The Universal Coefticient Theorem in the Category of Fuzzy Soft Modules

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Abstract. This paper begins with the basic concepts of chain comlexes of fuzzy soft modules. Later, we introduce short exact sequence of fuzzy soft modules and prove that split short exact sequence of fuzzy soft chain complex. Naturally, we want to investigate whether or not the universal coefficient theorems are satisfied in category of fuzzy soft chain complexes. However, in the proof of these theorems in the category of chain complexes, exact sequence of homology modules of chain complexes is used. Generally, sequence of fuzzy soft homology modules is not exact in fuzzy chain complexes. Therefore in this study, we construct exact sequence of fuzzy soft homology modules under some conditions. Universal coefficients theorem is proven by making use of this idea.

Keywords: Universal Coefticient Theorem, Soft Module, Fuzzy Soft Module, Chain Complex Of Fuzzy Soft Modules, Short Exact Sequence Fuzzy Soft Modules.

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1. Introduction

The concept of fuzzy sets was introduced by Lotfi A. Zadeh in 1965 [22]. Since then tht fuzzysets and fuzzy logic have been applied in many real life problems in uncertain, ambiguous environment. The idea of extending the concepts of fuzzy sets to algebra dates back to the introduction in 1971 by Rosenfeld of fuzzy subgroups of a group [11]. Later several researchers have studied fuzzy modules and then Lopez-Permouth and Malik introduced the category of $R-fz\,modof$ fuzzy left $R-$ modules over a ring R [21]. Ameri and Zahedi defined the concept of fuzzy exact sequence in the category of fuzzy modules, and obtained some results related to these notions [15]. Same researchers have previously introduced the category of fuzzy chain complexes and determined fuzzy homology functor in the category. It was proved that this functor is invariant with respect to fuzzy homotopy given in [14]. Molodtsov [10] initiated the concept of soft set theory as a new mathematical tool for dealing with uncertainties. Later, work on the soft set theory is progressing rapidly. Maji et al. [8,9] have published a detailed theoretical study on soft sets. After Molodtsov's work, some different applications of soft sets were studied in [9]. H. Aktaş and N. Cagman [2] has established a connection betwen soft sets and fuzzy sets and they introduced soft groups. At the same time, they gave a definition of soft groups, soft rings and derived their basic properties ([1,4,7]). Qiu-Mei Sun et al. [20] defined soft modules and investigated their basic properties.

L. Jin-Liang [7] presented fuzzy soft sets and fuzzy soft groups. C. Gunduz and S. Bayramov [5] presented fuzzy soft and intuitionistic fuzzy soft modules. Universal coefficient theorem in the category of fuzzy and intuitionistic fuzzy modules proved in [13,19]. Naturally , we want to investigate whether or not the universal coefficient theorems are satisfied in category of fuzzy chain complexes. However, in the proof of these theorems in the category of chain complexes, exact sequence of homology modules of chain complexes is used. Generally, sequence of fuzzy homology modules is not exact in fuzzy chain complexes. Therefore, in this study, we construct exact sequence of fuzzy homology modules under some conditions. Universal coefficients theorem is proven by making use of this idea.

2. Preminilaries

In this section, we recall necessary information commonly used in intuitionistic fuzzy soft module.

Definition 2.1. ([17]). Let X be an initial universe set and E be a set of parameters. A pair (F,E) is called a soft set over X if only if F is a mapping from E into the set of all subsets of the set, X i.e., $F:E\to P(X)$, whereP(X) is the power set of X.

In other words, the soft set is a parameterized family of subsets of the set X. Every set F(e), for every $e\in E$, may be considered as the set of $e-$elements of the soft set (F,E), or as the set of $e-$approximate elements of the soft set.

According to this manner, a soft set (F,E) is given as consisting of collection of approximations:

$$(F,E)=\{F(e):\ e\in E\}.$$

Definition 2.2 ([4, 11]). Let I^X denote the set of all fuzzy sets on X and $A\subset E$. A pair (f,A) is called a fuzzy soft set over X, where f is a mapping from A into I^X. That is, for each $a\in A$, $f(a)=f_a:X\to I$, is a fuzzy set on X .

Definition 2.3 ([4, 11]). Union of two fuzzy soft sets (f,A) and (g,B) over a common universe X is the fuzzy soft set (h,C), where $C=A\bigcup B$ and

$$h(c)=\begin{cases} f(c), & \text{if } c\in A-B \\ g(c), & \text{if } c\in B-A \\ f(c)\vee g(c), & \text{if } c\in A\bigcap B \end{cases}, \quad \forall c\in C$$

It is denoted as $(f,A)\bigcup(g,B)=(h,C)$.

Definition 2.4 ([4, 11]). Intersection of two fuzzy soft sets (f,A) and (g,B) over a common universe X is the fuzzy soft set (h,C), where $C=A\bigcap B$ and $h(c)=f(c)\wedge g(c)$, $\forall c\in C$.

