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Third Order Hamiltonian for a Binary System with Varying Masses Including Preastron Effect.

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Abstract

This work concerns of the effects of the variation in the masses for two attracting bodies on the orbiter orbital elements. The formulation of the problem was done in the frame of Hamiltonian mechanics. Moreover, constructing the Hamiltonian function of the varying masses of a binary system including, periastron effect, in canonical form in the extended phase space, up to third order of the small parameter α_i, to be able to solve using canonical perturbation techniques. The Hamiltonian is explicit function of time through the variable masses, so we will extend the phase space by introducing two canonical variables ℓ_4 *and* ℓ_5 represents the change of masses while L_4 *and* L_5 are represent their conjugate momentum. Finaly we will drive the new Hamiltonian in the extended phase space.

Keywords: Binary-System, Tow-Body Problem, Varying Mass, Pertubation, Periastron Effects**.**

1. Introduction

The problem of the two bodies with varying mass has roots going back in the history since the middle of the 19th century. The comprehensive work done by many scientists in this problem was using the Newtonian frame of work. Dufour (1866) he is the first one who examine the astronomical phenomenon of variable mass by relating the secular variation of lunar acceleration with the increase of the Earth mass due to the impact of meteorites. After that, Gylden (1884) set out the solution to the system of differential equations which describes the two-body motion when the masses are subject to variations.

Rahoma et al. (2009) was introduced paper concerned with the two-body problem with varying mass in case of isotropic mass loss from both components of the binary system. The law if mass variation used gives rise to a perturbed Keplerian problem depending on two small parameters. The problem is treated analytically in the Hamiltonian framework and the equations of motion are integrated using the Lie series developed and applied separately by Delva (1984) and Hanslmeier (1984). A second order theory of the two bodies eject mass was also constructed, returning the terms of the rate of change of mass up to second order in the small parameters of the problem.

M.I.EL-Saftawy, Amirah R. AL-Gethami(2014), in their work, the Hamiltonian of the two body problem with varying mass was developed in the extended phase space taking into consideration the periastron effect. The short period solution was obtained through constructing a second order canonical transformation using "Hori's" method (Hori, 1966) developed by "Kamel" (Kamel, 1969). The element s of the transformation as well as the invers transformation were obtained too. The final solution of the problem was derived using "Delva-Hanslmeier" method.

M.I.EL-Saftawy, F.A.Abd EL-Salam,(2017), the model of varying mass function, including periastron effect in terms of Delaunay variables was expanded. The Hamiltonian of the problem is developed in the extended phase space by introducing a new canonical pair of variables (q_4, Q_4). The first "q_4"is defined as explicit function of time and the initial mass of the system. The conjugate momenta "Q_4" is assigned as the momenta raises from the varying mass. The short-period analytical solution through a second-order canonical transforming using "Hori's" method developed by "Kamel" is obtained. The variation equation for the orbital elements are obtained too. The result of the effect of the varying mass and the periastron effect in the case of $n = 2$ are analyzed.

2. Formulation of the problem.

The Hamiltonian for the two-body problem expressed in term Delaunay variables, which derived firstly by Deprit, A. (1983), is:

$$H\left(\ell_1.L_1.L_2;t\right) = -\frac{\mu^2}{2L_1^2} + \frac{\dot{\mu}}{\mu}L_1 e\sin E,$$

where the usual Delaunay Variable defined by:

$$\ell_1 = Mean\,anamoly\ ,\ell_2 = \omega\ ,\ell_3 = \Omega, L_1 = \sqrt{\mu a}\ ,L_2 = L_1\sqrt{1-e^2} \text{ and } L_3 = L_2 cosI$$

Where, *a, e, I,* ω, and Ω are the calsical orbital elements. $\ell_i^{'}s$ are considered as the coordinates, while $L_i^{'}s$ are their corresponding conjugate momenta. The variation of μ be retained from the two masses m_1 and m_2 .

The Hamiltonian H is depending implicitly on time through the variable mass μ and its time derivative ($\dot{\mu}$). Andrade and Docobo, in 2004, introduces a law for the rate of change of mass including the periastron effect which is modified to be in terms of orbital elements as:

$$\dot{\mu}\left(r,t,\mu\right) = -\alpha\mu^n - \beta\frac{L_2\mu^2}{L_1^4}\left(\frac{a}{r}\right)^2$$

Where α and β are real numbers positive proximate to zero while *n* varying between 1.4 and 4.4.

