Some Remarks on a Class of Finite Projective Klingenberg Planes

Atilla Akpinar[a], İsa Doğan[b], Elif Demirci[b], Zeynep Sena Gürel[b], Bercem Boztemur[b],

[a]University of Uludağ, Faculty of Science and Art, Department of Mathematics, 16059, Bursa, Turkey

[b]University of Uludağ, Institute of Science, Department of Mathematics, 16059, Bursa, Turkey

Abstract

In this article, we deal with a class of projective Klingenberg planes constructed over a plural algebra of order m. Thanks to this, the incidence matrices for some special cases of the class are obtained. Next, the number of collineations of the certain classes are found. Besides, an example of a collineation for these classes are given. Finally, we achieve to carry the obtained results to more general case.

Indexing terms/Keywords: Plural Algebra, Local Ring, Projective Klingenberg Plane, Projective Collineation

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Type (Method/Approach): We take the finite field $\mathbb{Z}_p$ and the finite local ring $\mathbb{Z}_{p^r}$ instead of the field of real number in the definition of real plural algebra of order m. Such an obtained algebra has the structure of a local ring. We know that a plane coordinatized by the local ring is a projective Klingenberg plane. So, we obtain some numerical results by studying the incidence matrices and collineations on classes of the planes.

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Introduction

Jukl, in [1], study on the real plural algebra of order m and so investigated the linear forms on a free finite dimensional module M, especially their kernel. Jukl continued to deal with free finite dimensional modules in [2]. In [3], Erdogan et. al. examined some properties of the modules constructed over the real plural algebra and later, in [4], Ciftci and Erdogan established an n-dimensional projective coordinate space over (n+1)-dimensional module constructed by the help of this real plural algebra. For more detailed information on modules, see [5]. For the algebraic notions that will be used throughout this article, we refer to [6] and [7].

In this article we will study on a class of projective Klingenberg (PK) plane $\mathrm{M}(\mathbf{A})$ constructed over the algebra $\mathbf{A}:=F+F\eta+F\eta^2+\cdots+F\eta^{m-1}$ such that $\eta^m=0$ for $\eta\notin F$ (where F is a field). So, the incidence matrices for some special cases of the class are obtained, by taking the field $\mathbb{Z}_p$ (where p is a prime) instead of F. Also, the number of collineations of these classes are found. Besides, an example of a collineation for the classes are given. Finally, the obtained results are carried over PK plane $\mathrm{M}(\boldsymbol{\mathcal{A}})$, which is more general case than $\mathrm{M}(\mathbf{A})$, constructed over the algebra $\boldsymbol{\mathcal{A}}$ of order m obtained by taking the local ring $\mathbb{Z}_{p^r}$ (where r is a positive integer) instead of F.

Preliminaries

In this section we will give some definitions and results from [1], [8] and [9], which will be the basis of this study.

Definition 1 ([1, Def. 1]) The real plural algebra of order m is every linear algebra A on $\mathbb{R}$ having as a vector space over $\mathbb{R}$ a basis $\{1,\eta,\eta^2,\cdots,, \eta^{m-1}\}$ where $\eta^m=0$ for $\eta\notin\mathbb{R}$.

By Definition 1, we see that an element x of A is of the form $x=a_0+a_1\eta+a_2\eta^2+\cdots+a_{m-1}\eta^{m-1}$ where $a_i\in\mathbb{R}$ for $0\leq i\leq m-1$.

Now we can state the following two results without proof.

Proposition 2 ([1, Prop 1.3]) An element $x=a_0+a_1\eta+a_2\eta^2+\cdots+ a_{m-1}\eta^{m-1}\in A$ is a unit if and only if $a_0\neq 0$.

Proposition 3 ([1, Prop 1.5]) A is a local ring with the maximal ideal ηA. The subsets $\eta^j A$, $1\leq j\leq m$, are all ideals in A. In [1, Prop 1.7], it is stated that A is isomorphic to the linear algebra of matrix $M_{mm}(\mathbb{R})$ of the form

$$\begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_{m-2} & a_{m-1} \\ 0 & a_0 & a_1 & \cdots & \cdots & a_{m-2} \\ 0 & 0 & a_0 & a_1 & \cdots & a_{m-3} \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & \ddots \\ & & & & \ddots & \ddots \\ \vdots & & & & \ddots & a_1 \\ 0 & \cdots & \cdots & & 0 & a_0 \end{bmatrix}$$

where $b_i\in\mathbb{R}$ for $0\leq i\leq m-1$ (for the detailed proof of this fact, see [3].

Now, we will recall some information from [8].

Definition 4 Let $\mathrm{M}=(\mathbf{P},\mathbf{L},\in,\sim)$ consist of an incidence structure $(\mathbf{P},\mathbf{L},\in)$ (points, lines, incidence) and an equivalence relation '~' (neighbour relation) on **P** and on **L**. Then M is called a projective Klingenberg plane (PK-plane), if it satisfies the following axioms:

(PK1) If Q and S are non-neighbour points, then there is a unique line QS through Q and S.