It is written as $(f,A)\bigcap(g,B)=(h,C)$.

Definition 2.5 ([4, 11]). If (f,A) and (g,B) are two soft sets, then (f,A) and (g,B) is denoted as $(f,A)\wedge(g,B)$. $(f,A)\wedge(g,B)$ is defined as $(h,A\times B)$ where $h(a,b)=f(a)\wedge g(b)$, $\forall(a,b)\in A\times B$.

Now, let M be a left $R-$module, A be any nonemply set. $F:A\to P(M)$ refer to a set-valued function and the pair (F,A) is a soft set over M.

Definition 2.6 ([20]). Let (F,A) be a soft set over M. (F,A) is said to be a soft module over M if and only if $F(x)<M$ for all $x\in A$.

Definition 2.7 ([20]). Let (F,A) and (G,B) be two soft modules over M and N respectively. Then $(F,A)\times(G,B)=(H,\ A\times B)$ is defined as $H(x,y)=F(x)\times G(y)$ for all $(x,y)\in A\times B$.

Proposition 2.8 ([20]). Let (F,A) and (G,B) be two soft modules over M and N respectively. Then $(F,A)\times(G,B)$ is soft module over $M\times N$.

Definition 2.9 ([20]). Let (F,A) and (G,B) be two soft modules over M and N respectively, $f:M\to N$, $g:A\to B$ be two functions. Then we say that (f,g) is a soft homomorphism if the following conditions are satisfied:

(1) f is a homomorphism from M onto N,

(2) g is a mapping from A onto B, and

(3) $f\left(F(x)\right)=G\left(g(x)\right)$ for all $x\in A$.

Definition 2.10 ([11]). Let (F,A) be a fuzzy soft set over M. Then (F,A) is said to be a fuzzy soft module over M iff for each $a\in A$, $F(a)$ is a fuzzy submodule of M

Definition 2.11. Let μ_A be a right fuzzy Λ - module and let v_B be a left fuzzy Λ-module. Given a fuzzy projective presentation

$$\bar{0}\to \mu_{0_R} \overset{\bar{f}}{\to} \mu_{0_r} \overset{\bar{g}}{\to} \mu_A \to \bar{0}$$

Of μ_A, we define

$$\mathrm{F}-\mathrm{Tor}_{\bar{\mathrm{g}}}{}^{\wedge}\quad(\mu_{\mathrm{A}},\mathrm{v}_{\mathrm{B}})=\ker(\ \tilde{\mathrm{f}}=\tilde{\mathrm{f}}\ \otimes\ \tilde{1}:(\mu_0\ \otimes\ \mathrm{v})_{\mathrm{R}\otimes_{\wedge}}\ \mathrm{B}\to(\mu_0\ \otimes\ \mathrm{v})_{\mathrm{r}\otimes_{\wedge}}\mathrm{B}).$$

Thus, the fuzzy sequence

$$\bar{0}\to\mathrm{F}-\mathrm{Tor}_{\bar{\mathrm{g}}}{}^{\wedge}\quad(\mu_{\mathrm{A}},\mathrm{v}_{\mathrm{B}})\xrightarrow{\bar{\mathrm{I}}}(\mu_0\ \otimes\ \mathrm{v})_{\mathrm{R}\otimes_{\wedge}}\ \mathrm{B}\xrightarrow{\bar{\mathrm{f}}\otimes\bar{\mathrm{I}}}(\mu_0\ \otimes\ \mathrm{v})_{\mathrm{P}\otimes_{\wedge}\mathrm{B}})\xrightarrow{\bar{\mathrm{g}}\otimes\bar{\mathrm{I}}}(\mu\otimes\mathrm{v})_{\mathrm{A}\otimes\wedge\mathrm{B}})\to\bar{0}.$$

is exact.

3.Chain complexes of fuzzy soft modules.

Let be for $\forall n\in Z$ (F_n,A)- is fuzzy soft module over module M_n and

$(\partial_n,1_A):(F_n,A)\to(F_{n-1},A)$ is homomorphism of fuzzy soft modules.

Definition 3.1. If for all $a\in A$

$$\{(M_n\,,F_n(a)\ \ ,\partial_n:(M_n,F_n(a))\to(M_{n-1}\,,F_{n-1}(a)\}$$

is a chain complexes of fuzzy modules, then the following sequence is said to be a chain complex of fuzzy soft modules

$$\{(\,F_n,A),(\,\partial_n,(I_A):(\,F_n,A)\to(\,F_{n-1}A)\}\qquad\qquad(1)$$

Definition 3.2. If the condition $J_m\partial_n=ker\partial_{n-1}$ is satisfied at the chain complex $\{(M_n\,,F_n(a)\,,\partial_n:(M_n,F_n(a))\to(M_{n-1}\,,F_{n-1}(a)\}$, then the sequence (1) is said to be an exact sequence of fuzzy soft modules.

Now, let us define morphisms of the chain complexes of fuzzy soft modules.

