How to Cite: Li, Q., & Yang, X. (2018). On Suprememum of a set in a Dedikind Complete Topological Space. JOURNAL OF ADVANCES IN MATHAMTICS 14 (2), 8025 - 8029. Retrieved from https://cirworld.com/index.php/jam/article/view/7807. This work is licensed under a Creative Commons Attribution 4.0 International License.

ON SUPREMUM OF A SET IN A DEDIKIND COMPLETE TOPOLOGICAL SPACE

QINGXIA LI AND XINYAO YANG

Abstract. The supremum for a set in a multi-dimensional, Dedikind complete topological space is defined. The example is given to illustrate that the condition of Dedikind complete is necessary for the existence of supremum.

1. Introduction

In this paper, we will examine the superemum for a set in a multi-dimensional, Dedikind complete space, which is partially ordered by any pointed, closed and convex cones. There are several types of supremum for a set in multi-dimensional Euclidean space which were introduced by Nieuwenhuis [4], Tanino [3], Zowe [5], Kawasaki [6]. The definition in this paper will be an extension of the regular definition of supremum in Euclidean space to any Dedikind complete topological space.

First, in Section 2, several types definition of supremum and maximum in Dedikind complete topological space are defined. The relationships among these different kinds of definitions of supremum are also examined. In Section 3, a counterexample is given to illustrate that Dedikind complete is necessary for the existence of supremum for a set in a multi-dimensional topological space.

2. Preliminaries

In this section, we will discuss the definition of supremum and maximum for sets in a multi-dimensional, Dedikind complete topological space Y, which is partially ordered by any pointed, closed and convex cone K with nonempty interior $Int\ K$ in Y. We henceforth assume that Y is Dedikind complete topological space and K is a pointed, closed and convex cone throughout this paper.

A set D together with a relation $\leqslant$ is both transitive and reflexive such that for any two elements $a, b \in D$, there exists an element $c \in D$, such that $a \leqslant c$ and $b \leqslant c$. Then the relation $\leqslant$ is said to *direct* the set D. We say D converges to z, if for any open set U with $z \in U$, there exists a $d_0 \in D$ such that $d \in U$ whenever $d \geqslant d_0$. A topological space Y is called *Dedekind complete* if for every directed set $D \subseteq Y$ which is bounded above, the least upper bound $\sup D$ of D exists, and the directed set D converges to $\sup D$. We say that $y \geqslant y'$ if and only if $y - y' \in K$ and $y > y'$ if and only if $y - y' \in \text{Int } K$. Given a set $D \subset Y$, we set that $\text{Sup } D = +\infty$ if D is not bounded above and $\text{Sup } D = -\infty$ if D is empty.

Definition 2.1. *Assume that Y is partially ordered by any pointed, closed and convex cone K. For any set $Z \subset Y$, we say that*

Subject Classification: Optimization, Convex Analysis

Key words and phrases: Supremum, Dedikind, Necessary Conditions, Multi-dimensional, Set-valued Mappings.

Received 27 September 2018; Accepted 31 October 2018

Journal:Journal of Advances in Mathematics

Acknowledgements: QL was supported by the NSF grant HRD-CREST-1547757, by MAA Tensor-SUMMA, and XY was supported by the RDF at XJTLU.

(i) $\hat{z} = Max\ Z$ *if and only if* $\hat{z} \in Z$ *and* $Z \subset \hat{z} - K$.
(ii) $\hat{z} \in PMax\ Z$ *if and only if* $\hat{z} \in Z$ *and* $[\hat{z} + (K - \{\vec{0}\})] \cap Z = \emptyset$.
(ii) $\hat{z} \in WPMax\ Z$ *if and only if* $\hat{z} \in Z$ *and* $(\hat{z} + Int\ K) \cap Z = \emptyset$.