(PK2) If g and h are non-neighbour lines, then there is a unique point $g \wedge h$ on both g and h.

(PK3) There is a projective plane $\mathrm{M}^* = (\mathbf{P}^*, \mathbf{L}^*, \in)$ and incidence structure epimorphism Ψ: $\mathrm{M} \to \mathrm{M}^*$, such that the conditions

$$\Psi(Q)=\Psi(S) \Leftrightarrow Q \sim S, \ \Psi(g)=\Psi(h) \Leftrightarrow g \sim h$$

hold for all $Q,S \in \mathbf{P}$, $g,h \in \mathbf{L}$.

An incidence structure automorphism preserving and reflecting the neighbour relation is called a collineation of M.

A ring R with identity element 1 is called local if the set I of its non-unit elements is an ideal.

Let $\mathcal{R}$ be a local ring with the maximal ideal $\mathbf{I}$. Then $\mathrm{M}(\mathcal{R}) = (\mathbf{P}, \mathbf{L}, \in, \sim)$ is the incidence structure with neighbour relation defined as follows:

$$\mathbf{P} = \{(x, y, 1): x, y \in \mathcal{R}\} \cup \{(1, y, z): y \in \mathcal{R}, y \in \mathbf{I}\} \cup \{(w, 1, z): w, z \in \mathbf{I}\},$$

$$\mathbf{L} = \{[m, 1, k]: m, k \in \mathcal{R}\} \cup \{[1, n, t]: t \in \mathcal{R}, n \in \mathbf{I}\} \cup \{[q, n, 1]: q, n \in \mathbf{I}\},$$

$$[m, 1, k] = \{(x, xm + k, 1): x \in \mathcal{R}\} \cup \{(1, zk + m, z): z \in \mathbf{I}\},$$

$$[1, n, t] = \{(yn + t, y, 1): y \in \mathcal{R}\} \cup \{(zt + n, 1, z): z \in \mathbf{I}\},$$

$$[q, n, 1]\{(1, y, yn + q): y \in \mathcal{R}\} \cup \{(w, 1, wq + n): w \in \mathbf{I}\},$$

$$S = (x_{1,} x_2, x_3) \sim (y_{1,} y_2, y_3) = Q \Leftrightarrow x_i - y_i \in \mathbf{I} \ (i = 1,2,3) \text{ for } \forall\, S, Q \in \mathbf{P}.$$

$$g = [x_{1,} x_2, x_3] \sim [y_{1,} y_2, y_3] = h \Leftrightarrow x_i - y_i \in \mathbf{I} \ (i = 1,2,3) \text{ for } \forall\, g, h \in \mathbf{L}.$$

Now it is time to give the following theorem from [8].

Theorem 5 $\mathrm{M}(\mathcal{R})$ is a PK-plane, and each desarguesian PK-plane is isomorphic to some $\mathrm{M}(\mathcal{R})$.

Now, from [9] we give the definition of an n-gon, which is meaningful when $n \geq 3$: An n-tuple of pairwise non-neighbour points is called an (ordered) n-gon if no three of its elements are on neighbour lines.

The Main Results

Let F be a field. Let $\eta^m = 0$ for $\eta \notin F$. Consider $A := F(\eta) = F + F\eta + F\eta^2 + \cdots + \eta^{m-1}$ with componentwise addition and multiplication as follows:

$$ab = (a_0 + a_1\eta + a_2\eta^2 + \ldots + a_{m-1}\eta^{m-1})(b_0 + b_1\eta + b_2\eta^2 + \ldots + b_{m-1}\eta^{m-1})$$

$$= a_0b_0 + (a_1b_0 + b_1a_0)\eta + (a_2b_0 + a_1b_1 + a_0b_2)\eta^2 + \ldots + (a_{m-1}b_0 + a_0b_{m-1})\eta^{m-1}$$

$$= \sum_{k=0}^{m-1} \sum_{i+j=k} (a_i b_j)\eta^k \quad (a_i, b_j \in F \text{ and } i,j = 0,1,2,3,\ldots,m-1).$$

Then $\mathbf{A}$ is a (unital, commutative and associative) local ring with the maximal ideal $\mathbf{I} = \mathbf{A}\eta$ of non-units. So, by Theorem 5 we have that $\mathrm{M}(\mathbf{A})$ is a PK-plane. Also, by Definition 1, $\mathbf{A}$ can be called as plural F-algebra of order m.

Now we will obtain some results on this finite class of PK-plane by taking $\mathbb{Z}_p$ instead of F. If it is taken $\mathbb{Z}_p$ instead of F, then $|\mathbf{A}|=p^m$ and $|\mathbf{I}|=p^{m-1}$. The number of non-neighbour points and lines in $M(\mathbf{A})$ is p^2+p+1 and the number of points (or lines) in the neighborhood of any point (or any line) is $(p^{m-1})^2$, respectively. Thus, the total number of points and lines in $M(\mathbf{A})$ is $(p^{m-1})^2(p^2+p+1)$. From this results, we see that the plane M^* stated in (PK3) of Definition 4 is a projective plane of order p.