Definition 3.3. Let $\{(\,F_n,A),\partial_n\},\{(G_n,B),\partial_n'\}$ be chain complexes of soft modules over $\{M_n\}$ and $\{N_n\}$, respectively, $\{f_n:M_n\to N_n\}_n$ be homomorphism of modules and $g:A\to B$ is a mapping of sets. If the following diagram is commutative, for each $a\in A$

$$\begin{array}{ccc}(M_n\,,F_n(a)) & \xrightarrow{\partial_n} & (M_{n-1}\,,F_{n-1}(a))\\ \downarrow f_n & & \downarrow f_{n-1}\\ (N_n,G_n(g(a))) & \xrightarrow[\partial_n']{} & (N_{n-1},G_{n-1}(g(a)))\end{array}$$

then $(\{f_n\},g):\{(\,F_n,A),\partial_n\}\to\{(G_n,B),\partial_n'\}$ is said to be morphism of chain complexes of fuzzy soft modules.

Chain complexes of fuzzy soft modules and morphisms of their forms a category. This category is denoted by *CCSM*.

Definition 3.4. Let $(\{\varphi_n\},g):(\,\{\psi_n\},g):\{(F_n,A),\partial_n\}\to\{(G_n,B),\partial_n'\}$ be morphisms of chain complex of fuzzy soft modules and let $D=\{(D_n,g):(F_n,A)\to(G_{n+1},B)\}$ be a family of homomorphisms of fuzzy soft modules. If the equation $\varphi_n-\psi_n=D_{n-1}\circ\partial_n+\partial_{n+1}'\circ D_n$ is satisfied, then the family of homomorphisms of modules $D=\{(D_n,g):M_n\to N_{n+1}\}_{n\in Z}$ is said to be a chain homotopy morphisms, $(\{\varphi_n\},g),(\{\psi_n\},g)$ is said to be a chain homotopy morphisms and denoted by $(\{\varphi_n\},g)\sim(\{\psi_n\},g)$.

Teorem 3.5. *Chain homotopy relation in the category of fuzzy soft modules is a equivalence relation and is invariant according to composition.*

*Proof***.** Primarily, we show that chain homotopy relation is an equivalence relation.

1) Let $(\varphi,g)=(\{\varphi_n\},g):\{(F_n,A),\partial_n\}\to\{(G_n,B),\partial'_n\}$ be an arbitrary morphism. If $D_n=0$ then

$\varphi_n-\varphi_n=0$. That is $(\varphi,g)\sim(\varphi,g)$.

2) Let (φ,g) with (ψ,g) be a chain homotopy. That is,

$$D_{n-1}\circ\partial_n+\partial'_{n+1}\circ D_n=\varphi_n-\psi_n$$

if $\overline{D}_{n=}-D_n$, in this case

$$\overline{D}_{n-1}\partial_n+\partial'_{n+1}\overline{D}_n=-D_{n-1}\partial_n-\partial'_{n+1}D_n=-(,D_{n-1}\partial_n+\partial'_{n+1}D_n)=-(\varphi_n-\psi_n)$$

and then (φ,g) whit (ψ,g) is a chain homotopy.

3. Let (φ,g) with (ψ,g) and (ψ,g) with (γ,g) be a chain homotopy. We want to show that (φ,g) with (γ,g) is a chain homotopy. If (φ,g) with (ψ,g) is a chain homotopy

$\exists D_n \Rightarrow D_{n-1}\partial_n+\partial_{n+1}D_n=\psi_n-\gamma_n$. If (ψ,g) with (γ,g) is a chain homotopy,

$\exists D'_n \Rightarrow D'_{n-1}\partial_n+\partial'_{n+1}\ D'_n=\psi_n-\gamma_n$.

Let define the homomorphism $D''_n\ as\ D''_n=D_n+D'_n$

$$D''_n\partial_n+\partial'_{n+1}D''_n=$$

$$\left(D_{n-1}+D'_{n-1}\right)\partial_n+\partial'_{n+1}\left(D_n+D'_n\right)=$$

$$D_{n-1}\partial_n+D'_{n-1}\partial_n+\partial'_{n+1}D_n+\partial'_{n+1}D'_n=$$

$$D_{n-1}\partial_n+\partial'_{n+1}D_n+D'_{n-1}\partial_n+\partial'_{n+1}D'_n=$$

$$(\varphi_n-\psi_n)+(\psi_n-\gamma_n)=$$

$$(\varphi_n-\gamma_n)$$

Now, we show the invariance of composition.

