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# A unique Solution of Stochastic Partial Differential Equations with Non-Local Initial condition

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**Abstract**

In this paper, we shall discuss the uniqueness "pathwise uniqueness" of the solutions of stochastic partial differential equations (SPDEs) with non-local initial condition,

$$du(x,t) = \sum_{|q|\leq 2m} a_q(x,t)D^q u(x,t)dt + b(u(x,t))dt + \sigma(u(x,t))dB(t)$$

$$u(x,0) = \phi(x) + \sum_{i=1}^{p} c_i u(x,t_i) \tag{1}$$

We shall use the Yamada-Watanabe condition for "pathwise uniqueness" of the solutions of the stochastic differential equation; this condition is weaker than the usual Lipschitz condition. The proof is based on Bihari's inequality.

**Keywords**: Stochastic partial differential equation, Pathwise uniqueness, Bihari's inequality.

## 1 Introduction

Our main result is using the Yamada-Watanabe condition, which relaxes the Lipschitz condition for the pathwise uniqueness of the solutions of stochastic differential equation in [3],[4] in the proof the pathwise uniqueness of (1). Before starting the main theorem, we start with some definitions and theorems necessary for the sequel.

## 2 Materials and Methods

**Definition 1.** *The triple* $(\Omega, \Im, \mathbb{P})$ *consisting of a sample space* $\Omega$*, the* $\sigma$*-algebra* $\Im$ *of subsets of* $\Omega$ *and a probability measure* $\mathbb{P}$ *defined on* $\Im$ *is known as a probability space.*

**Definition 2.** *A filtration is a family* $\{\Im_t\}_{(t>0)}$ *of increasing sub-*$\sigma$*-algebra of* $\Im$ $(i.e., \Im_t \subset \Im_s \subset \Im, \quad \forall \quad 0 \leq t < s < \infty)$.

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**Remark 1.** *The probability space together with its family of increasing sub-$\sigma$-algebra denoted by $(\Omega, \Im, \Im_t, \mathbb{P})$ is called a standard filtration space.*

**Definition 3.** *Let $(\Omega, \Im, \mathbb{P})$ be a probability space. A real-valued function* $\mathrm{X} : \Omega \to R$ *is called $\Im$-measurable or random variable, if for all $a \in R$, $\{\omega \in \Omega : \mathrm{X}(\omega) \leq a\} \in \Im$.*

**Definition 4.** *A family of random variables $\mathrm{X}_t, t \in I$ , where $I \subset R$ is an interval defined on a probability space $(\Omega, \Im, \mathbb{P})$ and indexed by a parameter $t$ takes all possible values of $I$ is called a stochastic process.*

**Definition 5.** *Let $(\Omega, \Im, \Im_t, \mathbb{P})$ be a standard filtration space and $I \subset R$ be an interval. The stochastic process $\mathrm{X}_t$ is said to be $\Im_t$-adapted if for all $t \in I$, the random variable $\mathrm{X}_t$ is $\Im_t$-measurable.*

We further define the expectation $\mathbb{E}[\mathrm{X}] = \int_\Omega \mathrm{X} d\mathbb{P}$, for any random variable X.

**Theorem 1** (Bihari's inequality)**.** *Let $I$ denote an interval of the real line of the form $[a, \infty)$,$[a, b]$ or $[a, b)$ with $a < b$. Let $\beta, \upsilon : I \to [0, \infty)$ and $\gamma : [0, \infty) \to [0, \infty)$ be three functions, where $\upsilon$ and $\gamma$ are continuous on $I$, $\beta$ is continuous on the interior of $I$ with $\int_a^t \beta(s)ds < \infty$ for all $t \in I$ and $\gamma$ is non-decreasing and strictly positive on $(0, \infty)$,*

**a.** *If, for some $\alpha > 0$, the function $\upsilon$ satisfies the inequality*

$$\upsilon(t) \leq \alpha + \int_a^t \beta(s)\gamma(\upsilon(s))ds, \qquad t \in I \tag{2}$$

*then*

$$\upsilon(t) \leq F^{-1}\left(\int_a^t \beta(s)ds\right), \qquad t \in [a, T]$$

*where $F^{-1}$ is the inverse function of*

$$F(x) = \int_a^x \frac{dy}{\gamma(y)}, \qquad x > 0.$$

*and $T = sup\{t \in I | \int_a^t \beta(s)ds < \int_\alpha^\infty \frac{dy}{\gamma(y)}\}$*