For example, for p=2 and m=2, we can give the following results about the PK-plane $M(\mathbb{Z}_2+\mathbb{Z}_2\eta)=(\mathbf{P},\mathbf{L},\in,\sim)$:

The neighbour classes for points are

$N_1=(0,0,1)\sim(0,\eta,1)\sim(\eta,0,1)\sim(\eta,\eta,1)$,

$N_2=(0,1,1)\sim(\eta,1,1)\sim(0,1+\eta,1)\sim(\eta,1+\eta,1)$,

$N_3=(1,0,1)\sim(1,\eta,1)\sim(1+\eta,0,1)\sim(1+\eta,\eta,1)$,

$N_4=(1,1,1)\sim(1,1+\eta,1)\sim(1+\eta,1,1)\sim(1+\eta,1+\eta,1)$,

$N_5=(1,0,0)\sim(1,0,\eta)\sim(1,\eta,0)\sim(1,\eta,\eta)$,

$N_6=(1,1,0)\sim(1,1,\eta)\sim(1,1+\eta,0)\sim(1,1+\eta,\eta)$,

$N_7=(0,1,0)\sim(0,1,\eta)\sim(\eta,1,0)\sim(\eta,1,\eta)$ where $|\mathbf{P}|=28$.

Similarly the neighbour classes for lines are

$d_1=[0,1,0]\sim[0,1,\eta]\sim[\eta,1,0]\sim[\eta,1,\eta]$,

$d_2=[0,1,1]\sim[0,1,1+\eta]\sim[\eta,1,1]\sim[\eta,1,1+\eta]$,

$d_3=[1,1,0]\sim[1,1,\eta]\sim[1+\eta,1,0]\sim[1+\eta,1,\eta]$,

$d_4=[1,1,1]\sim[1,1,1+\eta]\sim(1+\eta,1,1)\sim(1+\eta,1,1+\eta)$,

$d_5=[1,0,0]\sim[1,0,\eta]\sim[1,\eta,\eta]\sim[1,\eta,0]$,

$d_6=[1,0,1]\sim[1,0,1+\eta]\sim[1,\eta,1]\sim[1,\eta,1+\eta]$,

$d_7=[0,0,1]\sim[0,\eta,1]\sim[\eta,0,1]\sim[\eta,\eta,1]$ where $|\mathbf{L}|=28$.

The incidence relation ∈ with the above results is given by the following incidence matrix (see Table 1):