$(\{\varphi_{0n}\},g)\sim(\{\psi_{0n}\},g):[\{(F_n,A),\partial_n\}\to\{(G_n,B),\partial'_n\}] \Rightarrow D_{n-1}\partial_n+\partial'_{n+1}D_n=\varphi_{0n}-\psi_{0n}$

$(\{\varphi_{1n}\},h)\sim(\{\psi_{1n}\},h):[\{(G_n,B),\partial'_n\}\to\{(P_n,C),\partial''_n\}] \Rightarrow D'_{n-1}\partial_n+\partial''_{n+1}D'_n=\varphi_{1n}-\psi_{1n}$

For $(\{\varphi_{1n}\},h)\circ(\{\varphi_{0n}\},g),(\{\psi_{1n}\},h)\circ(\{\psi_{0n}\},g):[\{(F_n,A),\partial_n\}\to\{(P_n,C),\partial''_n\}]$ to be a chain homotopy we have to define a homomorphism in following the form

$(D''_n,\omega):[\{(F_n,A),\partial_n\}\to\{(H_n,C),\partial''_n\}]$, $D''_n=D'_{n-1}(\varphi_{0n-1})$

$$D'_{n-1}(\varphi_{0n-1},g)\partial_n+\partial''_{n+1}D'_n(\varphi_{0n},g)=$$

$$D'_{n-1}\left(\partial'_n(\varphi_{0n},g)\right)+\partial''_{n+1}D'_n(\varphi_{0n},g)=$$

$$(D'_{n-1}\partial'\ +\partial''_{n+1}D'_n)(\varphi_{0n},g)=$$

$$(\varphi_{1n},h)(\varphi_{0n},g)-(\psi_{1n},h)(\varphi_{0n},g)$$

Now, we show that $(\{\psi_{1n}\},h)\circ(\{\varphi_{0n}\},g)$ with $(\{\psi_{1n}\},h)\circ(\{\psi_{0n}\},g)$ are chain homotopy. We are look at $(\psi_{1n+1},h)\circ D_n: F_n(a)\to P_{n+1}\left(h\big(g(a)\big)\right)$.

$(\psi_{1n},h)D_{n-1}\partial_n + \partial''_{n+1}(\psi_{1n+1},h)D_n = (\psi_{1n},h)D_{n-1}\partial_n + (\psi_{1n},h)\partial'_{n+1}D_n = (D_{n-1}\partial_n + \partial'_{n+1}D_n)\ (\psi_{1n},h) = \big((\varphi_{0n},g)-(\psi_{0n},g)\big)(\psi_{1n},h) = (\varphi_{0n},g)(\psi_{1n},h) - (\psi_{0n},g)(\psi_{1n},h)$

Then $(\varphi_{0n},g)^\circ(\psi_{1n},h)$ with $(\psi_{1n},g)^\circ(\psi_{0n},h)$

is a chain homotopy. Hence, from the two equalities, $(\{\varphi_{1n}\},h)^\circ(\{\varphi_{0n}\},g)$ with $(\{\psi_{1n}\},h)^\circ(\{\psi_{0n}\},g)$ is a chain homotopy.

Let $(\mathcal{F},A)=\{(F_n,A),\partial_n\}$ be a chain complex of fuzzy soft modules over $\{M_n\}$. We obtain the homology module $H_n\big(M_n,F_n(a)\big)=(ker\partial_n\ /\ Im\partial_{n+1},\widetilde{F_n}(a)$ for the chain complex $\{(M_n,F_n(a)),\partial_n:(M_n,F_n(a))\to(M_{n-1},F_{n-1}(a))\}$ and $\forall a\in A$. Here $\widehat{F_n}(a)$ degree function of quotient fuzzy modules in M_n module. If the exist an one-to-one and covered connection with every submodule of quotient module of M_n and submodule of $M_n$we can think the module $H_n\big(M_n,F_n(a)\big)$ as a fuzzy submodule of M_n. Thus $H_n(F_n,-):A\to P(M_n)$ is a fuzzy soft module.

Definition 3.6. Fuzzy soft module $H_n(\mathcal{F},A)$ is said to be n-dimensional homology fuzzy soft module of chain complexes of fuzzy soft modules $\{(F_n,A),\partial_n\}$.

Now, we show that homology fuzzy soft module is functor. Let $(\varphi=(\{\varphi_n\},g):\{(F_n,A),\partial_n\}\to\{(G_n,B),\partial'_n\}$ be morphisms of chain complexes of fuzzy soft modules. Since $\{\varphi_n:\big(M_n,F_n(a)\big)\to N_n,G_n\big(g(a)\big)\}$ is morphism of chain complexes of fuzzy modules for all $a\in A$, mapping $\varphi_{n*}:H_n(F,a)\to H_n\big(G,g(a)\big)$, defined by $\varphi_{n*}[x]=[\varphi_n(x)]$ for all $[x]\in H_n(F,a)$, is a homomorphism of fuzzy modules, and the following diagram is commutative

$$\begin{array}{ccc} A & \xrightarrow{H_n(F_n,-)} & P(M_n) \\ g\downarrow & & \downarrow f_{n*} \\ B & \xrightarrow[H_n(G_n,-)]{} & P(N_n) \end{array}$$

Then $(\varphi_{n*},g):H_n(F_n,-),A)\to\ (H_n(G_n,-),B)$ is homomorphism of fuzzy soft modules.

