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A Parabolic Transform and Averaging Methods for General Partial Differential Equations

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Abstract

Averaging method of the fractional partial differential equations and a special case of these equations are studied, without any restrictions on the characteristic forms of the partial differential operators. We use the parabolic transform, existence and stability results can be obtained.

Keywords: Averaging method, fractional partial differential equation, parabolic transform, existence and uniqueness of solutions.

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1 Introduction

Consider the following fractional partial differential equations:

$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} = \varepsilon L(x,t,D)u(x,t), \tag{1}$$

$$u(x,0) = \varphi(x), \tag{2}$$

where

$$L(x,t,D) = \sum_{|q|\leq m} a_q(x,t)D^q,$$

where $0 < \alpha \leq 1$, $\varepsilon > 0$, $D^q = D_1^{q_1}...D_n^{q_n}$, $D_j = \frac{\partial}{\partial x_j}$, $j = 1,...,n$, $q = (q_1,...,q_n)$ is an n-dimensional multi index, $|q| = q_1 + ... + q_n$, $x = (x_1,...,x_n) \in \Re^n$, $\Re^n$ is the $n-$dimensional Euclidean space, $0 < t < T$.
Let $C_b(\Re^n)$ be the set of all bounded continuous functions on $\Re^n$. Consider the following Cauchy problem [8]:

$$\frac{\partial u(x,t)}{\partial t} = (D_1^2 + ... + D_n^2)^{2N+1}u(x,t), \tag{3}$$

$$u(x,0)=\varphi(x)\in C(\Re^n), \tag{4}$$

where $C(\Re^n)$ is the set of all continuous functions on $\Re^n$, N is a sufficiently large positive integer.
The solution of the Cauchy problem (3), (4) is given by

$$u(x,t)=\int_{\Re^n} G(x-y,t)\varphi(y)dy,$$

where $dy=dy_1...dy_n$ and the function G is the fundamental solution of the Cauchy problem (3), (4).
For sufficiently large N, we find $\gamma\in(0,1)$ and a constant $M>0$ such that:

$$\max_x|D^q u(x,t)|\leq\frac{M}{t^\gamma}\max_x|\varphi(x)|,$$

for all $t>0$, $|q|\leq m$.
A parabolic transform of a function W is a function $\tilde{W}$ defined by [8]

$$\tilde{W}(x,t_1,...,t_r,c_1t+c_2)=\int_{\Re^n} G(x-y,c_1t+c_2)W(y,t_1,...,t_r)dy, \tag{5}$$

where $c_1\geq 0$, $c_2\geq 0$, t_j, $t\in[0,T]$, $j=1,...,r$ and $W(y,t_1,...,t_r)$ is a continuous bounded function on $\Re^n\times[0,T]^\gamma$.
By using the equations (1), (2), we get

$$u(x,t)=\varphi(x)+\frac{\varepsilon}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}L(x,s,D)u(x,s)ds, \tag{6}$$

where Γ is Gamma function.
In section 2, we study the averaging of the linear operator by using the parabolic transform where we generalize some known results due to Krol [3]. In section 3, we study a special case for the problem (1), (2) when $\alpha=1$. Compare [1, 2, 4, 5, 6, 7, 8, 9, 10, 11, 13].

2 Averaging a linear operator

Consider the following equation [8]:

$$v(x,t)=\tilde{\varphi}(x,c_1t)+\frac{\varepsilon}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\sum_{|q|\leq m}\tilde{a}_q(x,s,c_1t)D^q\tilde{v}(x,s,c_2-c_1s)ds, \tag{7}$$

where $c_2\geq c_1T$, c_1,c_2 are positive constants.
Let

$$\tilde{L}(x,t,c_1t,D)=\sum_{|q|\leq m}\tilde{a}_q(x,t,c_1t)D^q, \tag{8}$$

and

$$V(x,t)=\frac{\varepsilon}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\tilde{L}(x,s,c_1t,D)\tilde{v}(x,s,c_2-c_1s)ds, \tag{9}$$

