Asymptotic Behaviour of Solutions of Second Order Difference Equations with Damping Term

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Abstract

The author presents some sufficient conditions for second order difference equation with damping term of the form

$$\Delta(a_n\Delta(x_n + cx_{n-k})) + p_n\Delta x_n + q_n f(x_{n+1-l}) = 0$$

An example is given to illustrate the main results.

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1 Introduction

In this paper, we study the asymptotic behaviour of solutions of second order difference equation with damping term of the form

$$\Delta(a_n\Delta(x_n + cx_{n-k})) + p_n\Delta x_n + q_n f(x_{n+1-l}) = 0 \qquad (1.1)$$

Where $n \in N(n_0), \{a_n\}$ is a positive sequence, $\{p_n\}$ and $\{q_n\}$ are nonnegative real sequences, k and l are nonnegative integers, c is a real number, and $f : R \to R$ is continuous and nondecreasing with $uf(u) > 0$ for $u \neq 0$.

Let $\theta = \max\{k, l\}$: By a solution of equation (1.1), we mean a real sequence $\{x_n\}$ which is defined for $n \geq n_0 - \theta$ and satisfies equation (1.1) for all $n \in N(n_0)$. A solution of equation (1.1) is said to be almost oscillatory if $\{x_n\}$ is oscillatory or $\{\Delta x_n\}$ is oscillatory for $n \in N(n_0)$.

In recent years there has been great attention denoted to the asymptotic behaviour of second order difference equations with damping term, see [1-10, 12], and the references cited there in. Therefore, in this paper we establish some sufficient conditions for the almost oscillation of equation (1.1) and give an example for the main results.

2 Main Results

In this section, we establish sufficient conditions for the almost oscillation of equation (1.1). We begin with the following theorem.

Theorem 2.1.1 Let $0 < c < 1, a_n - p_n > 0$ for $n \in N(n_0)$, and

$$\sum_{n=n_0}^{\infty} \tfrac{1}{a_n} \prod_{s=n_0}^{n-1} (1 - \tfrac{p_s}{a_s}) = \infty \qquad (2.1)$$

If

$$\sum_{n=n_0}^{\infty} q_n = \infty \qquad (2.2)$$

then every solution of equation (1.1) is almost oscillatory.

Proof.

Let $\{x_n\}$ be an eventually positive solution of equation (1.1) say $x_n > 0, x_{n-k} > 0$ and $x_{n-l} > 0$ for $n \geq n_1 \in N(n_0)$. There are two possibilities to consider:

(i) $\Delta x_n > 0$ eventually;

(ii) $\Delta x_n < 0$ eventually.

Case (1) Suppose $\Delta x_n > 0$ eventually. Then equation (1.1) leads to

$$\Delta(a_n\Delta z_n)+q_n f(x_{n+1-l})\le 0, n\ge n_1 \quad (2.3)$$

where $z_n = x_n + cx_{n-k}$, $\Delta z_n > 0$ eventually. Then

$$x_n = z_n - cx_{n-k} \ge (1\text{-c})z_n$$

Combining the last inequality with (2.3), we obtain

$$\Delta(a_n\Delta z_n)+q_n f((1-c)z_{n+1-l})\le 0$$

For all $n\ge n_1$. Since $z_n > 0, \Delta z_n > 0$ there exists a constant k > 0 such that $z_{n+1-l}\ge k$ for all $n\ge n_2 \ge n_1 + l$. Hence

$$\Delta(a_n\Delta z_n)+q_n f(1-c))\le 0, n\ge n_2$$

Summing the last inequality from n_2 to n, we have

$$a_{n+1}\Delta z_{n+1}\le a_{n2}\Delta z_{n2} - f(k(1-c))\sum_{s=n_2}^{n} q_s$$

Now from (2.2), it follows that $a_n\Delta z_n \to -\infty$, a contradiction to $\Delta z_n > 0$ eventually. Case (2)

Suppose $\Delta x_n < 0$ eventually. Then from equation (1.1) we have

$$\Delta(a_n\Delta z_n)+p_n\Delta x_n = -q_n f(x_{n+1-l})$$

or

$$\Delta(a_n\Delta z_n)+p_n\Delta x_n < 0,\ n\ge n_1 \in N(n_0+l) \quad (2.4)$$

Since $\Delta z_n = \Delta x_n + c\Delta x_{n-k}$, we have $\Delta z_n < \Delta x_n < 0$ and from (2.4), we obtain

$$\Delta(a_n\Delta z_n)+p_n\Delta z_n < 0,\ n\ge n_1$$

Let $u_n = -a_n\Delta z_n$. Then we have

$$\Delta u_n + \tfrac{p_n}{a_n}u_n \ge 0.$$

Summing the last inequality from n_1 to n -1, we have

$$u_n \ge u_{n1}\prod_{s=n_1}^{n-1}\left(1-\tfrac{p_s}{a_s}\right)$$

Or

$$\Delta z_n \le -\frac{u_{n1}}{a_n} u_n \prod_{s=n_1}^{n-1}\left(1-\frac{p_s}{a_s}\right)$$

Again summing the last inequality from n_1 to n-1, we have

$$\Delta z_n \le -u_{n1} \sum_{s=n1}^{n-1} \frac{1}{a_n} u_n \prod_{t=n_1}^{n-1}\left(1-\frac{p_t}{a_t}\right)$$

However condition (2.1) leads to $z_n \to -\infty \ as \ n \to \infty$, a contradiction. The proof for the case $\{x_n\}$ eventually negative is similar. This completes the proof of the theorem.