Substitute frome the second equation into the first one we get:

$$H = -\frac{1}{2}\frac{\mu^2}{L_1^2} + \frac{\dot{\mu}}{\mu}L_1 esinE - \beta\frac{L_2 e\mu}{L^3}\left(\frac{a}{r}\right)^2 sinE \qquad (1)$$

where, $\dot{\mu} = \dot{m}_1 + \dot{m}_2$.

With the help of Jeans law (Jeans 1924,1925)

$\dot{m}_k = -\alpha_k m_k^{n_k}$, ($k$=1,2),

which yields:

$\dot{\mu} = -\alpha_1 m_1^{n_1} - \alpha_2 m_2^{n_2}$.

We can expand the function μ about its value at time t_0, up to 3$^{ed.}$ order, to be:

$$\mu = \mu_0 + \frac{1}{1}\dot{\mu}\Big|_{t=t_0}(t-t_0) + \frac{1}{2!}\ddot{\mu}|_{t=t_0}(t-t_0)^2 + \frac{1}{3!}\dddot{\mu}|_{t=t_0}(t-t_0)^3 + \cdots \qquad (2)$$

Where the required derivatives are,

$$\dot{\mu}|_{t=t_0} = -\alpha_1 m_{10}^{n_1} - \alpha_2 m_{20}^{n_2}$$

$$\ddot{\mu}|_{t=t_0} = \alpha_1^2 n_1 m_{10}^{2n_1-1} + \alpha_2^2 n_2 m_{20}^{2\,n_2-1}$$

$$\dddot{\mu}|_{t=t_0} = -\alpha_1^3 n_1(2\,n_1 - 1) m_{10}^{3n_1-2} - \alpha_2^3 n_2(2\,n_2 - 1)\, m_{20}^{3n_2-2}$$

then Eqn. (2) can written as:

$$\mu = \mu_0 - \left(\alpha_1 m_{10}^{n_1} + \alpha_2 m_{20}^{n_2}\right)(t - t_0) + \frac{1}{2}\left(\alpha_1^2 n_1 m_{10}^{2n_1-1} + \alpha_2^2 n_2 m_{20}^{2n_2-1}\right)(t - t_0)^2 + \frac{1}{6}\left[-\alpha_1^3 n_1(2\,n_1 - 1) m_{10}^{3n_1-2} - \alpha_2^3 n_2(2\,n_2 - 1)\, m_{20}^{3n_2-2}\right](t - t_0)^3 + \cdots \quad (3)$$

From which we can calculate μ^2 and $\frac{\dot{\mu}}{\mu}$ needed in Eqn.(1). Squireing Eqn. (3), retaining termes up to $O(\alpha_i^3)$, we get:

$$\therefore \mu^2 = \mu_0^2 - 2\mu_0\left(\alpha_1 m_{10}^{n_1} + \alpha_2 m_{20}^{n_2}\right)(t - t_0) + \alpha_1^2\left(n_1 \mu_0 m_{10}^{2n_1-1} + m_{10}^{2n_1}\right)(t - t_0)^2 + \alpha_2^2\left(n_2 \mu_0 m_{20}^{2n_2-1} + m_{20}^{2n_2}\right)(t - t_0)^2 + 2\alpha_1\alpha_2 m_{10}^{n_1} m_{20}^{n_2}\,(t - t_0)^2 + \alpha_1^3\left(n_1 m_{10}^{3n_1-1}\right)(t - t_0)^3 + \alpha_2^3\left(n_2 m_{20}^{3n_2-1}\right)(t - t_0)^3 + \alpha_1^2\,\alpha_2\left(n_1 m_{10}^{2n_1-1} m_{20}^{n_2}\right)(t - t_0)^3 + \alpha_2^2\,\alpha_1\left(n_2 m_{20}^{2n_2-1} m_{10}^{n_1}\right)(t - t_0)^3 + O(> \alpha_i^3) \quad (4)$$

Now, calculate the quantity $\frac{\dot{\mu}}{\mu}$ for the second term of the Hamiltonian.