we have

$$v(x,t)=\tilde{\varphi}(x,c_1t)+V(x,t), \tag{10}$$

$$\frac{\partial v(x,t)}{\partial t}=\frac{\partial\tilde{\varphi}(x,c_1t)}{\partial t}+\frac{\partial V(x,t)}{\partial t}, \tag{11}$$

$$v(x,0)=\tilde{\varphi}(x,0), \tag{12}$$

By averaging the coefficients $a_q(x,t)$, $\tilde{a}_q(x,t,c_1t)$ over t, we can average the operator $L(x,t,D)$, $\tilde{L}(x,t,c_1t,D)$,

$$\bar{a_q}(x) = \frac{1}{T}\int_0^T a_q(x,t)dt, \tag{13}$$

$$\bar{\tilde{a}}_q(x) = \frac{1}{T}\int_0^T \tilde{a}_q(x,t,c_1t)dt, \tag{14}$$

for all $(x,t),\ x \in \Re^n$ producing the averaged operator $\bar{L}(x,D)$, $\bar{\tilde{L}}(x,D)$, as an approximating problem for (1), (2) and (7) we take

$$\frac{\partial^\alpha u^*(x,t)}{\partial t^\alpha} = \varepsilon \bar{L}(x,D)u^*(x,t), \tag{15}$$

$$u^*(x,0) = \varphi(x), \tag{16}$$

and

$$v^*(x,t) = \tilde{\varphi}(x,c_1t) + \frac{\varepsilon}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\bar{\tilde{L}}(x,D)\tilde{v}^*(x,s,c_2-c_1s)ds, \tag{17}$$

where

$$\bar{\tilde{L}}(x,D) = \frac{1}{T}\int_0^T \sum_{|q|\le m} \tilde{a}_q(x,t,c_1t)dtD^q, \tag{18}$$

let

$$V^*(x,t) = \frac{\varepsilon}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\bar{\tilde{L}}(x,D)\tilde{v}^*(x,s,c_2-c_1s)ds, \tag{19}$$

we have

$$v^*(x,t) = \tilde{\varphi}(x,c_1t) + V^*(x,t), \tag{20}$$

$$\frac{\partial v^*(x,t)}{\partial t} = \frac{\partial \tilde{\varphi}(x,c_1t)}{\partial t} + \frac{\partial V^*(x,t)}{\partial t}, \tag{21}$$

$$v^*(x,0) = \tilde{\varphi}(x,0). \tag{22}$$

By using the equations (15), (16), we get

$$u^*(x,t) = \varphi(x) + \frac{\varepsilon}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\bar{L}(x,D)u^*(x,s)ds. \tag{23}$$

Another straightforward analysis shows the existence and uniqueness of the solutions of the problems (1), (2), (7), (15), (16) and (17) on the time-scale $\frac{1}{\varepsilon}$.

The norm $\| \ . \ \|_\infty$ is defined by the supremum norm on the spatial domain and on the time-scale $\frac{1}{\varepsilon}$ and denoted by $\| \ u \ \|_\infty = Sup_{x\in\Re^n}|u(x)|$.

Notice that there exist a dense set $E \in C_b(\Re^n)$.

Theorem 2.1 *Let u be the solution of the initial value problems (1), (2) and u^* be the solution of the initial value problems (15), (16) then we have the estimate $\| \ u - u^* \ \|_\infty = O(\varepsilon)$ on the time-scale $\frac{1}{\varepsilon}$.*

Proof. We introduce $\hat{v}(x,t)$ by the near-identity transformation:

$$\hat{v}(x,t) = v^*(x,t) + \varepsilon\int_0^t (\tilde{L}(x,s,c_1t,D) - \bar{\tilde{L}}(x,D))v^*(x,t)ds, \tag{24}$$

and suppose that the derivatives of $v^*(x,t)$, $\tilde{L}(x,t,c_1t,D)$ and $\bar{\tilde{L}}(x,D)$ are bounded.
If $nT \le t \le (n+1)T$ we obtain