To prove our next result, we need the following condition:

$$-f(xy) \ge f(xy) \ge f(x) f(y) \text{ for xy > 0} \qquad (2.5)$$

Theorem 2.2. Let 0 < c < 1 and conditions (2.1) and (2.5) hold. If the equation

$$\Delta(a_n \Delta z_n) + q_n f(1-c) f(z_{n+1-l}) \le 0 \qquad (2.6)$$

is oscillatory then equation (1.1) is almost oscillatory.

Proof. Let $\{x_n\}$ be an eventually positive solution of equation (1.1). We consider the two cases:

(i) $\Delta x_n > 0$ eventually;

(ii) $\Delta x_n < 0$ eventually;

Case (1) Assume that $\Delta x_n > 0$ eventually. Then equation (1.1) leads to

$$\Delta(a_n \Delta(x_n + c x_{n-k})) + q_n f(x_{n+1-l}) \le 0 \qquad (2.7)$$

Set

$$z_n = x_n + c x_{n-k} \qquad (2.8)$$

Then inequality (2.7) takes the form

$$\Delta(a_n \Delta z_n) + q_n f(x_{n+1-l}) \le 0 \qquad (2.9)$$

eventually, and clearly $\Delta z_n > 0$ eventually. Thus, from (2.8) we find

$$x_n \ge (1-c) z_n \text{ eventually} \qquad (2.10)$$

Using (2.10) in equation (2.9), we obtain

$$\Delta(a_n \Delta z_n) + q_n f((1-c) z_{n+1-l}) \le 0$$

and by (2.5), this inequality reduces to

$$\Delta(a_n \Delta z_n) + q_n f(1-c) f(z_{n+1-l}) \le 0$$

eventually. But inview of a lemma in [11], it follows that equation (2.6) has an eventually positive solution, which is a contradiction.

Case (2) Assume that $\Delta x_n < 0$ eventually. Then $\Delta z_n < 0$ eventually, and $\Delta z_n < \Delta x_n$ eventually. From equation (1.1), we see that inequality

$$\Delta(a_n \Delta z_n) + p z_n \le 0$$

eventually has an eventually negative solution. The rest of the proof is similar to that of Case(2) of Theorem 2.1. This completes the proof.

Theorem 2.3. Let c > 1; k is a negative integer, and conditions (2.1) and (2.5) hold. If the equation

$$\Delta(a_n \Delta z_n) + f\left(\tfrac{c-1}{c^2}\right) q_n f(z_{n+1+k-l}) = 0 \qquad (2.11)$$

is oscillatory, then equation (1.1) is almost oscillatory.

Proof. Let $\{x_n\}$ be an eventually positive solution of equation (1.1). We consider two Cases (1) and (2) as in Theorem 2.2.

Case (1) Assume that $\Delta x_n > 0$ eventually. Then as in the proof of Theorem 2.2, we obtain the inequality (2.9). Proceeding as in the proof of Case (2) of Theorem 2.2, we arrive at the desired contradiction. The proof of Case (2) is similar to the proof of Case (2) of Theorem 2.2. The proof is now complete.

We conclude this paper with the following example.

Example 2.4. Consider the difference equation

$$\Delta^2\left(x_n + \tfrac{1}{2} x_{n-1}\right) + \tfrac{1}{n} \Delta x_n + \tfrac{2}{(n+1)^2 (n+3)} x_{n+1} = 0 \quad , n \ge 2 \qquad (2.12)$$

Here $a_n = 1, c = \frac{1}{2}, p_n = \frac{1}{n}$, and $q_n = \frac{2}{(n+1)^2 (n+3)}$. It is easy to verify that all the conditions of the Theorem 2.2 are satisfied except that on the oscillatory behaviour of the equation

$$\Delta^2 z_n + \tfrac{1}{(n+1)^2 (n+3)} z_{n+1} = 0 \quad , n \ge 2$$

Equation (2.12) has a nonconciliatory solution $\{x_n\} = \{\frac{n}{n+1}\}$.

References

[1] R.P. Agarwal, M. Bohner, S.R. Grace, and D.O. Regan, Discrete Oscillation Theory, Hindawi Publ. Comp., New York, 2005.

[2] Ignore and Glades, Oscillation Theory of Delay Differential Equations with Applications, Clarendon Press, Oxford, 1991.

[3] W.T. Li and X.L. Fan, Oscillation criteria for second order nonlinear difference equations with damped term, Comput. Math. with Appl., 37(1999), 17-30.

[4] Y. Li and L. Zhao, Oscillation of a second order difference equations with a nonlinear damped term in archimedean space, Communi. Math. Anal., 5(1)(2008), 36-43.

[5] S.H. Saker and S.S. Cheng, Oscillation criteria for difference equations with damping term, Appl. Math. Comput., 148(2) (2004), 421-442.

[6] Sh. Salem and K.R. Raslam, Oscillation of some second order damped difference equations, Inter. J. Nonlinear Sci., 5(3) (2008), 246-254.

[7] X.H. Tang and J.S. Yu, Oscillation of nonlinear dealy difference equations, J. Math. Anal. Appl., 249 (2000), 476-490.

[8] E. Thandapani, Oscillation theorems for second order damped nonlinear difference equations, Czech. Math. J., 45(1995), 327-335.

[9] E. Thandapani and B.S. Lalli, Oscillation criteria for a second order damped difference equations, Appl. Math. Lett., 8(1) (1995), 1-6.

[10] P. Wang and M. Mu, Oscillation of certain second order nonlinear damped difference equations with continuous variable, Appl. Math. Lett., 20(2007), 637-644.

[11] G. Zhang, Oscillation for nonlinear neutral difference equations, Appl. Math. E-Notes, 2(2002), 22-24.

[12] Zhuang and B. Ping, Linear oscillation of second order nonlinear difference equations with damping term, Comput. Math. with Appl., 41(2001), 659-667.