$$\frac{\dot{\mu}}{\mu} = \frac{\dot{m}_1 + \dot{m}_2}{m_1 + m_2} = -\frac{1}{m_1 + m_2}\left(\alpha_1 m_1^{n_1} + \alpha_2 m_2^{n_2}\right) \quad (5)$$

Let us expand $\frac{\dot{\mu}}{\mu}$ as function of t around $t = t_0$, as:

$$\frac{\dot{\mu}}{\mu} = \left.\frac{\dot{\mu}}{\mu}\right|_{t=t_0} + \sum_{i=1}^{n} \frac{1}{i!} \frac{d^i}{dx^i} \left.\frac{\dot{\mu}}{\mu}\right|_{t=t_0} (t - t_0)^i$$

Calculating the required derivatives and retaining termes up to $O(\alpha_i^3)$, we get:

$$\frac{\dot{\mu}}{\mu} = -\frac{1}{\mu_0}\left(\alpha_1 m_{10}^{n_1} + \alpha_2 m_{20}^{n_2}\right) + \frac{1}{\mu_0^2}(t - t_0)\left[\alpha_1^2\left(\mu_0 n_1 m_{10}^{2n_1-1} - m_{10}^{2n_1}\right) + \alpha_2^2\left(\mu_0 n_2 m_{20}^{2\,n_2-1} - m_{20}^{2n_2}\right) - \left(2\alpha_1\alpha_2 m_{10}^{n_1} m_{20}^{n_2}\right)\right] + \frac{1}{2\mu_0^4}\,(t - t_0)^2\left\{\alpha_1^2\left(-2\mu_0^2 m_{10}^{2n_1}\right) + \alpha_2^2\left(-2\mu_0^2 m_{20}^{2n_2}\right) + \alpha_1\alpha_2\left(-4\mu_0^2 m_{10}^{n_1} m_{20}^{n_2}\right) + \alpha_1^3\left[-\mu_0^2 n_1 m_{10}^{3n_1-1} - \mu_0^3 n_1(2\,n_1 - 1) m_{10}^{3n_1-2} + 2\mu_0^2 n_1 m_{10}^{2n_1-1} - 2\mu_0 m_{10}^{n_1}\right] + \alpha_2^3\left[-\mu_0^2 n_2 m_{20}^{3n_2-1} - \mu_0^3 n_2(2\,n_2 - 1) m_{20}^{3n_2-2} + 2\mu_0^2 n_2 m_{20}^{2n_2-1} - 2\mu_0 m_{20}^{n_2}\right] + \alpha_1^2\alpha_2 m_{20}^{n_2}\left[-\mu_0^2 n_1 m_{10}^{2n_1-1} + 2\mu_0^2 n_1 m_{10}^{2n_1-1} - 2\mu_0 m_{10}^{n_1} - 4\mu_0 m_{10}^{2n_1}\right] + \alpha_2^2\alpha_1 m_{10}^{n_1}\left[-\mu_0^2 n_2 m_{20}^{2n_2-1} + 2\mu_0^2 n_2 m_{20}^{2n_2-1} - 2\mu_0 m_{20}^{n_2} - 4\mu_0 m_{20}^{2n_2}\right]\right\}$$

Last, we substitute the value of $\frac{\dot{\mu}}{\mu}$ and μ^2 and μ into the Hamiltonian we get:

$$H = -\frac{1}{2L_1^2}\left\{\mu_0^2 - 2\mu_0\left(\alpha_1 m_{10}^{n_1} + \alpha_2 m_{20}^{n_2}\right)(t - t_0) + \alpha_1^2\left(n_1 \mu_0 m_{10}^{2n_1-1} + m_{10}^{2n_1}\right)(t - t_0)^2 + \alpha_2^2\left(n_2 \mu_0 m_{20}^{2n_2-1} + m_{20}^{2n_2}\right)(t - t_0)^2 + 2\alpha_1\alpha_2 m_{10}^{n_1} m_{20}^{n_2}\,(t - t_0)^2 + \alpha_1^3\left(n_1 m_{10}^{3n_1-1}\right)(t - t_0)^3 + \alpha_2^3\left(n_2 m_{20}^{3n_2-1}\right)(t - t_0)^3 + \alpha_1^2\,\alpha_2\left(n_1 m_{10}^{2n_1-1} m_{20}^{n_2}\right)(t - t_0)^3 + \alpha_2^2\,\alpha_1\left(n_2 m_{20}^{2n_2-1} m_{10}^{n_1}\right)(t - t_0)^3\right\} + \left\{-\frac{1}{\mu_0}\left(\alpha_1 m_{10}^{n_1} + \alpha_2 m_{20}^{n_2}\right) + \frac{1}{\mu_0^2}(t - t_0)\left[\alpha_1^2\left(\mu_0 n_1 m_{10}^{2n_1-1} - m_{10}^{2n_1}\right) + \alpha_2^2\left(\mu_0 n_2 m_{20}^{2\,n_2-1} - m_{20}^{2n_2}\right) - \left(2\alpha_1\alpha_2 m_{10}^{n_1} m_{20}^{n_2}\right)\right] + \frac{1}{2\mu_0^4}\,(t - t_0)^2\left\{\alpha_1^2\left(-2\mu_0^2 m_{10}^{2n_1}\right) + \alpha_2^2\left(-2\mu_0^2 m_{20}^{2n_2}\right) + \alpha_1\alpha_2\left(-4\mu_0^2 m_{10}^{n_1} m_{20}^{n_2}\right) + \alpha_1^3\left[-\mu_0^2 n_1 m_{10}^{3n_1-1} - \mu_0^3 n_1(2\,n_1 - 1) m_{10}^{3n_1-2} + 2\mu_0^2 n_1 m_{10}^{2n_1-1} - 2\mu_0 m_{10}^{n_1}\right] + \alpha_2^3\left[-\mu_0^2 n_2 m_{20}^{3n_2-1} - \mu_0^3 n_2(2\,n_2 - 1) m_{20}^{3n_2-2} + 2\mu_0^2 n_2 m_{20}^{2n_2-1} - 2\mu_0 m_{20}^{n_2}\right] + \alpha_1^2\alpha_2 m_{20}^{n_2}\left[-\mu_0^2 n_1 m_{10}^{2n_1-1} + 2\mu_0^2 n_1 m_{10}^{2n_1-1} - 2\mu_0 m_{10}^{n_1} - 4\mu_0 m_{10}^{2n_1}\right] + \alpha_2^2\alpha_1 m_{10}^{n_1}\left[-\mu_0^2 n_2 m_{20}^{2n_2-1} + 2\mu_0^2 n_2 m_{20}^{2n_2-1} - 2\mu_0 m_{20}^{n_2} - 4\mu_0 m_{20}^{2n_2}\right]\right\}\right\} L_1 e \sin E -$$

$\beta \frac{L_2 e}{L_1^3}\Big\{\mu_0 - (\alpha_1 m_{10}^{n_1} + \alpha_2 m_{20}^{n_2})(t-t_0) + \frac{1}{2}(\alpha_1^2 n_1 m_{10}^{2n_1-1} + \alpha_2^2 n_2 m_{20}^{2n_2-1})(t-t_0)^2 + \frac{1}{6}[-\alpha_1^3 n_1(2\,n_1 - 1)m_{10}^{3n_1-2} - \alpha_2^3 n_2 (2\,n_2-1)\, m_{20}^{3n_2-2}](t-t_0)^3\Big\}\left(\frac{a}{r}\right)^2 sinE$ (6)

Finaly, if we assume $\alpha_1, \alpha_2 and\ \beta$, have the same order of magnitude, then we will write the Hamiltonian, up to order 2, in summation form:

$$\boldsymbol{H} = \sum_{i=0}^{3} \boldsymbol{H_i} \quad (7)$$

$$H_0 = -\frac{\mu_0^2}{2L_1^2} \quad (7.1)$$

$$H_1 = \sum_{i=1}^{3} H_{1i} \quad (7.2)$$

$$H_{11} = \alpha_1\left[\frac{\mu_0\, m_{10}^{n_1}}{L_1^2}(t-t_0) - \frac{m_{10}^{n_1}}{\mu_0}\, L_1 esinE\right] \quad (7.2.1)$$

$$H_{12} = \alpha_2\left[\frac{\mu_0\, m_{20}^{n_2}}{L_1^2}(t-t_0) - \frac{m_{20}^{n_2}}{\mu_0}\, L_1 esinE\right] \quad (7.2.2)$$