Definition 2.2. *Assume that* Y *is partially ordered by any pointed, closed and convex cone* K. *For any set* $Z \subset Y$, *we say that*

(i) $\hat{z} = Sup\ Z$ *if and only if it satisfies:*
  (1) $Z \subset \hat{z} - K$;
  (2) *if* $Z \subset z' - K$, *then* $\hat{z} \in z' + K$.
(ii) $\hat{z} \in PSup\ Z$ *if and only if it satisfies:*
  (1) $[\hat{z} + (K - \{\vec{0}\})] \cap Z = \emptyset$;
  (2) *if* $\hat{z} \in z' + (K - \{\vec{0}\})$, *then there exists* $z \in Z$ *such that* $z \in z' + (K - \{\vec{0}\})$.
(ii) $\hat{z} \in WPSup\ Z$ *if and only if it satisfies:*
  (1) $(\hat{z} + Int\ K) \cap Z = \emptyset$;
  (2) *if* $\hat{z} \in z' + Int\ K$, *then there exists* $z \in Z$ *such that* $z \in z' + Int\ K$.

Remark 2.3. In the definition of Supremum of a set, the part 2) in (ii) and (iii) can be simplified as $\hat{z} - (K - \{\vec{0}\}) \subset Z - (K - \{\vec{0}\})$ and $\hat{z} - Int\ K \subset Y - Int\ K$, which were introduced by Nieuwenhuis [4] and Tanino [3].

Note that $Max\ Z$ and $Sup\ Z$ are both single point sets, while $PMax\ Z$ and $PSup\ Z$ are sets that may contain more than one point. It is obvious that

$$Max\ Z \in PMax\ Z \subset WPMax\ Z,\ PSup\ Z \subset WPSup\ Z,$$

while $Sup\ Z$ is not generally contained in $PSup\ Z$. The following example will serve as a counterexample for this. Let

$$Y = \{y \in \mathbb{R}^2 | |y_1| < 1, |y_2| < 1\} \cup \{y \in \mathbb{R}^2 | -1 \leqslant y_1 \leqslant 0, y_2 = 1\}$$

and $K = \mathbb{R}^2_+$. It is easy to check that

$$Sup\ Y = \{(1,1)\}, PSup\ Y = \{(0,1)\}.$$

3. Supremum in Dedikind Complete Space

We will study some properties of supremum defined in a Dedikind complete topological space Y, which is partially ordered by any pointed, closed and convex cone K.

Proposition 3.1. (1) $Sup\ Z \cap Z = Max\ Z$.
(2) $PMax\ Z = Z \cap PSup\ Z$.
(3) $WPMax\ Z = Z \cap WPSup\ Z$.

Proof. This proposition is obvious from the definition. □

Theorem 3.2. *For a nonempty set* Z *in* Y,

(1) $Sup\ Z \neq \emptyset$ *if and only if* $Z \subset \bar{z} - K$ *for some* $\bar{z} \in Y$.
(2) *If* $Z - K \subsetneq Y$, *then* $PSupZ$ *has at least one element.*
(3) $WPSupZ \neq \emptyset$ *if and only if* $Z - K \neq Y$.

Proof. (1) Assume that $Sup\ Z \neq \emptyset$. Then there exists $\bar{z}$ such that $Z \subset \bar{z} - K$ from the definition. Conversely suppose that $Z \subset \bar{z} - K$ for some $\bar{z} \in Y$. Thus Z is bounded above by $\bar{z}$. Since Y is Dedikind complete space, it follows that $Sup\ Z$ exists. (2) and (3): These results are proved in Nieuwenhuis [4]. □

Let $\overline{Y} = Y \cup \{\infty\} \cup \{ -\infty\}$. Given any set $Z \subset \overline{Y}$, we define the set $A(Z)$ which consists of all points above Z and the set $B(Z)$ containing all points below Z. That is to say that

$$A(Z) = \{y \in \overline{Y} | y > y' \text{ for some } y' \in Z\}$$

and

$$B(Z) = \{y \in \overline{Y} | y < y' \text{ for some } y' \in Z\}$$

respectively. The following two definitions were introduced by Tanino in [2].