Theorem 3.7. *The corresponding* $(\mathcal{F},A)\mapsto H_n(\mathcal{F},A),(\{\varphi_n\},g)\to(\varphi_n)_*$ *is a covariant functor from category CCSM to the category FSM.*

Theorem 3.8. *Homology functor of chain complexes of fuzzy soft modules is invariant according to chain homotopy. That is, if* $\{\varphi_n\}\sim\{\psi_n\}:\{(F_n,A),\partial_n\}\to\{(G_n,B),\partial'_n\}$ *then* $\varphi_{n*}=\psi_{n*}=H_n(\mathcal{F},A)\to\ H_n(G,A)$

Proof. Since $\{\varphi_n\}$ with $\{\psi_n\}$ is a chain homotopy for all $a\in A$, then

$\exists D_n:(F_n,A)\to(G_{n+1},B)$ such that the equation

$D_{n-1}\partial_n+\partial'_{n+1}D_n=\varphi_n-\psi_n$ is satisfied.

Now, we show that $\varphi_{n*}=\ \psi_{n*}:H_n(\mathcal{F},A)\to\ H_n(G,A)$ is satisfied. For $\forall a\in A$ and

$\forall[z]=z+\ Im\partial_{n+1}\in H_n\big(M_n,F_n(a)\big)$ we want to show that

$$\varphi_{n*}(z+\ Im\partial_{n+1})=\psi_{n*}(z+\ Im\partial_{n+1}).$$

That is $\varphi_{n*}\,(z + Im\,\partial_{n+1}) = \varphi_n(z) + Im\,\partial`_{n+1}$

We show that

$$\psi_{n*}(z + Im\,\partial_{n+1}) = \psi_n(t) + Im\,\partial`_{n+1}$$

Since $z \in ker\partial_n$, and from the equation

$$D_{n-1}\partial_n\,(z) + \partial`_{n+1}\,D_n(z) = \partial`_{n+1}\big(D_n(z)\big) = \varphi_n(z) - \psi_n(z)$$

$$\Rightarrow a = \varphi_n(z) - \psi_n(z) \quad \mathcal{I}b \in D_n(z) \quad \partial`_{n+1}\,(b) = a$$

$$\Rightarrow a\epsilon\, J_m\partial`_{n+1}$$

$$\Rightarrow \varphi_n(z) - \psi_n(z)\epsilon J_m\partial`_{n+1}$$

$$[\varphi_n(z)] = \varphi_n(z) + J_m\partial`_{n+1}$$

$$[\psi_n(z)] = \psi_n(z) + J_m\partial`_{n+1}$$

Let $(F`, A)$ is fuzzy soft module over $M`$ and $(F``, A)$ is fuzzy soft module over $M``$

Definition 3.9 If for each $a\epsilon A$ sequence of fuzzy modules

$$0 \to (M`, F`(a)) \overset{i_a}{\to} (M, F(a)) \overset{p_a}{\to} (M``, F``(a)) \to 0$$

is short exact, then the sequence

$$0 \to (F`, A) \to (F, A) \to (F``, A) \to 0$$

of fuzzy soft modules is said short exact sequence

Definition 3.10 If for each $a\epsilon A$ sequence of fuzzy modules

$$0 \to (M`, F`(a)) \to (M, F(a)) \to (M``, F``(a)) \to 0$$

is splitting short exact sequence then

$$0 \to (F`, A) \to (F, A) \to (F``, A) \to 0$$

is called splitting short exact sequence of fuzzy soft modules.

Teorem 3.11. *If the sequence* $0 \to \big(F_n`, A\big) \to \big(F_{n,}\, A\big) \to \big(F_n``A\big) \to 0 \qquad (2)$

is short exact sequence of fuzzy soft chain complexes, then the following sequence of fuzzy soft homology modules

$$\ldots \leftarrow H_{n-1}(F_n`, A) \overset{\partial_*}{\leftarrow} H_n\big(F_n``, A\big) \leftarrow H_n\big(F_{n,}\, A\big) \leftarrow H_n(F_n`, A) \ldots \qquad (3)$$

is exact.

Proof: Firstly, we prove that the sequence of homology modules of chain complexes is exact and homomorphism is fuzzy homomorphism. Since the homomorphism