$$H_{13} = -\beta\frac{\mu_0 L_2 e}{L_1^3}\left(\frac{a}{r}\right)^2 sinE \quad (7.2.3)$$

$$H_2 = \sum_{i=1}^{5} H_{2i} \quad (7.3)$$

$$H_{21} = \alpha_1^2\Big\{-\frac{1}{2L_1^2}(n_1\mu_0 m_{10}^{2n_1-1} + m_{10}^{2n_1})(t-t_0)^2 + \frac{1}{\mu_0^2}\big[(\mu_0 n_1 m_{10}^{2n_1-1} - m_{10}^{2n_1})(t-t_0) - m_{10}^{2n_1}(t-t_0)^2\big]L_1 esinE\Big\} \quad (7.3.1)$$

$$H_{22} = \alpha_2^2\Big\{-\frac{1}{2L_1^2}(n_2\mu_0 m_{20}^{2n_2-1} + m_{20}^{2n_2})(t-t_0)^2 + \frac{1}{\mu_0^2}\big[(\mu_0 n_2 m_{20}^{2\,n_2-1} - m_{20}^{2n_2})(t-t_0) - m_{20}^{2n_2}(t-t_0)^2\big]L_1 esinE\Big\} \quad (7.3.2)$$

$$H_{23} = -\alpha_1\alpha_2(2m_{10}^{n_1}m_{20}^{n_2})\Big\{\frac{1}{2L_1^2}(t-t_0)^2 + \frac{1}{\mu_0^2}[(t-t_0) + (t-t_0)^2]L_1 esinE\Big\} \quad (7.3.3)$$

$$H_{24} = \beta\alpha_1\frac{L_2 e}{L_1^3} m_{10}^{n_1}(t-t_0)\left(\frac{a}{r}\right)^2 sinE \quad (7.3.4)$$

$$H_{25} = \beta\alpha_2\frac{L_2 e}{L_1^3} m_{20}^{n_2}(t-t_0)\left(\frac{a}{r}\right)^2 sinE \quad (7.3.5)$$

$$H_3 = \sum_{i=1}^{5} H_{3i} \quad (7.4)$$

$$H_{31} = \alpha_1^3\Big\{-\frac{1}{2L_1^2}(n_1 m_{10}^{3n_1-1})(t-t_0)^3 + \frac{1}{2\mu_0^4}(t-t_0)^2[-\mu_0^2 n_1 m_{10}^{3n_1-1} - \mu_0^3 n_1(2\,n_1-1)m_{10}^{3n_1-2} + 2\mu_0^2 n_1 m_{10}^{2n_1-1} - 2\mu_0 m_{10}^{n_1}]L_1 esinE\Big\} \quad (7.4.1)$$

$$H_{32} = \alpha_2^3\Big\{-\frac{1}{2L_1^2}(n_2 m_{20}^{3n_2-1})(t-t_0)^3 + \frac{1}{2\mu_0^4}(t-t_0)^2[-\mu_0^2 n_2 m_{20}^{3n_2-1} - \mu_0^3 n_2(2\,n_2-1)m_{20}^{3n_2-2} + 2\mu_0^2 n_2 m_{20}^{2n_2-1} - 2\mu_0 m_{20}^{n_2}]L_1 esinE\Big\} \quad (7.4.2)$$

$$H_{33} = \alpha_1\alpha_2^2\Big\{-\frac{1}{2L_1^2}(n_2 m_{20}^{2n_2-1} m_{10}^{n_1})(t-t_0)^3 + \frac{1}{2\mu_0^4}(t-t_0)^2 m_{10}^{n_1}[-\mu_0^2 n_2 m_{20}^{2n_2-1} + 2\mu_0^2 n_2 m_{20}^{2n_2-1} - 2\mu_0 m_{20}^{n_2} - 4\mu_0 m_{20}^{2n_2}]L_1 esinE\Big\} \quad (7.4.3)$$

$$H_{34} = \alpha_2 \alpha_1^2 \left\{ -\frac{1}{2L_1^2} \left(n_1 m_{10}^{2n_1-1} m_{20}^{n_2} \right) (t - t_0)^3 + \frac{1}{2\mu_0^4} (t - t_0)^2 m_{20}^{n_2} \left[-\mu_0^2 n_1 m_{10}^{2n_1-1} + 2\mu_0^2 n_1 m_{10}^{2n_1-1} - 2\mu_0 m_{10}^{n_1} - 4\mu_0 m_{10}^{2n_1} \right] L_1 e \sin E \right\} \quad (7.4.4)$$