$$\begin{aligned}
\| \hat{v}(x,t) - v^*(x,t) \|_\infty &= \varepsilon \| \int_{nT}^{(n+1)T} (\tilde{L}(x,s,c_1t,D) - \bar{\tilde{L}}(x,D))v^*(x,t)ds \|_\infty \\
&= \varepsilon \| \int_{nT}^{(n+1)T} [\sum_{|q|\le m} \tilde{a}_q(x,s,c_1t)D^q \\
&\quad -\frac{1}{T}\int_0^T \sum_{|q|\le m} \tilde{a}_q(x,t,c_1t)dtD^q]v^*(x,t)ds \|_\infty \\
&= O(\varepsilon) \quad \text{on the time-scale } \frac{1}{\varepsilon}.
\end{aligned}$$

Differentiation of the near-identity transformation (24) and using the equations (21, 24) repeatedly, we have

$$\begin{aligned}
\frac{\partial \hat{v}(x,t)}{\partial t} &= \frac{\partial v^*(x,t)}{\partial t} + \varepsilon(\tilde{L}(x,t,c_1t,D) - \bar{\tilde{L}}(x,D))v^*(x,t) \\
&\quad +\varepsilon \int_0^t (\tilde{L}(x,s,c_1t,D) - \bar{\tilde{L}}(x,D))\frac{\partial v^*(x,t)}{\partial t}ds \\
&= \frac{\partial V^*(x,t)}{\partial t} + \frac{\partial \tilde{\varphi}(x,c_1t)}{\partial t} + \varepsilon(\tilde{L}(x,t,c_1t,D) - \bar{\tilde{L}}(x,D))v^*(x,t) \\
&\quad +\varepsilon \int_0^t (\tilde{L}(x,s,c_1t,D) - \bar{\tilde{L}}(x,D))[\frac{\partial V^*(x,t)}{\partial t} + \frac{\partial \tilde{\varphi}(x,c_1t)}{\partial t}]ds \\
&= \varepsilon\tilde{L}(x,t,c_1t,D)v^*(x,t) + \frac{\partial V^*(x,t)}{\partial t} - \varepsilon\bar{\tilde{L}}(x,D)v^*(x,t) \\
&\quad +\varepsilon \int_0^t (\tilde{L}(x,s,c_1t,D) - \bar{\tilde{L}}(x,D))\frac{\partial V^*(x,t)}{\partial t}ds \\
&\quad +\frac{\partial \tilde{\varphi}(x,c_1t)}{\partial t} + \varepsilon \int_0^t (\tilde{L}(x,s,c_1t,D) - \bar{\tilde{L}}(x,D))\frac{\partial \tilde{\varphi}(x,c_1t)}{\partial t}ds \\
&= \varepsilon\tilde{L}(x,t,c_1t,D)[\hat{v}(x,t) - \varepsilon \int_0^t (\tilde{L}(x,s,c_1t,D) - \bar{\tilde{L}}(x,D))v^*(x,t)ds] \\
&\quad +\frac{\partial V^*(x,t)}{\partial t} - \varepsilon\bar{\tilde{L}}(x,D)v^*(x,t) \\
&\quad +\varepsilon \int_0^t (\tilde{L}(x,s,c_1t,D) - \bar{\tilde{L}}(x,D))\frac{\partial V^*(x,t)}{\partial t}ds \\
&\quad +\frac{\partial \tilde{\varphi}(x,c_1t)}{\partial t} + \varepsilon \int_0^t (\tilde{L}(x,s,c_1t,D) - \bar{\tilde{L}}(x,D))\frac{\partial \tilde{\varphi}(x,c_1t)}{\partial t}ds \\
&= \varepsilon\tilde{L}(x,t,c_1t,D)\hat{v}(x,t) + \frac{\partial V^*(x,t)}{\partial t} - \varepsilon\bar{\tilde{L}}(x,D)v^*(x,t) \\
&\quad +\varepsilon[\int_0^t (\tilde{L}(x,s,c_1t,D) - \bar{\tilde{L}}(x,D))\frac{\partial V^*(x,t)}{\partial t}ds \\
&\quad -\varepsilon\tilde{L}(x,t,c_1t,D) \int_0^t (\tilde{L}(x,s,c_1t,D) - \bar{\tilde{L}}(x,D))v^*(x,t)ds] \\
&\quad +\frac{\partial \tilde{\varphi}(x,c_1t)}{\partial t} + \varepsilon \int_0^t (\tilde{L}(x,s,c_1t,D) - \bar{\tilde{L}}(x,D))\frac{\partial \tilde{\varphi}(x,c_1t)}{\partial t}ds,
\end{aligned}$$