$\partial_{*n}: H_n(M_n``, F_n''(a)\ ) \to H_{n-1}(M'_{n-1}, F'_{n-1}(a))$ is not homomorphism of fuzzy modules. Sequence of fuzzy homology modules (3) is not generally exact. Since fuzzy short exact sequence (2) is fuzzy splitting, there exist fuzzy homomorphisms

$$J_n : \big((F_n^*, A) \to (F'_n, A\big), q_n : \big(F_n'', A\big) \to (F_n, A), \forall n \in Z$$

such as

$$J_n \circ i_n = \big(1_{(F_n,A)}\big), P_n \circ q_n = 1_{(F_n'',A)}, i_n \circ j_n +, q_n \circ p_n = 1_{(F_n\ \ ,A)}$$

then

$$\bar{d}_n = j_{n-1} \circ\ \partial_n \circ p_n : \big(F_n``\ \ , A\big) \to (F'_{n-1}, A)\ \ \forall\ n \in Z$$

is a fuzzy soft homomorphism of fuzzy soft modules, and the family

$$\bar{d}_n = \{d_n: (F_n``, A) \to (F_n`, A)\}$$

is fuzzy soft homomorphism of fuzzy soft chain complexes having a degree of " -1". Indeed, for the homomorphisms $\bar{d}_n = \{d_n: (F_n``, A) \to (F_n`, A)\}$, the following

$$i_{n-2}(\partial`_{n-1} d_n) = (i_{n-2}\partial`_{n-1})\, J_{n-1}\, \partial_n q_n =$$

$$\partial_{n-1}(i_{n-1}\ J_{n-1})\partial_n q_n\ = \partial_{n-1}(i_{n-1} - q_{n-1}\, p_{n-1})\partial_n q_n =$$

$$\partial_{n-1}\partial_n q_n - \partial_{n-1} q_{n-1}\, P_{n-1}\, \partial_n q_n = -\partial_{n-1} q_{n-1}\, p_{n-1}\partial_n q_n =$$

$$-\partial_{n-1} q_{n-1}(p_{n-1}\, \partial_n) q_n = -\partial_{n-1} q_{n-1}\, \partial_n``P_n q_n =$$

$$-\partial_{n-1} q_{n-1}\partial``_n\ 1_{c``n} = -\partial_{n-1} q_{n-1}\partial``_n\ =$$

$$-(i_{n-2}\ J_{n-2} + q_{n-2}\, p_{n-2})\partial_{n-1} q_{n-1}\partial_n``\ =$$

$$-i_{n-2}(J_{n-2}\partial_{n-1} q_{n-1}\,)\partial``_n\ - q_{n-2}\,(p_{n-2}\, \partial_{n-1}) q_{n-1}\, \partial_n``\ =$$

$$-i_{n-2}(d_{n-1}\ \partial``_n) - q_{n-2}\, \partial``_{n-1}\ (P_{n-1}\, q_{n-1})\partial``_n\ =$$

$$-i_n(d_{n-1}\ \partial``_n)$$

imply that $\partial`_{n-1} d_n = d_{n-1}\ \partial_n''$ is satisfied since i_{n-2} is a monomorphism, that is , the family $\{d_n\}$ is a morphism of chain complexes [2] since $d_n: (F_n``, A) \to (F_{n-1}`\ \ , A)$ is a fuzzy soft homomorphism , $\tilde{d}_n: \big(F_n``, A\big) \to \big(F_n`, A\big)$ is fuzzy homomorphism of fuzzy chain complexes

For each $[z] \epsilon H_n(c``)$

$\partial_{*n}(z) = \big[{i_{n-1}}^{-1} \circ \partial \circ {j_n}^{-1}(z)\big] = [J_{n-1} \circ]\partial_n \circ q_n(z)] = [d_n(z)] = d_{n*}(z),$

These $\bar{\partial}_n: H_n(M_n`, F`(a)) \to H_{n-1}(M_n`, F`(a))$ is a fuzzy homomorphism of fuzzy modules. Therefore, the sequence (2) is exact.

Proposition 3.12. *For each split fuzzy soft short exact sequence of fuzzy soft modules*

$$0 \to (F_, A) \overset{\alpha}{\to} (P_, A) \overset{\beta}{\to} (Q_, A) \to 0$$

over M , and each fuzzy soft module (G, B) over N the sequence

$$0 \to (F_, A) \otimes (G, B) \to (P, A) \otimes (G, B) \to (Q, A) \otimes (G, B) \to 0$$

is fuzzy split short exact sequence.

Proof: In order to prove proposition, it is enough to demonstrate that the fuzzy soft homomorphism

$\bar{\alpha} \otimes \bar{1}$ has a left inverse. The fuzzy soft homomorphism $\bar{\alpha}$ has a left inverse $\bar{\alpha}$. Hence the fuzzy homomorphism $\bar{\alpha} \otimes \tilde{1}$ is the left inverse of the fuzzy soft homomorphism $\bar{\alpha}` \otimes \bar{1}$. Since tensor product in category of fuzzy soft modules is a functor for each fuzzy chain complex

$(\mathcal{F}_{n,} A) = \{(\mathcal{F}_{n,} A), \tilde{\partial}_n\}$ and each fuzzy soft (G, B) over module N the family

$$(F_{n,} A) \otimes (G, B) \to \{(F_{n,} A) \otimes (G, B), \tilde{\partial}_n \otimes \bar{1}_{(G,B)}\}$$

Is a fuzzy soft chain complexes of fuzzy soft modules?