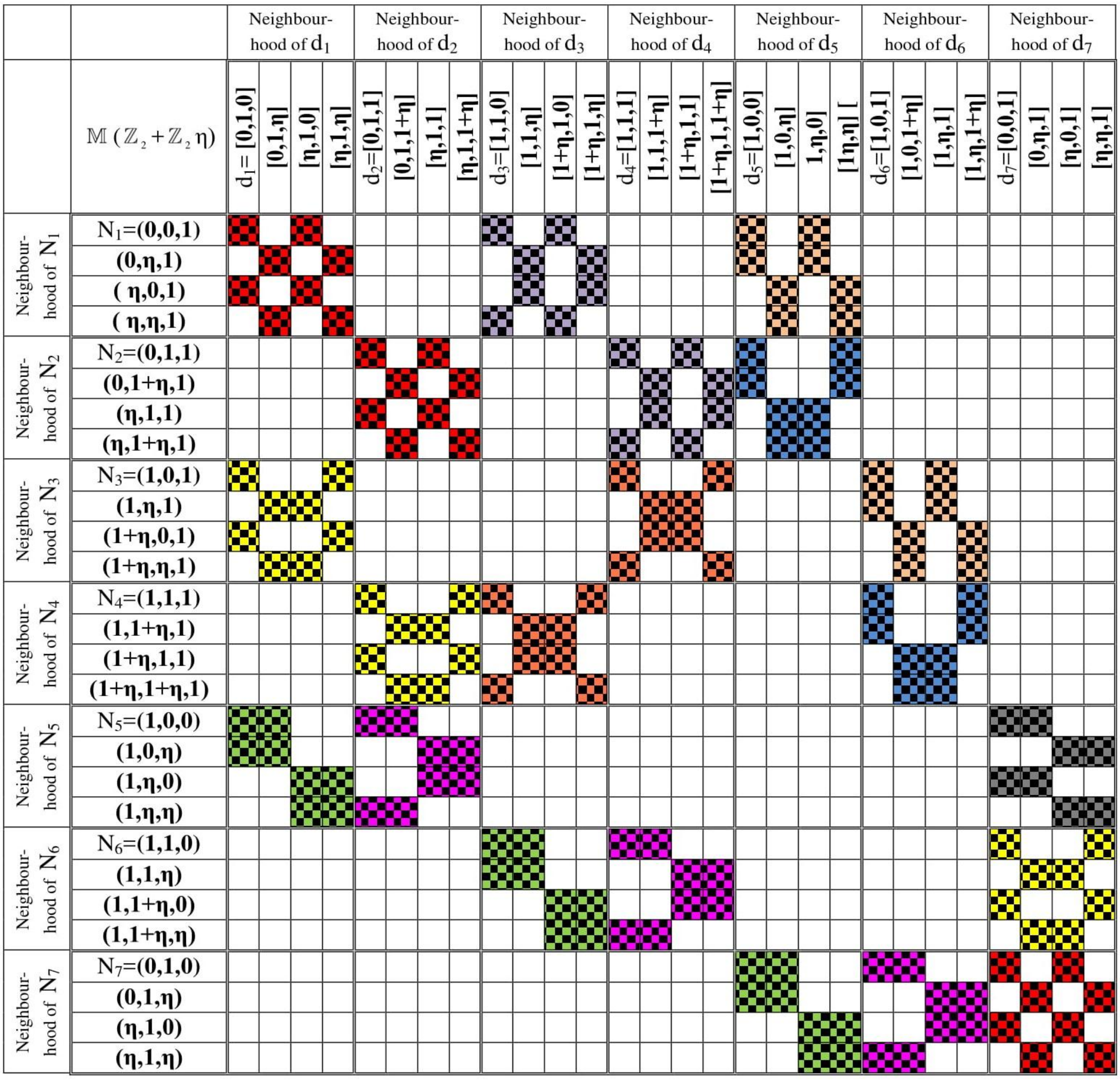


Table 1. The incidence matrix for the plane $\mathbb{M}(\mathbb{Z}_2+\mathbb{Z}_2\eta)$

As a second example, for p=2 and m=3, we can give the following results about the PK-plane $\mathbb{M}(\mathbb{Z}_2+\mathbb{Z}_2\eta+\mathbb{Z}_2\eta^2)$=(**P**,**L**,∈,~):

The neighbour classes for points are

N_1=(0,0,1)~(0,η,1)~(0,η^2,1)~(0,η+η^2,1)~(η,0,1)~(η,η,1)~(η,η^2,1)~(η,η+η^2,1)~(η^2,0,1)~(η^2,η,1)~(η^2,η^2,1)~(η^2,η+η^2,1)~(η+η^2,0,1)~(η+η^2,η,1)~(η+η^2,η^2,1)~(η+η^2,η+η^2,1),

N_2=(0,1,1)~(η,1,1)~(η^2,1,1)~(η+η^2,1,1)~(0,1+η,1)~(η,1+η,1)~(η^2,1+η,1)~(η+η^2,1+η,1)~(0,1+η^2,1)~(η,1+η^2,1)~(η^2,1+η^2,1)~(η+η^2,1+η^2,1)~(0,1+η+η^2,1)~(η,1+η+η^2,1)~(η^2,1+η+η^2,1)~(η+η^2,1+η+η^2,1),

N_3=(1,0,1)~(1,η,1)~(1,η^2,1)~(1,η+η^2,1)~(1+η,0,1)~(1+η,η,1)~(1+η,η^2,1)~(1+η,η+η^2,1)~(1+η^2,0,1)~(1+η^2,η,1)~(1+η^2,η^2,1)~(1+η^2,η+η^2,1)~(1+η+η^2,0,1)~(1+η+η^2,η,1)~(1+η+η^2,η^2,1)~(1+η+η^2,η+η^2,1),

N_4=(1,1,1)~(1,1+η,1)~(1,1+η^2,1)~(1,1+η+η^2,1)~(1+η,1,1)~(1+η,1+η,1)~(1+η,1+η^2,1)~

$(1+\eta,1+\eta+\eta^2,1)\sim(1+\eta^2,1,1)\sim(1+\eta^2,1+\eta,1)\sim(1+\eta^2,1+\eta^2,1)\sim(1+\eta^2,1+\eta+\eta^2,1)\sim(1+\eta+\eta^2,1,1)\sim$

$(1+\eta+\eta^2,1+\eta,1)\sim(1+\eta+\eta^2,1+\eta^2,1)\sim(1+\eta+\eta^2,1+\eta+\eta^2,1)$,

$N_5=(1,0,0)\sim(1,0,\eta)\sim(1,0,\eta^2)\sim(1,0,\eta+\eta^2)\sim(1,\eta,0)\sim(1,\eta,\eta)\sim(1,\eta,\eta^2)\sim(1,\eta,\eta+\eta^2)\sim(1,\eta^2,0)\sim(1,\eta^2,\eta)\sim$

$(1,\eta^2,\eta^2)\sim(1,\eta^2,\eta+\eta^2)\sim(1,\eta+\eta^2,0)\sim(1,\eta+\eta^2,\eta)\sim(1,\eta+\eta^2,\eta^2)\sim(1,\eta+\eta^2,\eta+\eta^2)$,