with initial value $\hat{v}(x,0) = \tilde{\varphi}(x,0)$.
Let

$$\frac{\partial}{\partial t} - \varepsilon\tilde{L}(x,t,c_1t,D) = \mathcal{L},$$

we get

$$\mathcal{L}\ (\hat{v} - v^*) = O(\varepsilon) \quad \text{on the time-scale } \tfrac{1}{\varepsilon}.$$

Moreover $(\hat{v} - v^*)(x, 0) = 0,\ x \in \Re^n$.
To complete the proof we use the barrier functions and the Phragmèn-Lïndelöf principle see [12].
Suppose that the barrier function:

$$\begin{aligned} B(x,t) &= \varepsilon \parallel M(x,t) \parallel_\infty t \\ &+ \parallel \frac{\partial V^*(x,t)}{\partial t} - \frac{\partial V(x,t)}{\partial t} + \varepsilon \tilde{L}(x,t,c_1t,D)v(x,t) - \varepsilon \bar{\tilde{L}}(x,D)v^*(x,t) \\ &+ \varepsilon \int_0^t (\tilde{L}(x,s,c_1t,D) - \bar{\tilde{L}}(x,D)) \frac{\partial \tilde{\varphi}(x,c_1t)}{\partial t} ds \parallel_\infty t \\ &+ \frac{1}{2}\varepsilon \parallel \tilde{L}(x,t,c_1t,D)[\frac{\partial V^*(x,t)}{\partial t} - \frac{\partial V(x,t)}{\partial t} \\ &+ \varepsilon \tilde{L}(x,t,c_1t,D)v(x,t) - \varepsilon \bar{\tilde{L}}(x,D)v^*(x,t) \\ &+ \varepsilon \int_0^t (\tilde{L}(x,s,c_1t,D) - \bar{\tilde{L}}(x,D)) \frac{\partial \tilde{\varphi}(x,c_1t)}{\partial t} ds] \parallel_\infty t^2, \end{aligned}$$

where

$$\begin{aligned} M(x,t) &= \int_0^t (\tilde{L}(x,s,c_1t,D) - \bar{\tilde{L}}(x,D)) \frac{\partial V^*(x,t)}{\partial t} ds \\ &\quad -\varepsilon \tilde{L}(x,t,c_1t,D) \int_0^t (\tilde{L}(x,s,c_1t,D) - \bar{\tilde{L}}(x,D))v^*(x,t) ds, \end{aligned}$$

and the functions (we omit the arguments)

$$Z_1(x,t) = \hat{v}(x,t) - v(x,t) - B(x,t),\ Z_2(x,t) = \hat{v}(x,t) - v(x,t) + B(x,t).$$

We obtain

$$\mathcal{L}\ Z_1(x,t) = (\tfrac{\partial}{\partial t} - \varepsilon \tilde{L}(x,t,c_1t,D))[\hat{v}(x,t) - v(x,t) - B(x,t)]$$

$$= \tfrac{\partial V^*(x,t)}{\partial t} - \tfrac{\partial V(x,t)}{\partial t} + \varepsilon \tilde{L}(x,t,c_1t,D)v(x,t) - \varepsilon \bar{\tilde{L}}(x,D)v^*(x,t)$$

$$+\varepsilon \textstyle\int_0^t (\tilde{L}(x,s,c_1t,D) - \bar{\tilde{L}}(x,D)) \tfrac{\partial \tilde{\varphi}(x,c_1t)}{\partial t} ds$$