**b.** *If the function $\upsilon$ satisfies (2) with $\alpha = 0$ and $\int_0^x \frac{dy}{\gamma(y)} = +\infty \quad \forall x > 0$ then $\upsilon(t) = 0$ $t \in I$.*

*Proof.* for proof see [5]. □

**Definition 6.** *$\mathfrak{L}^2(\Omega, \mathcal{H})$; collection of all strongly measurable $\mathcal{H}$-valued random variables is a banach space equipped with the norm $\| \bullet \|_{\mathfrak{L}^2} := \left[\mathbb{E} \| \bullet \|^2_{\mathcal{H}}\right]^{1/2}$*

**Theorem 2.** *Consider the SPDE's*

$$du(x, t) = \sum_{|q| \leq 2m} a_q(x, t)D^q u(x, t)dt + b(u(x, t))dt + \sigma(u(x, t))dB(t)$$

*with non-local initial condition*

$$u(x, 0) = \phi(x) + \sum_{i=1}^{p} c_i u(x, t_i)$$

*where* $x \in R^n$, $B(t)$ *is a standard Brownian motion defined over the standard filtration space* $(\Omega, \Im, \Im_t, \mathbb{P})$, $D = (D_1, \cdots, D_n)$, $q = (q_1, \cdots, q_n)$, $D_i := \frac{\partial}{\partial x_i}$, $D^q = D_1^{q_1}, \cdots, D_n^{q_n}$ *and* $q$ *is a multi-index,* $|q| = q_1 + \cdots + q_n$, $0 \le t_1 < \cdots < t_p$.

*Equation (1) is called parabolic in the region* $\Gamma = \{(x,t) : x \in R^n, t \ge 0\}$, *if for any point* $(x,t) \in \Gamma$ *the real part of the* $\lambda$*-roots of the characteristic equation*

$$Det\left[(-1)^m \sum_{|q|=2m} a_q(x,t)\xi^q - \lambda \mathbf{I}\right] = 0,$$

*satisfy the inequality* $Re[\lambda(x,t,\xi)] \le -\delta|\xi|^m$ *where* $\delta$ *is a positive constant,* $\xi \in R^n, \xi^q = \xi_1^{q_1} \cdots \xi_n^{q_n}$, $\mathbf{I}$ *is the unit matrix. We suppose that the coefficients* $a_q$, $|q| \le 2m$ *are continuous and bounded on* $R^{n+1}$ *and satisfy the HÖlder condition with respect to* $x$. *Under these conditions, there exists a fundamental solution* $\Theta(x,t,y,\theta)$ *which satisfies*

1. $\frac{d\Theta}{dt} = \sum_{|q| \le 2m} a_q(x,t) D^q \Theta(x,t,y,\theta), \quad t > 0, \quad x, y \in R^n.$
2. $\frac{\partial\Theta}{\partial t}$ *and* $D^q\Theta \in \mathcal{C}(\Gamma_1)$ *such that* $\Gamma_1 = \{(x,t,y,\theta) \in R^{2n} \times (0,\infty) \times (0,\infty)\}, \quad |q| \le 2m.$
3. $\| D^q(x,t,y,\theta) \| \le [\frac{A_1}{t^\zeta}] e^{-A_2\zeta_1}, \zeta_1 = \sum_{i=1}^n |x_i - y_i|^{\frac{2m}{2m-1}} t^{\frac{-1}{2m-1}}, \zeta = -\frac{n+|q|}{2m}$ *and* $A_1, A_2$ *are positive constants.*

**Definition 7.** *By a solution of the equation (1), we mean a family of stochastic processes* $\Upsilon = \{u, B(t)\}$ *defined on a standard filtration space* $(\Omega, \Im, \Im_t, \mathbb{P})$ *such that*