Definition 3.3. [2] *Given a set $Z \subset \overline{Y}$, a point $\bar{y} \in \overline{Y}$ is said to be a maximal point of Z if $\bar{y} \in Z$ and there is no $y' \in Z$ such that $\bar{y} < y'$. The set of all maximal points of Z is called the maximum of Z and is denoted by Max Z. The minimum of Z, Min Z, is defined analogously.*

Definition 3.4. [2] *Given a set $Z \subset \overline{Y}$, a point $\bar{y} \in \overline{Y}$ is said to be a supremal point of Z if $\bar{y} \notin B(Z)$ and $B(\bar{y}) \subset B(Z)$, that is, there is no $y \in Z$ such that $\bar{y} < y$ and if the relation $y' < \bar{y}$ implies the existence of some $y \in Z$ such that $y' < y$. The set of all supremal points of Z is called the supremum of Z and is denoted by Sup Z. The infimum of Z, Inf Z, is defined analogously.*

In [2], Tanino proposed that $B(Z) = B(SupZ)$ for a set $Z \subset Y$. Unfortunately, this proposition will not be valid unless the topological space Y is Dedikind complete. The following example shows that it is necessary to require K to be dedekind complete.

Example 3.5. Let $Y = \mathcal{C}([-1, 1], \mathbb{R})$ be the space of all continuous functions from $[-1, 1]$ to $\mathbb{R}$. Let $f_n : [-1, 1] \to \mathbb{R}$ be defined as:

$$f_n(x) = \begin{cases} x^{\frac{1}{n}}, & x > 0 \\ 0 & x \leqslant 0 \end{cases}$$

Then it is clear that $f_n(x) \in \mathcal{C}([-1, 1], \mathbb{R})$. Let $Z = \{f_n : n \in Z\}$. Thus

$$f(x) = \begin{cases} 1, & x > 0 \\ 0, & x \leqslant 0 \end{cases}$$

is the supremum of f_n. But $f(x)$ is not continuous and thus does not belong to $\mathcal{C}([0, 1], \mathbb{R})$. Then Sup $Z = \emptyset$. Thus it follows that $B(\text{Sup } Z) = \{-\infty\}$. However, $B(Z)$ is not empty.

Lemma 3.6. *If the cone $K \subseteq Y$ is Dedekind complete and Int $K \neq \emptyset$, then*

(a) *For all $A \subseteq Y$, Inf A and Sup A exist and are nonempty.*
(b) *For every $x \in A$, there exist $u \in$ Inf A and $v \in$ Sup A such that $u \leqslant x \leqslant v$.*

Proof. We will prove this lemma in two cases:

Case I: If $A = \emptyset$, then Sup $A = \{-\infty\}$. If A is unbounded above, then Sup $A = \{+\infty\}$.

Case II: Suppose $A \neq \emptyset$ has an upper bound $b \in Y$. Let $x \in A$. By Zorn's lemma, there exists a maximal chain $M \subseteq A$ with $x \in M$. Then M is directed and bounded above by b, so M has a least upper bound $d = \sup M$ by Dedekind completeness of the cone K. We claim $d \in$ Sup A. By definition, $x \leqslant d$. Let $d \lneqq q$. If $q \in A$, then $\{q\} \cup M$ being a chain contained in A, larger than M, a contradiction with M is a maximal chain in A. Thus $q \notin A$. Let $p < d$. Then $p +$ Int K is an open set containing d. Since M converges to d, there exists $m \in M$ such that $m \in p +$ Int K. That is equivalent to say that $m - p \in$ Int K, which implies that $p < m \in M$. Thus $d \in$ Sup A. Since Inf $A = -$Sup $(-A)$, it follows for all $x \in A$ that there exists an $e \in$ Inf A such that $x \geqslant e$. This completes the proof. ∎

Theorem 3.7. *Assume two sets $A, B \subset Y$ ordered by a pointed, closed and convex cone K. Then*

$$\mathit{Sup}\ (A+B) \subseteq \mathit{Sup}\ A + \mathit{Sup}\ B.$$

Proof. By Proposition 2.6 in [2], it yields that

$$\text{Sup}\ (A + \text{Sup}\ B) = \text{Sup}\ (A+B).$$

Then it suffices to show that Sup $(A + \text{Sup}\ B) \subseteq \text{Sup}\ A + \text{Sup}\ B$. If $\bar{x} \in \text{Sup}\ (A + \text{Sup}\ B)$, then it satisfies the following two conditions:

(1) There is no $a \in A$ and $\bar{b} \in \text{Sup}\ B$ such that $a + \bar{b} > \bar{x}$.