$$+\varepsilon [\textstyle\int_0^t (\tilde{L}(x,s,c_1t,D) - \bar{\tilde{L}}(x,D)) \tfrac{\partial V^*(x,t)}{\partial t} ds$$

$$-\varepsilon \tilde{L}(x,t,c_1t,D) \textstyle\int_0^t (\tilde{L}(x,s,c_1t,D) - \bar{\tilde{L}}(x,D))v^*(x,t) ds]$$

$$-\varepsilon \parallel M(x,t) \parallel_\infty - \parallel \tfrac{\partial V^*(x,t)}{\partial t} - \tfrac{\partial V(x,t)}{\partial t} + \varepsilon \tilde{L}(x,t,c_1t,D)v(x,t)$$

$$-\varepsilon\bar{\bar{\tilde{L}}}(x,D)v^*(x,t)+\varepsilon\int_0^t(\tilde{L}(x,s,c_1t,D)-\bar{\bar{\tilde{L}}}(x,D))\tfrac{\partial\tilde{\varphi}(x,c_1t)}{\partial t}ds\parallel_\infty$$

$$-\varepsilon\parallel\tilde{L}(x,t,c_1t,D)[\tfrac{\partial V^*(x,t)}{\partial t}-\tfrac{\partial V(x,t)}{\partial t}+\varepsilon\tilde{L}(x,t,c_1t,D)v(x,t)$$

$$-\varepsilon\bar{\bar{\tilde{L}}}(x,D)v^*(x,t)+\varepsilon\int_0^t(\tilde{L}(x,s,c_1t,D)-\bar{\bar{\tilde{L}}}(x,D))\tfrac{\partial\tilde{\varphi}(x,c_1t)}{\partial t}ds]\parallel_\infty t$$

$$+\varepsilon^2\tilde{L}(x,t,c_1t,D)\parallel M(x,t)\parallel_\infty t$$

$$+\varepsilon\tilde{L}(x,t,c_1t,D)\parallel\tfrac{\partial V^*(x,t)}{\partial t}-\tfrac{\partial V(x,t)}{\partial t}+\varepsilon\tilde{L}(x,t,c_1t,D)v(x,t)$$

$$-\varepsilon\bar{\bar{\tilde{L}}}(x,D)v^*(x,t)+\varepsilon\int_0^t(\tilde{L}(x,s,c_1t,D)-\bar{\bar{\tilde{L}}}(x,D))\tfrac{\partial\tilde{\varphi}(x,c_1t)}{\partial t}ds\parallel_\infty t$$

$$+\tfrac{1}{2}\varepsilon^2\tilde{L}(x,t,c_1t,D)\parallel\tilde{L}(x,t,c_1t,D)[\tfrac{\partial V^*(x,t)}{\partial t}-\tfrac{\partial V(x,t)}{\partial t}+\varepsilon\tilde{L}(x,t,c_1t,D)v(x,t)$$

$$-\varepsilon\bar{\bar{\tilde{L}}}(x,D)v^*(x,t)+\varepsilon\int_0^t(\tilde{L}(x,s,c_1t,D)-\bar{\bar{\tilde{L}}}(x,D))\tfrac{\partial\tilde{\varphi}(x,c_1t)}{\partial t}ds]\parallel_\infty t^2$$

$$\leq 0,$$

$Z_1(x,0)=0,\ x\in\Re^n$ similarly, $\mathcal{L}\ Z_2(x,t)\geq 0,\ Z_2(x,0)=0,\ x\in\Re^n$.
$Z_1(x,t)$ and $Z_2(x,t)$ are bounded so we apply the Phragmèn-Lïndelöf principle, resulting in $Z_1(x,t)\leq 0$ and $Z_2(x,t)\geq 0$. We get

$$-B(x,t)\leq\hat{v}(x,t)-v(x,t)\leq B(x,t),$$

so we estimate

$$\parallel\hat{v}(x,t)-v(x,t)\parallel_\infty\leq\parallel B(x,t)\parallel_\infty=O(\varepsilon),$$

on the time-scale $\frac{1}{\varepsilon}$. We can use the triangle inequality to obtain

$$\begin{aligned}\parallel v(x,t)-v^*(x,t)\parallel_\infty &\leq \parallel\hat{v}(x,t)-v^*(x,t)\parallel_\infty+\parallel\hat{v}(x,t)-v(x,t)\parallel_\infty\\ &= O(\varepsilon)\text{ on the time-scale }\frac{1}{\varepsilon}.\end{aligned}\tag{25}$$