$$H_{35} = \beta \alpha_1^2 \left\{ -\frac{L_2 e}{2L_1^3} \left(n_1 m_{10}^{2n_1-1} \right) (t - t_0)^2 \right\} \left(\frac{a}{r} \right)^2 \sin E \quad (7.4.5)$$

$$H_{36} = \beta \alpha_2^2 \left\{ -\frac{L_2 e}{2L_1^3} \left(n_2 m_{20}^{2n_2-1} \right) (t - t_0)^2 \right\} \left(\frac{a}{r} \right)^2 \sin E \quad (7.4.6)$$

3. Development of the Hamiltonian:

Since the Hamiltonian H is explicitly time dependent, we will develop the Hamiltonian of the problem by introducing a new two pairs of canonical variables (ℓ_4, L_4) and (ℓ_5, L_5). ℓ_4 and ℓ_5 are the rate change of mass and L_4 and L_5 are represent their conjugate momentum. The new variable are defined as:

$$\ell_4 = m_{10}^{n_1} (t - t_0) \quad \rightarrow \quad \dot{\ell}_4 = m_{10}^{n_1}$$

and,

$$\ell_5 = m_{20}^{n_2} (t - t_0) \quad \rightarrow \quad \dot{\ell}_5 = m_{20}^{n_2}$$

From Hamilton's equations of motion, we have:

$$\dot{\ell}_4 = \frac{\partial K}{\partial L_4} = m_{10}^{n_1} \quad \rightarrow \quad K = m_{10}^{n_1} L_4 + F(\ell_i, L_i)$$

$$\dot{\ell}_5 = \frac{\partial K}{\partial L_5} = m_{20}^{n_2} \quad \rightarrow \quad K = m_{20}^{n_2} L_5 + F(\ell_i, L_i)$$

Where $F(\ell_i, L_i)$ is arbitrary function of the old variables and momenta. We can choose it to be the old Hamiltonian.

The new Hameltonian, K, in the extended phase space, up to second order, is given by:

$$K = \sum_{i=1}^{3} K_i \quad (8)$$

where,

$$K_0 = -\frac{\mu_0^2}{2L_1^2} + m_{10}^{n_1} L_4 + m_{20}^{n_2} L_5 \quad (8.1)$$

$$K_1 = \sum_{i=1}^{3} K_{1i} \quad (8.2)$$

$$K_{11} = \alpha_1 \left[\frac{\mu_0}{L_1^2} \ell_4 - \frac{m_{10}^{n_1}}{\mu_0} L_1 e \sin E \right] \quad (8.2.1)$$

$$K_{12} = \alpha_2 \left[\frac{\mu_0}{L_1^2} \ell_5 - \frac{m_{20}^{n_2}}{\mu_0} L_1 e \sin E \right] \quad (8.2.2)$$

$$K_{13} = -\beta \frac{\mu_0 L_2 e}{L_1^3} \left(\frac{a}{r} \right)^2 \sin E \quad (8.2.3)$$

$$K_2 = \sum_{i=1}^{5} K_{2i} \quad (8.3)$$

$$K_{21} = \alpha_1^2 \left\{ -\frac{1}{2L_1^2} \left(\frac{n_1 \mu_0}{m_{10}} + 1 \right) \ell_4^2 + \frac{1}{\mu_0^2} \left[\left(\frac{n_1 \mu_0}{m_{10}} - 1 \right) m_{10}^{n_1} \ell_4 - \ell_4^2 \right] L_1 e \sin E \right\} \quad (8.3.1)$$

$$K_{22} = \alpha_2^2 \left\{ -\frac{1}{2L_1^2} \left(\frac{n_2 \mu_0}{m_{20}} + 1 \right) \ell_5^2 + \frac{1}{\mu_0^2} \left[\left(\frac{n_2 \mu_0}{m_{20}} - 1 \right) m_{20}^{n_2} \ell_5 - \ell_5^2 \right] L_1 e sinE \right\} \quad (8.3.2)$$