$N_6=(1,1,0)\sim(1,1,\eta)\sim(1,1,\eta^2)\sim(1,1,\eta+\eta^2)\sim(1,1+\eta,0)\sim(1,1+\eta,\eta)\sim(1,1+\eta,\eta^2)\sim(1,1+\eta,\eta+\eta^2)\sim(1,1+\eta^2,0)\sim$

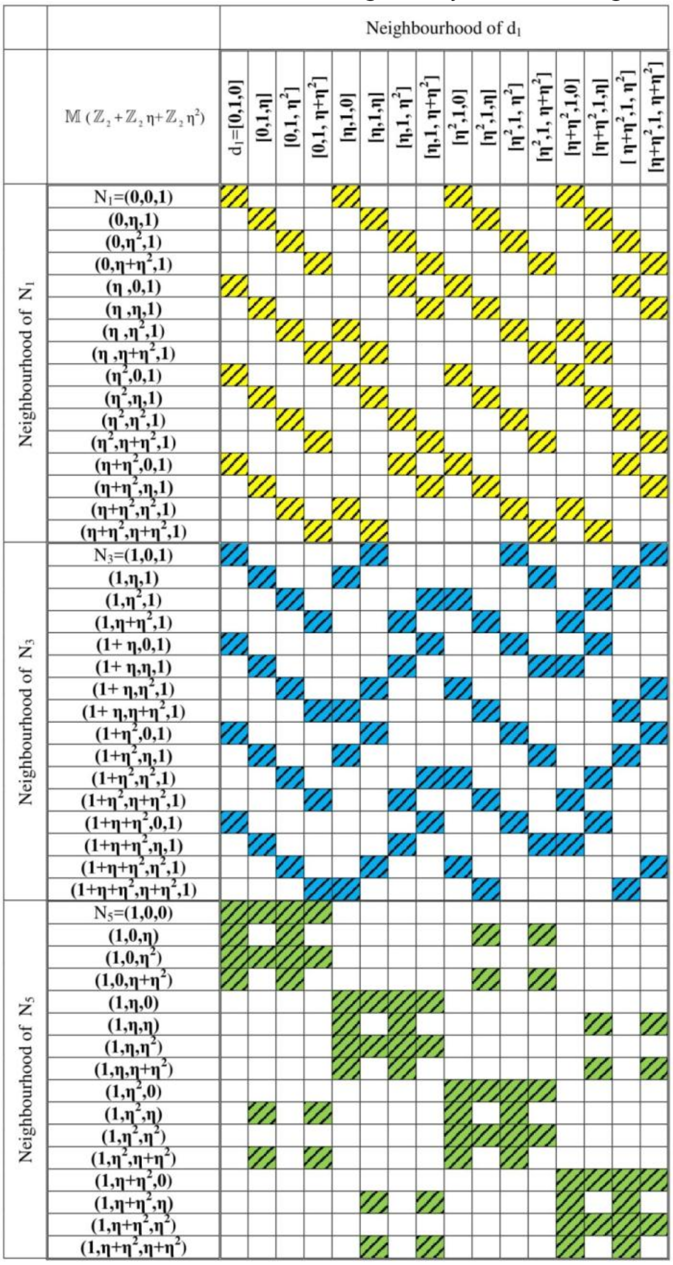
$(1,1+\eta^2,\eta)\sim(1,1+\eta^2,\eta^2)\sim(1,1+\eta^2,\eta+\eta^2)\sim(1,1+\eta+\eta^2,0)\sim(1,1+\eta+\eta^2,\eta)\sim(1,1+\eta+\eta^2,\eta^2)\sim$

$(1,1+\eta+\eta^2,\eta+\eta^2)$,

$N_7=(0,1,0)\sim(0,1,\eta)\sim(0,1,\eta^2)\sim(0,1,\eta+\eta^2)\sim(\eta,1,0)\sim(\eta,1,\eta)\sim(\eta,1,\eta^2)\sim(\eta,1,\eta+\eta^2)\sim(\eta^2,1,0)\sim(\eta^2,1,\eta)\sim$

$(\eta^2,1,\eta^2)\sim(\eta^2,1,\eta+\eta^2)\sim(\eta+\eta^2,1,0)\sim(\eta+\eta^2,1,\eta)\sim(\eta+\eta^2,1,\eta^2)\sim(\eta+\eta^2,1,\eta+\eta^2)$ where $|\mathbf{P}|=112$.