If $\varphi,\bar{a}_q\in C_b(\Re^n)$, $a_q\in C_b(\Re^n\times[0,T])$, then $v(x,t,\frac{1}{nT},\frac{1}{n})$, $v^*(x,t,\frac{1}{nT},\frac{1}{n})$ are the solutions of the equations (7), (17) with $c_1=\frac{1}{nT}$ and $c_2=\frac{1}{n}$.
Let

$$u_n(x,t)=\int_{\Re^n}G(x-y,\frac{1}{n}-\frac{t}{nT})v(y,t,\frac{1}{nT},\frac{1}{n})dy,$$
$$u_n^*(x,t)=\int_{\Re^n}G(x-y,\frac{1}{n}-\frac{t}{nT})v^*(y,t,\frac{1}{nT},\frac{1}{n})dy.$$

By using the semi-group property and the equations (7), (17), we find that u_n and u_n^* satisfy the following equations

$$u_n(x,t) = \varphi_n(x) + \frac{\varepsilon}{\Gamma(\alpha)} \sum_{|q|\leq m} \int_0^t (t-s)^{\alpha-1} a_{q,n}(x,s) D^q u_n(x,s) ds,$$

$$u_n^*(x,t) = \varphi_n(x) + \frac{\varepsilon}{\Gamma(\alpha)} \sum_{|q|\leq m} \int_0^t (t-s)^{\alpha-1} \bar{a}_{q,n}(x) D^q u_n^*(x,s) ds,$$

where $\varphi_n, a_{q,n}, \bar{a}_{q,n}$ are defined as in

$$v_{n+1}(x,t) = \tilde{\varphi}(x,c_1 t) + \frac{\varepsilon}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \sum_{|q|\leq m} \tilde{a}_q(x,s,c_1 t) D^q \tilde{v}_n(x,s,c_2 - c_1 s) ds,$$

$$v_{n+1}^*(x,t) = \tilde{\varphi}(x,c_1 t) + \frac{\varepsilon}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \sum_{|q|\leq m} \bar{\tilde{a}}_q(x) D^q \tilde{v}_n^*(x,s,c_2 - c_1 s) ds,$$

where

$$\tilde{v}_n(x,s,c_2 - c_1 s) = \int_{\Re^n} G(x-y, c_2 - c_1 s) v_n(y,s) dy,$$

and

$$\tilde{v}_n^*(x,s,c_2 - c_1 s) = \int_{\Re^n} G(x-y, c_2 - c_1 s) v_n^*(y,s) dy.$$

Hence the required result.

Notice that if the coefficients a_q's and $\bar{a}_q$'s do not depend on x for all $|q| \leq m$, then $a_{q,n} = a_q$ and $\bar{a}_{q,n} = \bar{a}_q$.

3 A special case

We study a special case for problem (1), (2) when $\alpha = 1$:

$$\frac{\partial}{\partial t} u(x,t) = \varepsilon L(x,t,D) u(x,t), \tag{26}$$

$$u(x,0) = \varphi(x). \tag{27}$$

We have

$$u(x,t) = \varphi(x) + \varepsilon \int_0^t L(x,s,D) u(x,s) ds, \tag{28}$$

let

$$v(x,t) = \tilde{\varphi}(x,c_1 t) + \varepsilon \int_0^t \sum_{|q|\leq m} \tilde{a}_q(x,s,c_1 t) D^q \tilde{v}(x,s,c_2 - c_1 s) ds, \tag{29}$$