Definition 3.13. Fuzzy soft homology module $H_n\,((\mathcal{F}, A) \otimes (G, B)$ is called homology module with coefficient (G, B) of fuzzy chain complex and is represented by

$H_n\,((\mathcal{F}, A); (G, B))$.

From proposition 2.11, for each split short exact sequence of fuzzy chain complexes

$$0 \to (\mathcal{F}`, A) \to (\mathcal{F}, A) \to (\mathcal{F}``, A\ \ ) \to 0$$

and each fuzzy soft module (G, B) , the sequence

$$0 \to (F`, A) \otimes (G, B) \to (F, A) \otimes (G, B) \to (F_n``, A\ \ ) \otimes (G, B) \to 0$$

is a split fuzzy soft short exact sequence. Then by using Theorem 2.9, we can easily prove the following theorem.

Teorem 3.14. *For each split short exact sequence of fuzzy soft chain complex*

$$0 \to (\mathcal{F}`, A) \to (\mathcal{F}, A) \to (\mathcal{F}``, A\ \ ) \to 0$$

and each fuzzy module (G, B) , the sequence of fuzzy soft homology modules.

$$\ldots \leftarrow H_{n-1}((\mathcal{F}`, A); (G, B)) \leftarrow H_n((F\ \ ``, A); (G, B)) \leftarrow H_n((\mathcal{F}, A); (G, B)) \leftarrow\leftarrow H_n((F_n`, C`); (G, B)) \leftarrow \cdots$$

is exact and functorial.

Let $(\mathcal{F}, A) = \{(F_{n,} A), \tilde{\partial}_n\}$ be fuzzy soft chain complexes and (G, B) be fuzzy soft module. We can show that the homomorphism of fuzzy soft module

$\varphi_n: H_n(\mathcal{F}, A) \otimes (G, B) \to H_n((\mathcal{F}, A); (G, B))$, for each $a \in A, b \in B$ $\varphi_n([z], g) = [z \otimes g]$

is a fuzzy homomorphism.

For each $[z] \otimes g \epsilon H_n(C) \otimes G$

$$\left((\tilde{F}_n(a)) \otimes (G(b)\right)([z] \otimes g) = \vee \left((\overline{F_n}(a)) \times (G(b)\right) \vee ([z], g) == \vee_{(z`,g) \in [z] \otimes g} ((F_n(a))z` \wedge (G,B)g)$$

$$(F_n(a)) \otimes (G(b))(\varphi_n[z] \otimes g) = ((F_n(a)) \otimes (G(b)) =$$

$$\vee_{(z`,g`) \in [z \otimes g]} \left((F_n(a))z`` \wedge (G(b))g`\right)$$

It is a $[z] \otimes g \subset [z \otimes g]$, then

$$\left((F_n(a)) \otimes (G(b))\right)([z] \otimes g) \leq (F_n(a)) \otimes (G(b)))((\varphi_n[z] \otimes g))$$

and $\bar{\varphi}_n$ is a fuzzy homomorphism

Let R be a principial ideal domain and R be a commutative ring [15, 17]

Teorem 3.15. *If* $(\mathcal{F}, A)$ *is a free fuzzy soft chain complex and (G, B) is a fuzzy soft module then there is a functorial fuzzy short exact sequence*

$$0 \to H_n((\mathcal{F}, A) \otimes (G, b) \overset{\bar{\varphi}_n}{\to} H_n((\mathcal{F}, A); (G, B) \to FS - Tor(H_{n-1}((\mathcal{F}, A), (G, B)) \to 0$$

and this sequence a split.

Proof. Let for $\forall\ a \epsilon A$

$$Z(M_n, F_n(a)) = \{Ker\ \bar{\partial}_n \subset (M_n, F_n(a))$$

$$B(M_{n+1,}F_{n+1}(a) = \{Im\ \bar{\partial}_n \subset (M_n, F_n(a))$$

be the subcomplex of $(\mathcal{F}, A)$. The operator "0" is boundary operator of these subcomplexes. Since R is a principial ideal domain, both $Z(M_n, F_n(a))$ and $B(M_{n+1,}F_{n+1}(a)$are free fuzzy chain complexes and there is a fuzzy short exact sequence.

$$0 \to Z\left(M_n, F_n(a)\right) \overset{\alpha}{\to} (M_n, F_n(a)) \overset{\beta}{\to} B(M_{n+1}, F_{n+1}(a)) \to 0 \qquad (4)$$

where the fuzzy homomorphisms

$\tilde{\alpha}_n: \left((F_n, Z(c)\right) \to (F_n C_n)$,$\tilde{\beta}_n: ((F_n, A) \to (F_n, B_{(n-1)(\varepsilon n)})$
are induced from the homomorphisms