The neighbour classes for lines (where $|\mathbf{L}|=112$) can be similarly written. The incidence relation $\in$ of the line [0,1,0] with the above results is given by the following incidence matrix (see Table 2):



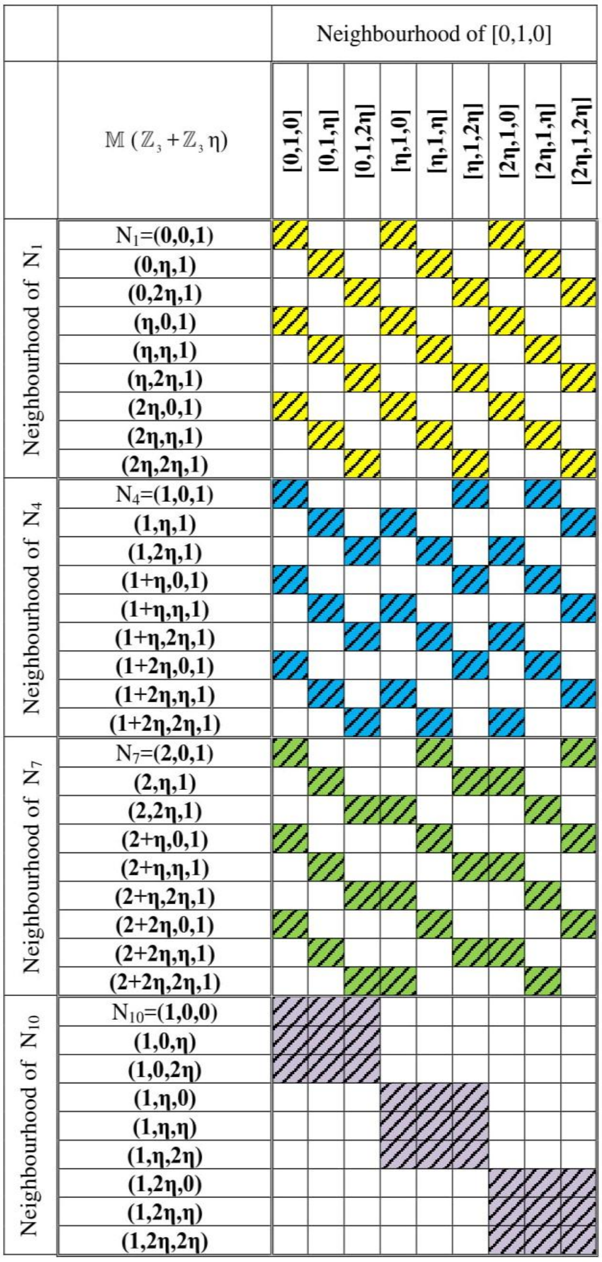


Table 2. The incidence matrix for the

Table 3. The incidence matrix for the line [0,1,0]

line [0,1,0] in the plane $M(\mathbb{Z}_2+\mathbb{Z}_2\eta+\mathbb{Z}_2\eta^2)$ in the plane $M(\mathbb{Z}_3+\mathbb{Z}_3\eta)$

As a last example, for p=3 and m=2, we can give the following results about the PK-plane $M(\mathbb{Z}_3+\mathbb{Z}_3\eta)=(\mathbf{P},\mathbf{L},\in,\sim)$:

The neighbour classes for points are

N_1=(0~0~1)~(0,η,1)~(0,2η,1)~(η,0,1)~(η,η,1)~(η,2η,1)~(2η,0,1)~(2η,η,1)~(2η,2η,1),
N_2=(0,1,1)~(0,1+η,1)~(0,1+2η,1)~(η,1,1)~(η,1+η,1)~(η,1+2η,1)~(2η,1,1)~(2η,1+η,1)~(2η,1+2η,1),
N_3=(0,2,1)~(0,2+η,1)~(0,2+2η,1)~(η,2,1)~(η,2+η,1)~(η,2+2η,1)~(2η,2,1)~(2η,2+η,1)~(2η,2+2η,1),
N_4=(1,0,1)~(1,η,1)~(1,2η,1)~(1+η,0,1)~(1+η,η,1)~(1+η,2η,1)~(1+2η,0,1)~(1+2η,η,1)~(1+2η,2η,1),

N_5=(1,1,1)~(1,1+η,1)~(1,1+2η,1)~(1+η,1,1)~(1+η,1+η,1)~(1+η,1+2η,1)~(1+2η,1,1)~

(1+2η,1+η,1)~(1+2η,1+2η,1),
N_6=(1,2,1)~(1,2+η,1)~(1,2+2η,1)~(1+η,2,1)~(1+η,2+η,1)~(1+η,2+2η,1)~(1+2η,2,1)~

(1+2η,2+η,1)~(1+2η,2+2η,1),
N_7=(2,0,1)~(2,η,1)~(2,2η,1)~(2+η,0,1)~(2+η,η,1)~(2+η,2η,1)~(2+2η,0,1)~(2+2η,η,1)~(2+2η,2η,1),

N_8=(2,1,1)~(2,1+η,1)~(2,1+2η,1)~(2+η,1,1)~(2+η,1+η,1)~(2+η,1+2η,1)~(2+2η,1,1)~

(2+2η,1+η,1)~(2+2η,1+2η,1),
N_9=(2,2,1)~(2,2+η,1)~(2,2+2η,1)~(2+η,2,1)~(2+η,2+η,1)~(2+η,2+2η,1)~(2+2η,2,1)~

