:$\sqrt{5}$ triangle and Pythagorean triangle, give much more precise relation, as $5\pi = 6\varphi^2$.

Similarly, if the Quadrilateral, formed by 1:2:$\sqrt{5}$ triangle and 3-4-5 triangle, is circumscribed in a circle, with midpoint of the common hypotenuse as the centre of the circle, as shown below in Figure 9;

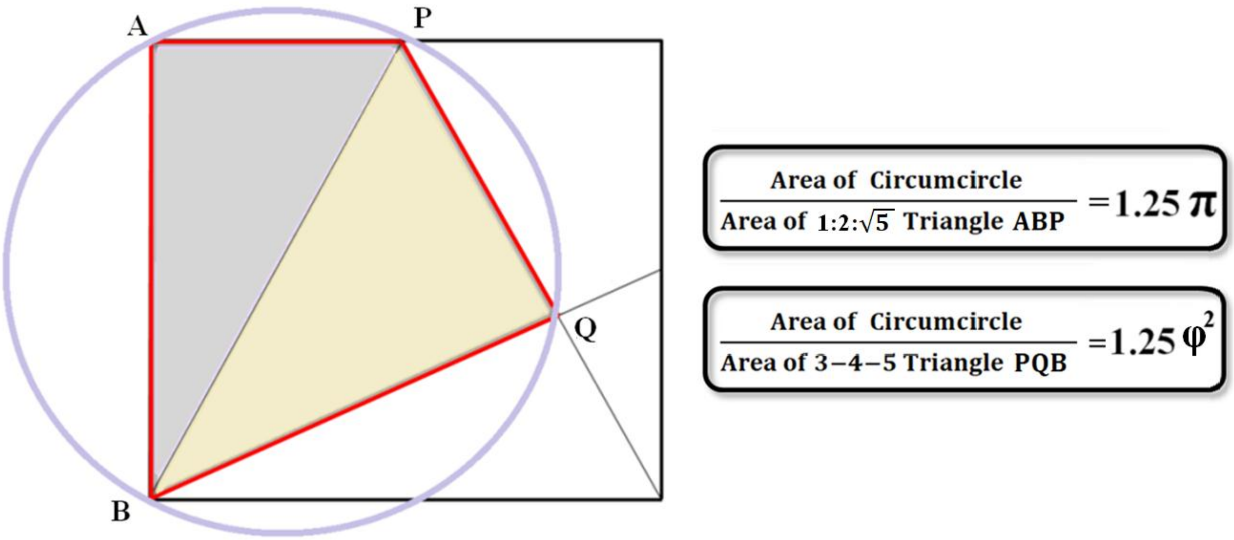


Figure 9: Circumcircle of the Qradrilateral formed by 1:2:$\sqrt{5}$ triangle and Pythagorean Triple

$$\frac{\textbf{Area of Circumcircle}}{\textbf{Area of } 1{:}2{:}\sqrt{5}\textbf{ triangle}} = \mathbf{1.25\ \pi}$$

and, $$\frac{\textbf{Area of Circumcircle}}{\textbf{Area of Pythagorean Triangle}} = \mathbf{1.25\ \varphi^2}$$

and, that naturally confirms the same Pi:Phi relation as $\frac{\varphi^2}{\pi} = \frac{5}{6}$. This is the General Pi:Phi Correlation, which is impressively precise up to unprecedented four decimal places, much more precise than the well-known relation $\frac{4}{\pi} = \sqrt{\varphi}$.

Noticeably, this $\frac{\varphi^2}{\pi} = \frac{5}{6}$ is the radius of such a circle whose circumference is exactly equal to the perimeter of a 1:2:$\sqrt{5}$ triangle; that is 1+2+$\sqrt{5}$ = 5.236…… And hence, the precision beyond four decimal places may not be achieved by this method, due to the classical and impossible problem of the quadrature of circle; or squaring the circle.

However, remarkably, the ratio between area of a 3-4-5 triangle and the area of a 1:2:$\sqrt{5}$ triangle, having hypotenuse of same length, equals the ratio between **π** and **φ²**, which further authenticates the complementary relationship between two triangles.