$$K_{23} = -2\alpha_1 \alpha_2 \left\{ \frac{1}{2L_1^2} \ell_4 \ell_5 + \frac{\ell_4}{\mu_0^2} \left[m_{20}^{n_2} + \ell_5 \right] L_1 e sinE \right\} \quad (8.3.3)$$

$$K_{24} = \beta \alpha_1 \frac{L_2 e}{L_1^3} \ell_4 \left(\frac{a}{r} \right)^2 sinE \quad (8.3.4)$$

$$K_{25} = \beta \alpha_2 \frac{L_2 e}{L_1^3} \ell_5 \left(\frac{a}{r} \right)^2 sinE \quad (8.3.5)$$

$$K_3 = \sum_{i=1}^{6} K_{3i} \quad (8.4)$$

$$K_{31} = \alpha_1^3 \left\{ -\frac{n_1}{2\, m_{10} L_1^2} \ell_4^3 + \frac{\ell_4^2 L_1 e}{2\mu_0^3} \left[\frac{\mu_0 n_1 (2 - m_{10}^{n_1})}{m_{10}} - \frac{\mu_0^2 n_1 (2\, n_1 - 1)\, m_{10}^{n_1}}{m_{10}^2} - \frac{2}{m_{10}^{n_1}} \right] sinE \right\} \quad (8.4.1)$$

$$K_{32} = \alpha_2^3 \left\{ -\frac{n_2}{2\, m_{20} L_1^2} \ell_5^3 + \frac{L_1 e}{2\mu_0^3} \ell_5^2 \left[\frac{\mu_0 n_2 (2 - m_{20}^{n_2})}{m_{20}} - \frac{\mu_0^2 n_2 (2\, n_2 - 1)\, m_{20}^{n_2}}{m_{20}^2} - \frac{2}{m_{20}^{n_2}} \right] sinE \right\} \quad (8.4.2)$$

$$K_{33} = \alpha_1 \alpha_2^2 \left\{ -\frac{n_2}{2\, m_{20} L_1^2} \ell_5^2 \ell_4 + \frac{L_1 e}{2\mu_0^3} \left[\mu_0 n_2 \frac{m_{10}^{n_1}}{m_{20}} \ell_5^2 - 2\, \ell_4 \ell_5 - 4\, m_{10}^{n_1} \ell_5^2 \right] sinE \right\} \quad (8.4.3)$$

$$K_{34} = \alpha_2 \alpha_1^2 \left\{ -\frac{n_1}{2\, m_{10} L_1^2} \ell_4^2 \ell_5 + \frac{L_1 e}{2\mu_0^3} \left[\mu_0 n_1 \frac{m_{20}^{n_1}}{m_{10}} \ell_5^2 - 2\ell_4 \ell_5 - 4\, m_{20}^{n_2} \ell_4^2 \right] sinE \right\} \quad (8.4.4)$$

$$K_{35} = \beta \alpha_1^2 \left\{ -\frac{L_2 n_1 e}{2\, m_{10}\, L_1^3} \ell_4^2 \right\} \left(\frac{a}{r} \right)^2 sinE \quad (8.4.5)$$

$$K_{36} = \beta \alpha_2^2 \left\{ -\frac{L_2\, n_2\, e}{2\, m_{20}\, L_1^3} \ell_5^2 \right\} \left(\frac{a}{r} \right)^2 sinE \quad (8.4.6)$$

Conclusions

We obtained in this work the third order Hamiltonian for the tow body problem with varying mass including periastron effects in the extended phase space in terms Delaunay varriabels. The Hamiltonian was developed to be able to solve the system using one of the canonical perturbation techniques introduced by many authors such as Von-Ziple, Horis, Depri, or Kamel. In the Hamiltonian, there is defferent parties determine different effects. The part, K_{31} is rising from the variation of mass of the body have the small parameter α_1 while K_{32} is rising from the variation of mass of the body have the small parameterα_2 . The Hamiltonian part K_{33} is rising from the coupling between second order variation of mass of the body have the small parameter $\alpha_1$and first order variation of mass of the body have the small parameter α_2 . . The Hamiltonian part K_{34} is rising from the coupling between second order variation of mass of the body have the small parameter $\alpha_2$and first order variation of mass of the body have the small parameter α_1 . Finally, K_{34} and K_{35} arising from the periastron effect with the second order of variation of mass for m_1 and m_2 respectivily.

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