1. *With probability one,* $u$ *and* $B(t)$ *are continuous in* $t$ *and* $B(0) = 0$.
2. *They are adapted to* $\Im_t$ , *i.e., for each* $t$, $u$ *and* $B(t)$ *are* $\Im_t$*-measurable.*
3. $B(t)$ *is a system of* $\Im_t$*-martingale such that* $\langle B^i, B^j \rangle = \delta_{ij} \cdot t$, $i, j = 1, 2, \cdots, n$.
4. *Theorem (2) holds.*
5. $\Upsilon = \{u, B(t)\}$ *satisfies*

$$\begin{aligned} u(x,t) &= \int_{R^n} \Theta(x,t,y,0)u(y,0)dy \\ &+ \int_0^t \int_{R^n} \Theta(x,t,y,s)b(u(y,s))dyds \\ &+ \int_0^t \int_{R^n} \Theta(x,t,y,s)b(u(y,s))dydB(s). \end{aligned} \tag{3}$$

*where the integral by* $dB(s)$ *is understood in the sense of the stochastic integral.*

**Definition 8** (Pathwise Uniqueness)**.** *We shall say that the pathwise (strong) uniqueness holds for (1) if, for any two solutions* $\Upsilon = \{u, B(t)\}$ *and* $\bar{\Upsilon} = \{\bar{u}, \bar{B(t)}\}$, *defined on a same filtration space* $(\Omega, \Im, \Im_t, \mathbb{P})$, $u(x,0) = \hat{u}(x,0)$ *and* $B(t) \equiv \hat{B}(t)$ *imply* $u \equiv \bar{u}$.

It supposed that $cM^* < 1$ where $c = \sum_{i=1}^{p} |c_i|$.

**Theorem 3.** *If $u \in \mathcal{C}([0,T];\mathcal{H})$ is an $\Im_t$-adapted stochastic process and satisfies equation (3), then $u(x,t)$ satisfies the following equation*

$$\begin{aligned} u(x,t) &= Z(t)\Lambda^{-1}\phi(x) + Z(t)\Lambda^{-1}\sum_{i=1}^{p} c_i \int_0^{t_i} Z(t_i)b(u(x,s))ds \\ &+ Z(t)\Lambda^{-1}\sum_{i=1}^{p} c_i \int_0^{t_i} Z(t_i)\sigma(u(x,s))dB(s) \\ &+ \int_0^t Z(t)b(u(x,t))ds + \int_0^t Z(t)\sigma(u(x,t))dB(s). \end{aligned} \tag{4}$$

*where $\Lambda = I - \sum_{i=1}^{p} c_i Z(t_i)$ and $Z(t)$ is an operator defined as*

$$Z(t)f = \int_{R^n} \Theta(x,t,y,0) f dy$$

*Proof.*

$$\begin{aligned} \sum_{i=1}^{p} c_i u(x,t_i) &= \sum_{i=1}^{p} c_i \int_{R^n} \Theta(x,t_i,y,0)[\phi(y) + \sum_{j=1}^{p} c_j u(y,t_i)] \\ &+ \sum_{i=1}^{p} c_i \int_0^{t_i}\int_{R^n} \Theta(x,t_i,y,s)b(u(y,s))dyds \\ &+ \sum_{i=1}^{p} c_i \int_0^{t_i}\int_{R^n} \Theta(x,t_i,y,s)\sigma(u(y,s))dydB(s) \end{aligned}$$

$$\begin{aligned} \sum_{i=1}^{p} c_i u(x,t_i) &- \sum_{i=1}^{p} c_i \sum_{j=1}^{p} c_j \int_{R^n} \Theta(x,t_i,y,0)u(y,t_i) \\ &= \sum_{i=1}^{p} c_i \int_{R^n} \Theta(x,t_i,y,0)\phi(y) \\ &+ \sum_{i=1}^{p} c_i \int_0^{t_i}\int_{R^n} \Theta(x,t_i,y,s)b(u(y,s))dyds \\ &+ \sum_{i=1}^{p} c_i \int_0^{t_i}\int_{R^n} \Theta(x,t_i,y,s)?(u(y,s))dydB(s) \end{aligned}$$

$$\begin{aligned} \Lambda\sum_{i=1}^{p} c_i u(y,t_i) &= \sum_{i=1}^{p} c_i Z(t_i)\phi(y) + \sum_{i=1}^{p} c_i \int_0^{t_i} Z(t_i)b(u(x,s))ds \\ &+ \sum_{i=1}^{p} c_i \int_0^{t_i} Z(t_i)\sigma(u(x,s))dB(s) \end{aligned}$$

using $u(x,0)=\phi(x)+\sum_{i=1}^{p} c_i u(x,t_i)$ and multiply with $Z(t)$,

$$\begin{aligned} Z(t)\phi(x)+Z(t)\sum_{i=1}^{p} c_i u(x,t_i) &= Z(t)\phi(x)+Z(t)\Lambda^{-1}\sum_{i=1}^{p} c_i Z(t_i)\phi(x) \\ &+ Z(t)\Lambda^{-1}\sum_{i=1}^{p} c_i\int_0^{t_i} Z(t_i)b(u(x,s))ds \\ &+ Z(t)\Lambda^{-1}\sum_{i=1}^{p} c_i\int_0^{t_i} Z(t_i)\sigma(u(x,s))dB(s), \end{aligned}$$