$\alpha_n(z) = z,\ \ \beta_n(c) = \partial_n(c)$ for each $n \in Z$. Since $\{B\left(M_{n+1}, F_{n+1}(a)\right)\}$ is a free fuzzy soft chain complex, the short exact sequence (4) is split. Hence from theorem 3.11, the following fuzzy soft exact sequence is obtain

$$\begin{aligned} \ldots \to H_n(Z\left(M_n, F_n(a)\right); (G, B)) \to H_n(\left(M_n, F_n(a)\right); (G, B)) \to H_n(B(\left(M_{n+1}, F_{n+1}(a)\right); (G, B)) \\ \to H_{n-1}(Z\left(M_{n-1}, F_{n-1}(a)\right); (G, B)) \to \cdots \qquad (5) \end{aligned}$$

Since the fuzzy soft chain complex $(B(M_{n+1,}F_{n+1}(a)$ have trivial boundary operators, the boundary operators of fuzzy soft chain complexes $(Z(M_n, F_n(a)) \otimes (G, B)$ and $(B(M_{n+1,}F_{n+1}(a) \otimes (G, B)$ are trivial too. Therefore, we have

$$H_n(Z\left(M_n, F_n(a)\right); (G, B)) \to H_n(\left(M_n, F_n(a)\right) \otimes (G, B)$$

$$H_n(B(\left(M_n, F_n(a)\right); (G, B)) \to H_n(B\left(M_{n-1}, F_{n-1}(a)\right) \otimes (G, B))$$

hence the fuzzy soft exact sequence (5) turns into the fuzzy soft exact sequence

$$\ldots \to (B(M_{n+1},F_{n+1}(a)) \otimes (G,B) \xrightarrow{\tilde{j}_n \otimes 1_{(G,B)}} (Z(M_n,F_n(a)) \otimes (G,B)$$
$$\to H_n\left(M_n,(F_n(a));(G,B)\right)$$
$$\to (B(M_n,F_n(a)) \otimes (G,B) \xrightarrow{j_{n-1} \otimes 1_{(G,B)}} Z(M_{n-1},F_{n-1}(a) \otimes (G,B) \to \cdots \quad (6)$$

Where $\tilde{j}_n : ((B(M_{n+1},F_{n+1}(a)) \to Z(M_n,F_n(a))$ is fuzzy soft embedding homomorphism. From sequence (6) we obtain the following fuzzy soft short exact sequence

$$0 \to coker(\tilde{j}_n \otimes 1_{(G,B)} \to H_n\left(M_n,(F_n(a));(G,B)\right) \to \ker(j_{n-1} \otimes 1_{(G,B)} \to 0 \quad (7)$$

Now, let us consider fuzzy soft short exact sequence of fuzzy modules

$$0 \to B\big(M_{n+1},F_{n+1}(a)\big) \to \big(M_n,F_n(a)\big) \to H_n\big(M_n,F_n(a)\big) \to 0$$

Since $Z(M_n,F_n(a))$is a fuzzy soft free module, there is the following fuzzy soft exact sequence

$$0 \to FS-TorH_n\big(M_n,F_n(a)\big);(G,B)) \to B(\big(M_{n+1},F_{n+1}(a)\big) \otimes (G,B)) \to\to H_n\left(\big(M_n,F_n(a)\big) \otimes (G,B)\right)$$
$$\to 0 \qquad (8)$$

From the sequence (8)

$$coker\left(\tilde{j}_n \otimes 1_{(G,B)}\right) = H_n\left(M_n,(F_n(a))\right) \otimes (G,B) = H_n\big(M_n,F_n(a)\big) \otimes (G,B)$$

$$ker\left(\tilde{j}_n \otimes 1_{(G,B)}\right) = FS-Tor\big(M_n,F_n(a)\big);(G,B)$$

Substituting these with (7), the fuzzy soft short exact sequence

$$0 \to H_n((\mathcal{F},A) \otimes (G,b) \overset{\overline{\varphi}_n}{\to} H_n((\mathcal{F},A);(G,B) \to FS-Tor(H_{n-1}((\mathcal{F},A),(G,B)) \to 0$$

Can be obtained.

If $\tilde{\tau}:(\mathcal{F},A) \to (\mathcal{F}',A)$ is a fuzzy soft morphism of fuzzy soft chain complexes, the following commutative diagram.

$0 \to H_n((\mathcal{F},A) \otimes (G,b) \to H_n((\mathcal{F},A);(G,B) \to FS-Tor(H_{n-1}((\mathcal{F},A),(G,B)) \to 0$ $\tilde{\tau}_* \otimes 1_{(G,B)}$

$(\tilde{\tau} \otimes 1_{(G,B)})_*$ $\tilde{\tau}_* * 1_{(G,B)}$

↓ ↓ ↓

$$0 \to H_n((\mathcal{F}',A) \otimes (G,b) \to H_n((\mathcal{F}',A);(G,B) \to FS-Tor(H_{n-1}((\mathcal{F}',A),(G,B)) \to 0$$

is obtained, since the fact that the operations H_n, $\otimes$, $FS-Tor$ are functors. There fore it is proved that fuzzy soft short exact sequence in the theorem in functorial.

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