and

$$V_1(x,t) = \varepsilon \int_0^t \sum_{|q|\leq m} \tilde{a}_q(x,s,c_1 t) D^q \tilde{v}(x,s,c_2 - c_1 s) ds, \tag{30}$$

we have

$$v(x,t) = \tilde{\varphi}(x,c_1 t) + V_1(x,t), \tag{31}$$

$$\frac{\partial v(x,t)}{\partial t} = \frac{\partial \tilde{\varphi}(x,c_1 t)}{\partial t} + \frac{\partial V_1(x,t)}{\partial t}, \tag{32}$$

$$v(x,0) = \tilde{\varphi}(x,0), \tag{33}$$

By averaging the coefficients $a_q(x,t)$, $\tilde{a}_q(x,t,c_1t)$ over t, we can average the operator $L(x,t,D)$, $\tilde{L}(x,t,c_1t,D)$, for all $(x,t),\ x \in \Re^n$ producing the averaged operator $\bar{L}(x,D)$, $\bar{\bar{L}}(x,D)$, as an approximating problem for (26), (27) and (29) we take

$$\frac{\partial u^*(x,t)}{\partial t} = \varepsilon \bar{L}(x,D)u^*(x,t), \tag{34}$$

$$u^*(x,0) = \varphi(x), \tag{35}$$

and

$$v^*(x,t) = \tilde{\varphi}(x,c_1t) + \varepsilon \int_0^t \bar{\bar{L}}(x,D)\tilde{v}^*(x,s,c_2-c_1s)ds, \tag{36}$$

let

$$V_1^*(x,t) = \varepsilon \int_0^t \bar{\bar{L}}(x,D)\tilde{v}^*(x,s,c_2-c_1s)ds, \tag{37}$$

we get

$$v^*(x,t) = \tilde{\varphi}(x,c_1t) + V_1^*(x,t), \tag{38}$$

$$\frac{\partial v^*(x,t)}{\partial t} = \frac{\partial \tilde{\varphi}(x,c_1t)}{\partial t} + \frac{\partial V_1^*(x,t)}{\partial t}, \tag{39}$$

$$v^*(x,0) = \tilde{\varphi}(x,0). \tag{40}$$

From the equations(34), (35), we have

$$u^*(x,t) = \varphi(x) + \varepsilon \int_0^t \bar{L}(x,D)u^*(x,s)ds, \tag{41}$$

Theorem 3.2 *Let u be the solution of the initial value problems (26), (27) and u^* be the solution of the initial value problems (34), (35) then we have the estimate $\| u - u^* \|_\infty = O(\varepsilon)$ on the time-scale $\frac{1}{\varepsilon}$.*

Proof. Consider the near-identity transformation:

$$\hat{v}(x,t) = v^*(x,t) + \varepsilon \int_0^t (\tilde{L}(x,s,c_1t,D) - \bar{\bar{L}}(x,D))v^*(x,t)ds, \tag{42}$$

we obtain

$$\| \hat{v}(x,t) - v^*(x,t) \|_\infty = O(\varepsilon) \quad \text{on the time-scale } \frac{1}{\varepsilon}.$$

By differentiating of the near-identity transformation (42) and using the equations (39, 42) repeatedly, we get

$$\begin{aligned}
\frac{\partial \hat{v}(x,t)}{\partial t} &= \varepsilon \tilde{L}(x,t,c_1t,D)\hat{v}(x,t) + \frac{\partial V_1^*(x,t)}{\partial t} - \varepsilon \bar{\bar{L}}(x,D)v^*(x,t) \\
&+\varepsilon[\int_0^t (\tilde{L}(x,s,c_1t,D) - \bar{\bar{L}}(x,D))\frac{\partial V_1^*(x,t)}{\partial t}ds \\
&-\varepsilon \tilde{L}(x,t,c_1t,D)\int_0^t (\tilde{L}(x,s,c_1t,D) - \bar{\bar{L}}(x,D))v^*(x,t)ds] \\
&+\frac{\partial \tilde{\varphi}(x,c_1t)}{\partial t} + \varepsilon \int_0^t (\tilde{L}(x,s,c_1t,D) - \bar{\bar{L}}(x,D))\frac{\partial \tilde{\varphi}(x,c_1t)}{\partial t}ds,
\end{aligned}$$

with initial value $\hat{v}(x,0) = \tilde{\varphi}(x,0)$.
We have

$$\mathcal{L}\ (\hat{v}-v^*)=O(\varepsilon)\quad \text{on the time-scale } \tfrac{1}{\varepsilon}.$$