(2+2η,2+η,1)~(2+2η,2+2η,1),
N_{10}=(1,0,0)~(1,0,η)~(1,0,2η)~(1,η,0)~(1,η,η)~(1,η,2η)~(1,2η,0)~(1,2η,η)~(1,2η,2η),
N_{11}=(1,1,0)~(1,1,η)~(1,1,2η)~(1,1+η,0)~(1,1+η,η)~(1,1+η,2η)~(1,1+2η,0)~(1,1+2η,η)~(1,1+2η,2η),

N_{12}=(1,2,0)~(1,2,η)~(1,2,2η)~(1,2+η,0)~(1,2+η,η)~(1,2+η,2η)~(1,2+2η,0)~(1,2+2η,η)~(1,2+2η,2η)

N_{13}=(0,1,0)~(0,1,η)~(0,1,2η)~(η,1,0)~(η,1,η)~(η,1,2η)~(2η,1,0)~(2η,1,η)~(2η,1,2^{η}) where $|\mathbf{P}|$=117.

The neighbour classes for lines (where $|\mathbf{L}|$=117) can be easily obtained as in the first example. The incidence relation ∈ of the line [0,1,0] with the above results is given by the following incidence matrix (see Table 3).

Notice that the plane M^* in (PK3) of Definition 4 for the plane $M(\mathbb{Z}_2+\mathbb{Z}_2\eta)$ is the Fano plane of order 2, the smallest finite projective plane. The obtained plane $M(\mathbb{Z}_2+\mathbb{Z}_2\eta)$ is not isomorphic to the finite projective Klingenberg plane $M(\mathbb{Z}_4)$ (see Table 4 for the incidence table) coordinatized by $\mathbb{Z}_4$={0,1,2,3} with the maximal ideal I={0,2} in [10]. For this, it is enough to check the incidence relations $N_3 \in d_4$ in Table 1 and Table 4 since the other incidence relations are the same. The fact that the planes $M(\mathbb{Z}_2+\mathbb{Z}_2\eta)$ and $M(\mathbb{Z}_4)$ are not isomorphic is not new and can be found in the literature at different places, see e.g. [11,12]. Moreover, it can be seen to the papers of [13,14,15] for more detailed information about such finite structures.