It is easy to see that $\Lambda^{-1}=I+\Lambda^{-1}\sum_{i=1}^{p} c_i Z(t_i)$ , then we get the result. □

# 3 Main Result

In this section, we state and discuss the main theorem for this paper.

**Theorem 4.** *Let* $\sigma(x)=\begin{bmatrix} \sigma_1(x_1) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_n(x_n) \end{bmatrix}$, $b(x)=(b_1(x),\cdots,b_n(x))$ *such that:*

1. *There exists a positive increasing function* $\rho(\varrho)$, $\varrho\in(0,\infty)$ *such that*

$$|\sigma_i(\tau)-\sigma_i(\eta)|\leq\rho(|\tau-\eta|),\quad \tau,\eta\in R,\quad i=1,2,\cdots,n.$$

*and*

$$\int_{0+}\rho^{-2}(\varrho)d\varrho=+\infty$$

2. *There exists a positive increasing concave function* $\kappa(\varrho)$, $\varrho\in(0,\infty)$ *such that*

$$|b_i(x)-b_i(y)|\leq\kappa(\| x-y \|),\quad x,y\in R^n,\quad i=1,2,\cdots,n.$$

*and*

$$\int_{0+}\kappa^{-1}(\varrho)d\varrho=+\infty$$

3. *Theorem (2) holds.*

*then the pathwise uniqueness of the solutions holds for (1).*

*Proof.* Let $a_0=1>a_1>a_2>\cdots>a_k\to 0$ be defined by

$$\int_{a_1}^{a_0}\rho^{-2}(\varrho)d\varrho=1,\int_{a_2}^{a_1}\rho^{-2}(\varrho)d\varrho=2,\cdots,\int_{a_k}^{a_{k-1}}\rho^{-2}(\varrho)d\varrho=k,\cdots.$$

then there exists a twice continuity differentiable function $\psi_k(\varrho)$ on $[0,\infty)$ such that $\psi_k(0)=0$ ,

$$\psi_k'(\varrho)=\begin{cases} 0 & , \quad 0\leq\varrho\leq a_k \\ between \quad 0 \quad and \quad 1 & , \quad a_k\leq\varrho\leq a_{k-1} \\ 1 & , \quad \varrho\geq a_{k-1} \end{cases}$$

and

$$\psi_k''(\varrho) = \begin{cases} 0 & , \quad 0 \le \varrho \le a_k \\ between \quad 0 \quad and \quad \frac{2}{k}\cdot\rho^{-2}(\varrho) & , \quad a_k \le \varrho \le a_{k-1} \\ 0 & , \quad \varrho \ge a_{k-1} \end{cases}$$

we extend $\psi_k(\varrho)$ on $(-\infty,\infty)$ symmetrically, i.e., $\psi_k(\varrho) = \psi_k(|\varrho|)$ clearly $\psi_k(\varrho)$ is a twice continuously differentiable function on $(-\infty,\infty)$ such that $\psi_k(\varrho) \uparrow |\varrho|$ as $k \to \infty$.

Now let $\{u, B(t)\}$ and $\{\bar{u}, \bar{B(t)}\}$ be two solutions of (1) on the same probability space such that $u(x,0) = \bar{u}(x,0)$ and $B(t) \equiv \bar{B(t)}$ then,

$$\begin{aligned} u^j(x,t) - \bar{u}^j(x,t) &= \int_0^t Z(t)\left[\sigma_j(u^j(y,s)) - \sigma_j(\bar{u}^j(y,s))\right] dB^j(s) \\ &+ Z(t)\Lambda^{-1}\sum_{i=1}^p c_i \int_0^{t_i} Z(t_i)\left[\sigma_j(u^j(y,s)) - \sigma_j(\bar{u}^j(y,s))\right] dB^j(s) \\ &+ Z(t)\Lambda^{-1}\sum_{i=1}^p c_i \int_0^{t_i} Z(t_i)\left[b_j(u(y,s)) - b_j(\bar{u}(y,s))\right] ds \\ &+ \int_0^t Z(t)\left[b_j(u(y,s)) - b_j(\bar{u}(y,s))\right] ds \end{aligned}$$