Moreover $(\hat{v}-v^*)(x,0)=0,\ x\in\Re^n$.
We can use the barrier functions and the Phragmèn-Lïndelöf principle see [12].
Let the barrier function:

$$\begin{aligned}
B_1(x,t) &= \varepsilon \parallel M_1(x,t) \parallel_\infty t \\
&+ \parallel \frac{\partial V_1^*(x,t)}{\partial t}-\frac{\partial V_1(x,t)}{\partial t}+\varepsilon\tilde{L}(x,t,c_1t,D)v(x,t)-\varepsilon\bar{\tilde{L}}(x,D)v^*(x,t) \\
&+ \varepsilon\int_0^t(\tilde{L}(x,s,c_1t,D)-\bar{\tilde{L}}(x,D))\frac{\partial\tilde{\varphi}(x,c_1t)}{\partial t}ds \parallel_\infty t \\
&+ \frac{1}{2}\varepsilon \parallel \tilde{L}(x,t,c_1t,D)[\frac{\partial V_1^*(x,t)}{\partial t}-\frac{\partial V_1(x,t)}{\partial t} \\
&+ \varepsilon\tilde{L}(x,t,c_1t,D)v(x,t)-\varepsilon\bar{\tilde{L}}(x,D)v^*(x,t) \\
&+ \varepsilon\int_0^t(\tilde{L}(x,s,c_1t,D)-\bar{\tilde{L}}(x,D))\frac{\partial\tilde{\varphi}(x,c_1t)}{\partial t}ds] \parallel_\infty t^2,
\end{aligned}$$

where

$$\begin{aligned}
M_1(x,t) &= \int_0^t(\tilde{L}(x,s,c_1t,D)-\bar{\tilde{L}}(x,D))\frac{\partial V_1^*(x,t)}{\partial t}ds \\
&\quad -\varepsilon\tilde{L}(x,t,c_1t,D)\int_0^t(\tilde{L}(x,s,c_1t,D)-\bar{\tilde{L}}(x,D))v^*(x,t)ds,
\end{aligned}$$

and the functions (we omit the arguments)

$$Z_3(x,t)=\hat{v}(x,t)-v(x,t)-B_1(x,t),\ Z_4(x,t)=\hat{v}(x,t)-v(x,t)+B_1(x,t).$$

We have
$\mathcal{L}\ Z_3(x,t)\le 0,\ Z_3(x,0)=0$ similarly, $\mathcal{L}\ Z_4(x,t)\ge 0,\ Z_4(x,0)=0$.
$Z_3(x,t)$ and $Z_4(x,t)$ are bounded so we apply the Phragmèn-Lïndelöf principle, resulting in $Z_3(x,t)\le 0$ and $Z_4(x,t)\ge 0$. We get

$$-B_1(x,t)\le\hat{v}(x,t)-v(x,t)\le B_1(x,t),$$

so we estimate

$$\parallel\hat{v}(x,t)-v(x,t)\parallel_\infty\le\parallel B_1(x,t)\parallel_\infty=O(\varepsilon),$$

on the time-scale $\frac{1}{\varepsilon}$. We can use the triangle inequality to get

$$\parallel v(x,t)-v^*(x,t)\parallel_\infty=O(\varepsilon)\text{ on the time-scale }\frac{1}{\varepsilon}. \tag{43}$$

Similar to section (2), we have the required result.

4 Conclusion

A fractional partial differential equation can be solved without any restrictions on the characteristic forms by using the parabolic transform and the averaging methods. As a special case Cauchy problem is solved for a fractional partial differential equation.

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