(2) If $x' < \bar{x}$, then there exists $a' \in A$ and $\bar{b}' \in \text{Sup}\ B$ such that $x' < a' + \bar{b}$.

Next, we will prove that $\bar{x} \in \text{Sup}\ A + \text{Sup}\ B$. First, it is clear that there is no $a \in A$ such that $a > \bar{x} - \bar{b}$ for any fixed $\bar{b} \in \text{Sup}\ B$. Otherwise it will contradict with condition (1). Second, for any $a_0 < \bar{x} - \bar{b}$, it yields that $a_0 + \bar{b} < \bar{x}$. Let $x' = a_0 + \bar{b}$. By condition (2), there exists $a' \in A$ and $\bar{b}' \in \text{Sup}\ B$ such that

$$a_0 + \bar{b} < a' + \bar{b}'.$$

a) If $\bar{b} = \bar{b}'$, then there exists $a' \in A$ such that $a_0 < a'$ holds. Thus $\bar{x} - \bar{b} \in \text{Sup}\ A$ and $\bar{x} \in \text{Sup}\ A + \text{Sup}\ B$ follows.

b) If $\bar{b} \neq \bar{b}'$, then there is no $a' \in A$ such that $a_0 + \bar{b} < a' + \bar{b}$. Thus $a_0 + \bar{b} \in \text{Sup}\ (A + \bar{b})$. Thus $a_0 \in \text{Sup}\ A$ and $a_0 + \bar{b} \in \text{Sup}\ A + \text{Sup}\ B$. Because $x' = a_0 + \bar{b} < \bar{x}$, we then can obtain that $\bar{x} \in \text{Sup}\ A + \text{Sup}\ B$. It completes the proof. □

Example 3.8. This example shows that the equality does not hold in Theorem 3.7. Let K be the positive orthant cone in $\mathbb{R}^2$. For any two vectors $x, y \in \mathbb{R}^2$, we define that $x \leqslant y$ if and only if $y \in x + K$. We let $A = \{\begin{bmatrix}4\\1\end{bmatrix}, \begin{bmatrix}1\\3\end{bmatrix}\}$, and $B = \{\begin{bmatrix}2\\1\end{bmatrix}, \begin{bmatrix}1\\5\end{bmatrix}\}$. Then $A = \text{Sup}A$ and $B = \text{Sup}B$. We calculate that

$$\text{Sup}A + \text{Sup}B = \{\begin{bmatrix}6\\2\end{bmatrix}, \begin{bmatrix}5\\6\end{bmatrix}, \begin{bmatrix}3\\4\end{bmatrix}, \begin{bmatrix}2\\8\end{bmatrix}\}.$$

However the set $\text{Sup}(A+B)$ will become:

$$\text{Sup}(A+B) = \{\begin{bmatrix}6\\2\end{bmatrix}, \begin{bmatrix}5\\6\end{bmatrix}, \begin{bmatrix}2\\8\end{bmatrix}\}.$$

Thus we can obtain that $\text{Sup}(A+B) \subsetneq \text{Sup}A + \text{Sup}B$.

References

[1] Tanino T, Sawaragi Y (1980) Conjugate maps and duality in multiobjective optimization. J Optim Theo Appl 31: 473-499.
[2] Tanino T (1992) Conjugate duality in vector optimization. J Math Anal Appl 167: 84-97.
[3] Tanino T (1988) On Supremum of a Set in a Multi-dimensional Space. J Math Anal Appl 130: 386-397.
[4] Nieuwenhuis JW (1980) Supremal points and generalized duality. Math Oper Optim 11: 41-59.
[5] Zowe J (1975) A duality theory for a convex programming in order-complete vector lattices. J Math Anal Appl 50: 273-287.
[6] Kawasaki H (1981) Conjugate relations and weak differentials. Math Oper Res 6: 593-607.

Department of Mathematics and Computer Science, Fisk University, Nashville,TN, USA
Email address: `qli@fisk.edu`

Department of Mathematical Sciences, Xi'an Jiaotong-Liverpool University, Suzhou, Jiangsu, P. R. China
Email address: `xinyao.yang@xjtlu.edu.cn`