According to theorem (3), there is a positive constant $M$ such that $\| Z(t) \|_{\mathcal{H}} \le M$, and by Ito's formula,

$$\begin{aligned} &\psi_k(u(x,t) - \bar{u}(x,t)) = \int_0^t \psi_k'(u^j - \bar{u}^j) Z(t)\left[\sigma_j(u^j(y,s)) - \sigma_j(\bar{u}^j(y,s))\right] dB^j(s) \\ &+ \int_0^t \psi_k'(u^j - \bar{u}^j)\left[Z(t)\Lambda^{-1}\sum_{i=1}^p c_i \int_0^{t_i} Z(t_i)\{\sigma_j(u^j(y,s)) - \sigma_j(\bar{u}^j(y,s))\}\right] dB^j(s) \\ &+ \int_0^t \psi_k'(u^j - \bar{u}^j)\left[Z(t)\Lambda^{-1}\sum_{i=1}^p c_i \int_0^{t_i} Z(t_i)\{b_j(u(y,s)) - b_j(\bar{u}(y,s))\}\right] ds \\ &+ 1/2\int_0^t \psi_k''(u^j - \bar{u}^j)\left[Z(t)\Lambda^{-1}\sum_{i=1}^p c_i \int_0^{t_i} Z(t_i)\{\sigma_j(u^j(y,s)) - \sigma_j(\bar{u}^j(y,s))\}\right]^2 ds \\ &+ \int_0^t \psi_k'(u^j - \bar{u}^j)\left[Z(t)\{b_j(u(y,s)) - b_j(\bar{u}(y,s))\}\right] ds \\ &+ 1/2\int_0^t \psi_k''(u^j - \bar{u}^j)\left[Z(t)\{\sigma_j(u^j(y,s)) - \sigma_j(\bar{u}^j(y,s))\}\right]^2 ds \\ &= I_1 + I_2 + I_3 + I_4 + I_5 + I_6 \end{aligned}$$

It is clear that $\mathbb{E}[I_1] = \mathbb{E}[I_2] = 0$ and since $\psi_k'$ is uniformly bounded, $\kappa$ is concave

$$\begin{aligned} \| \mathbb{E}[I_5] \| &\le k_1 \int_0^t \mathbb{E}\left[\kappa(\| u - \bar{u} \|)\right] ds \\ &\le k_1 \int_0^t \kappa(\mathbb{E} \| u - \bar{u} \|) ds \end{aligned}$$

by Jensen's inequality. Similarly for $I_3$.
We have, for $I_6$

$$\begin{aligned} \| I_6 \| &\leq 1/2 \int_0^t \psi_k''(u^j - \bar{u}^j) \| Z(t) \|^2 \rho^2(| u^i - \bar{u}^i |) ds \\ &\leq k_2 \cdot t \max_{a_k \leq |\varrho| \leq a_{k-1}} \left[\psi_k''(\varrho)\rho^2(\varrho)\right] \\ &\leq k_2 \cdot t \cdot \frac{2}{k} \to 0 \qquad as \quad k \to \infty \end{aligned}$$

Similarly for $I_4$.

Where $k_1$ and $k_2$ are positive constants. Also, $\psi_k(u^i - \bar{u}^i) \uparrow | u^i - \bar{u}^i |$ as $k \to \infty$,

$$\mathbb{E}(| u^i - \bar{u}^i |) \leq k_1 \int_0^t \kappa(\mathbb{E} \| u - \bar{u} \|) ds, \quad i = 1, 2, \cdots, n$$

and hence, we have

$$\mathbb{E}(\| u - \bar{u} \|) \leq k_3 \int_0^t \kappa(\mathbb{E} \| u - \bar{u} \|) ds,$$

where $k_3$ is positive constant.

By using theorem (2), this implies $\mathbb{E}(\| u - \bar{u} \|) = 0$ and therefore $u \equiv \bar{u}$ □

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