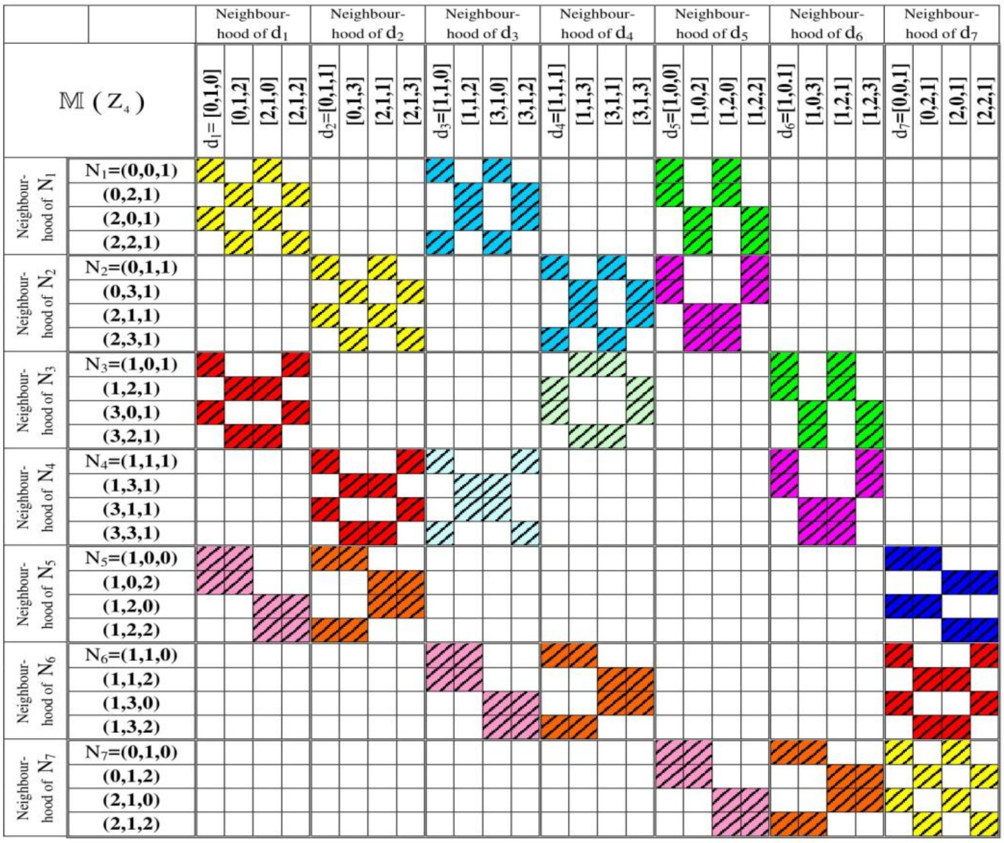


Table 4. The incidence matrix for the plane $\mathbb{M}(\mathbb{Z}_4)$

The number of the collineations of the planes $\mathbb{M}(\mathbb{Z}_2+\mathbb{Z}_2\eta)$ and $\mathbb{M}(\mathbb{Z}_4)$ are $4(7\cdot6\cdot4)=4(168)=672$ as the number of points in neighborhood of a point is 4 and a basis of the plane consists of the 3-gons. For an arbitrary m, we can generalize this result to $\mathbb{Z}_p$ as follows:

The number of the collineations of $\mathbb{M}(\mathbf{A})$ is $(p^{m-1})^2[v(v-1)(v-(p+1))]$ where $v:=p^2+p+1$ is total number of points of a projective plane of order p.

Now, we will give an example of a collineation of PK-plane $\mathbb{M}(\mathbf{A})$. This collineation is analogous to the collineation Gs given in [9]. To show that Gs is a collineation, some similar calculations to the proof of [9, Lemma 3] must be done.

For any $s\in\mathbf{A}$, the collineation Gs transforms points and lines as follows:

$(x,y,1) \rightarrow (x,y+xs,1)$,

$(1,y,z)\rightarrow(1,y+s,z)$,

$(w,1,z)\rightarrow((1+ws)^{-1}w,1,z-((1+ws)^{-1}w)(sz))$

$[m,1,k]\rightarrow[m+s,1,k]$,

$[1,n,t]\rightarrow[1,(1+ns)^{-1}n,t-(ts)((1+ns)^{-1}n)]$,

$[q,n,1]\rightarrow[q-sn,n,1]$.

Note that it can be found many collineations, which are similar to this collineation, of the PK-plane.

Further Results

It is possible to further generalize all results in this study by taking the local ring ($\mathbb{Z}_{p^r}$, I) instead of the field F where p is a prime and r is a positive integer. In this case, it is clear that $|\mathcal{A}| = (p^r)^m$ ve $|\mathbf{I}| = |I|\times(p^r)^{m-1} = p^{rm-1}$ where $\mathcal{A} := \mathbb{Z}_{p^r}+\mathbb{Z}_{p^r}\eta+\mathbb{Z}_{p^r}\eta^2+\cdots+\mathbb{Z}_{p^r}\eta^{m-1}$ since $|\mathbb{Z}_{p^r}| = p^r$ ve $|I| = p^{r-1}$. So, we can immediately give the following results on the plane $M(\mathcal{A})$:

1) The number of points (or lines) in the neighborhood of any point (or any line) is $|\mathbf{I}|^2=(p^{rm-1})^2$, respectively.

2) The number of points of type of $(x, y, 1)$ is $(p^r)^m\times(p^r)^m=[(p^r)^m]^2$,

The number of points of type of $(1, y, z)$ is $(p^r)^m\times|\mathbf{I}|$,

The number of points of type of $(w, 1, z)$ is $|\mathbf{I}|\times|\mathbf{I}|=|\mathbf{I}|^2$.

Moreover, the results is dually valid for the lines.

3) From 1) and 2) the number of non-neighbour points is:

The number of points of type of $(x, y, 1)$ is $\frac{[(p^r)^m]^2}{|\mathbf{I}|^2}=\frac{[(p^r)^m]^2}{(p^{rm-1})^2}=p^2$,

The number of points of type of $(1, y, z)$ is $\frac{(p^r)^m\times|\mathbf{I}|}{|\mathbf{I}|^2}=\frac{(p^r)^m}{|\mathbf{I}|}=\frac{(p^r)^m}{p^{rm-1}}=p$,

The number of points of type of $(w, 1, z)$ is $\frac{|\mathbf{I}|^2}{|\mathbf{I}|^2}=1$.

Moreover, the results is dually valid for the lines.

4) From 3) the number of non-neighbour points or lines of PK-plane $M(\mathcal{A})$ is p^2+p+1.

5) The total number of the points or the lines of PK-plane $M(\mathcal{A})$ is $|\mathbf{I}|^2\times(p^2+p+1)=(p^{rm-1})^2(p^2+p+1)$.

6) As a result of 4), M^* stated in (PK3) of Definition 4 is a projective plane of order p.

7) The number of collineations of PK-plane $M(\mathcal{A})$ is $(p^{rm-1})^2$ [v(v-1)(v-(p+1))] where $v:=p^2+p+1$ is total number of points of a projective plane of order p.

8) Gs is a collineation of PK-plane $M(\mathcal{A})$. Note that it is established similar collineations for the planes.

9) By choosing p, r and m, incidence matrix of any line can be obtained by following the previous examples.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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