More importantly, just as the side lengths of 1:2:$\sqrt{5}$ triangle provide the fractional expression for the Golden Ratio as $\frac{1+\sqrt{5}}{2}$, they can also give the value of Pi as;

$$(\sin 63.435^0 + \sin 26.565^0) + (\sin 63.435^0 + \sin 26.565^0)^2 = \pi$$

and it gives the exactly same value as the above equation **5π = 6φ²**

Incircle and the π:φ Special Relationship:

The above equation, **5π = 6φ²** is impressively precise up to four decimal places. However, the unique and magnificent features of 1:2:$\sqrt{5}$ triangle enable it to reveal **very precise and entirely accurate Pi:Phi correlation**. If the incircle is incorporated into the 1:2:$\sqrt{5}$ triangle, the following extremely precise relation is observed between the area of 1:2:$\sqrt{5}$ triangle and the area of its incircle;

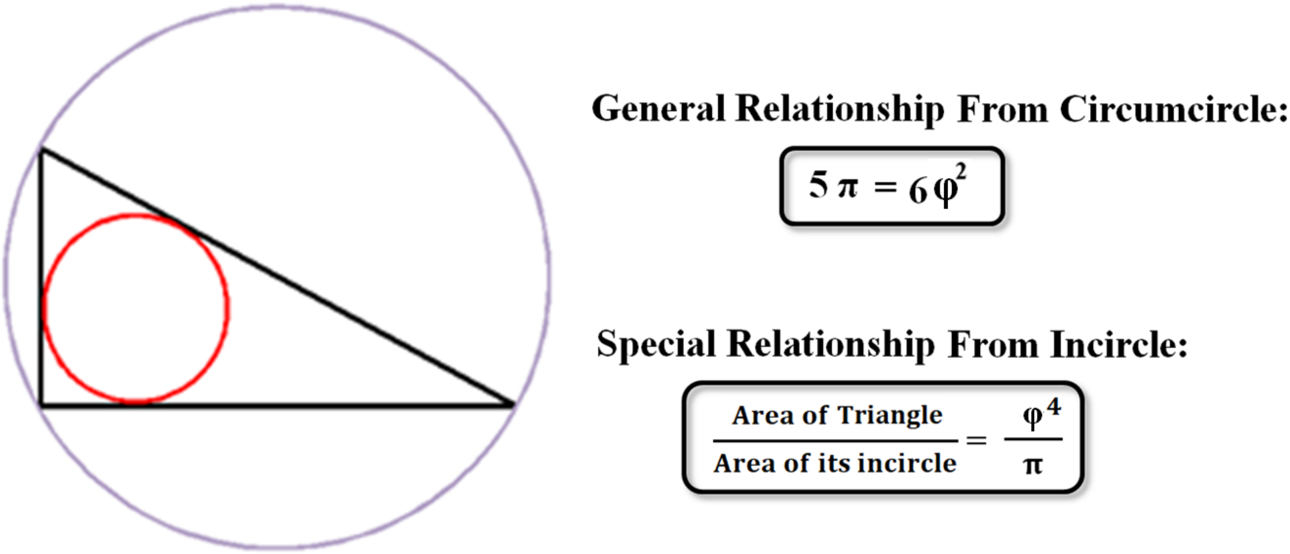


Figure 10: Circumcirle and Incircle of 1:2:$\sqrt{5}$ triangle; General and Special π:**φ** Relationships

Here, $\frac{\textbf{Area of 1:2:}\sqrt{5}\textbf{ triangle}}{\textbf{Area of Incircle}} = \frac{\varphi^4}{\pi}$

and this is **the ultimate geometric substantiation of φ**, and this is **the extremely precise Pi:Phi correlation, based upon classical geometry**, as revealed by the unique $\mathbf{1:2:\sqrt{5}}$ **triangle.**

This "1:2:$\sqrt{5}$ triangle specific" precise relationship is also observed with a couple of 1:2:$\sqrt{5}$ triangles, having one of their catheti in common. Two equal sized 1:2:$\sqrt{5}$ triangles, having **common longer cathetus AD**, show the following precise relation:

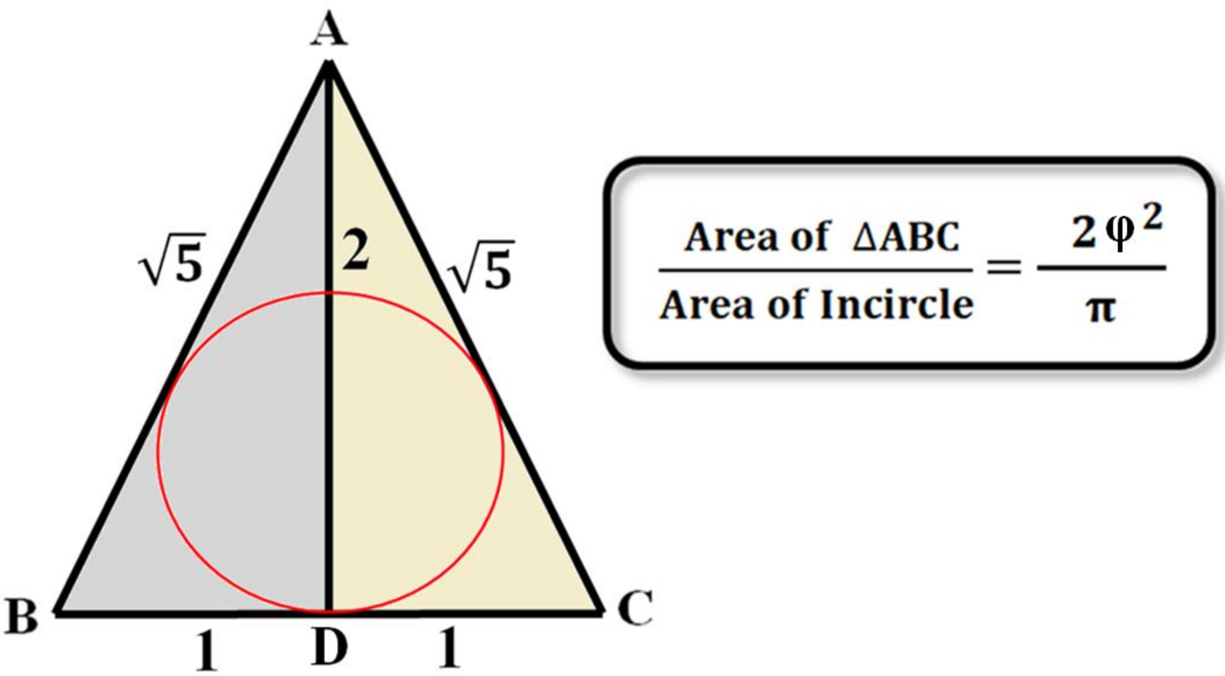


Figure 11: Two 1:2:$\sqrt{5}$ triangles, with Common Longer Leg AD

Here, $\frac{\textbf{Area of Triangle ABC}}{\textbf{Area of Incircle}} = \frac{2\varphi^2}{\pi}$

Finally, and most surprisingly, this "1:2:$\sqrt{5}$ triangle specific precision" is also observed in **the merger of 1:2:$\sqrt{5}$ triangle -Pythagorean triangle**, which is the paramount validation of the geometric intimacy between these two triangles.

As shown below in figure 12, the Pythagorean triangle ACD and the 1:2:$\sqrt{5}$ triangle ABD share their longer cathetus AD, and the precise relation observed as;

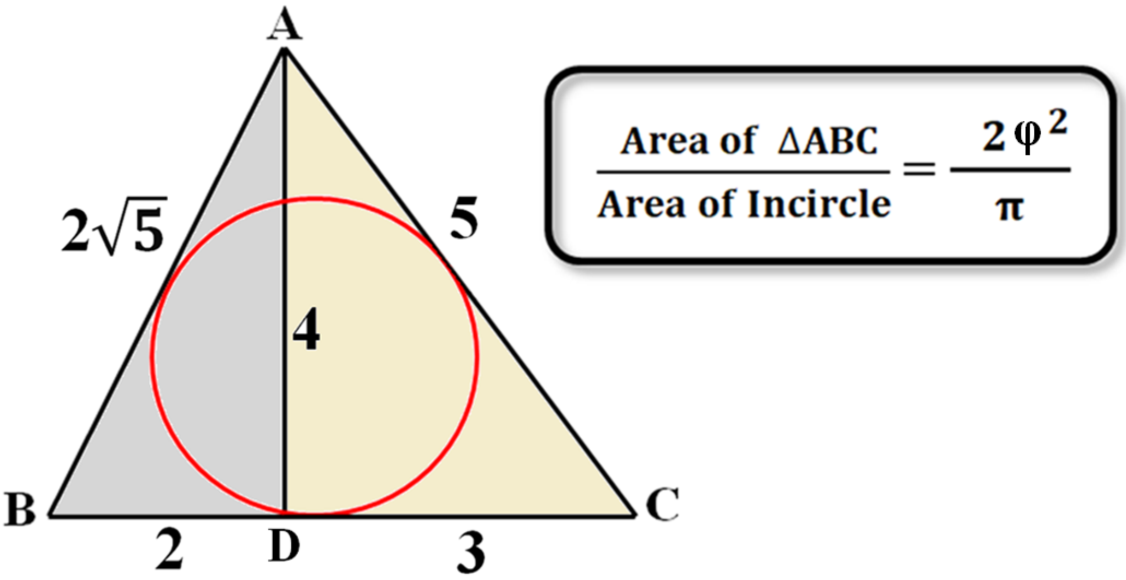


Figure 12: 1:2:$\sqrt{5}$ triangle and 3-4-5 Triangle, with Common Longer Leg AD; and The Incircle

In figure 12, $\frac{\textbf{Area of Triangle ABC}}{\textbf{Area of Incircle}} = \frac{2\varphi^2}{\pi}$

Also, 1:2:$\sqrt{5}$ triangle and Pythagorean triple, having **common shorter cathetus**, reveal similar kind of precise relation;

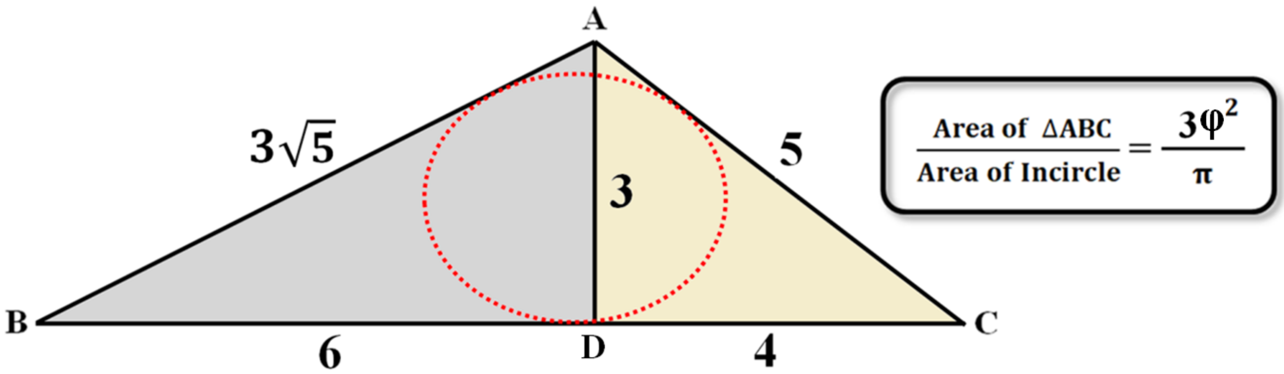


Figure 13: 1:2:$\sqrt{5}$ triangle ABD and 3-4-5 Triangle ACD, with Common Shorter Leg AD

In above diagram; $\frac{\textbf{Area of Triangle ABC}}{\textbf{Area of Incircle}} = \frac{3\varphi^2}{\pi}$

which finally corroborates the classical geometric relationship between 1:2: $\sqrt{5}$ triangle and the 3-4-5 Pythagorean triple. And, all these peculiar interrelations impart the extremely precise geometric substantiation of **π : φ** correlation.

Conclusion

This paper revealed the hidden link in geometry, namely, the precise and perfect complementary relationship between the 1:2:$\sqrt{5}$ triangle and the 3-4-5 Pythagorean triangle. The confluence of the 1:2:$\sqrt{5}$ triangle with 3-4-5 Pythagorean triple engendered the very origin of golden ratio in geometry.

Also, Circumcircles and Incircles of all geometric formations were incorporated. With advent of circles, naturally, the Pi(π) entered into picture, and then, by sheer geometric measurements, the 1:2:$\sqrt{5}$ triangle, in collaboration with the Pythagorean triple, revealed the General Pi:Phi Correlation : **5π = 6φ²**; and also disclosed the **"extremely precise Special Relationship"**, as the ratio of the area of 1:2:$\sqrt{5}$ triangle to the area of its Incircle, which exactly equals $\frac{\varphi^4}{\pi}$. This extremely precise Pi:Phi Correlation is also demonstrated by the merger of 1:2:$\sqrt{5}$ triangle with the 3-4-5 Pythagorean triangle, which is the paramount validation of the geometric intimacy between these two triangles.

Hence, the intimate and classical geometric relationship between 1:2:$\sqrt{5}$ triangle and 3-4-5 Pythagorean triple not only provided for the ultimate geometric substantiation of Golden Ratio, but it also revealed the Golden Link in Geometry; that is the ultimate Pi:Phi correlation, with an unprecedented level of precision, and which is firmly premised upon the classical geometric principles.

Supplementary Materials

Additional findings about the classical geometric relationship between these two triangles, and the Video graphic summary of the findings are provided along with this article, as the Supplementary Material.