ISSN 2347-1921

Optimal control of a fractional diffusion equation with delay

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Abstract

We study a homogeneous Dirichlet boundary fractional diffusion equation with delay in a bounded domain. The fractional time derivative is considered in the left Caputo sense. By means of a linear continuous operator, we first transform the fractional diffusion equation with delay into a an equivalent equation without delay. Then we show that the optimal control problem associate to the controlled equivalent fractional diffusion equation has a unique solution. Interpreting the Euler-Lagrange first order optimality condition with an adjoint problem defined by means of right fractional Caputo derivative, we obtain an optimality system.

Keywords: fractional differential equation; optimal control.

Mathematics Subject Classification: 49J20, 49K20, 34K05; 34A12, 26A33.

Council for Innovative Research

Peer Review Research Publishing System

Journal: Journal of Advances in Mathematics

Vol 6, No. 3

editor@cirworld.com

www.cirworld.com, member.cirworld.com

ISSN 2347-1921

1 introduction

Let $N \in \mathbb{N}^*$ and Ω be a bounded open subset of $\mathbb{R}^N$ with boundary $\partial\Omega$ of class $\mathcal{C}^2$. For a time $T>0$, we set $Q=\Omega\times(0,T)$ and $\Sigma=\partial\Omega\times(0,T)$. For any $\tau>0$, consider the following fractional differential equation with delay:

$$\begin{cases} \mathcal{D}_l^{\alpha} y(x,t) - Ay(x,t) + y(x,t-\tau) & = & f_1(x,t), \quad (x,t)\in(\tau,T)\times\Omega, \\ y(x,t) & = & g(x,t), t\in(0,\tau)\times\Omega, \\ y(x,t) & = & 0 \quad (x,t)\in\Sigma, \end{cases} \tag{1}$$

where f_1 is given in $L^2((\tau,T)\times\Omega)$ which is the set of all measurable functions defined on $(\tau,T)\times\Omega$ such that $\left(\int_\tau^T\int_\Omega |\rho(x,t)|^2\,dxdt\right)^{1/2} < +\infty$. The function g belongs to $W(0,\tau)$ with

$$W(0,\tau)=\left\{\rho, \rho\in L^2((0,\tau);H^2(\Omega)), \mathcal{D}_l^{\alpha}\rho\in L^2((0,\tau)\times\Omega)\right\}. \tag{2}$$

The fractional derivative of order α, $\mathcal{D}_l^{\alpha}$ is to be understood in the Caputo sense. The operator A is given by:

$$Ay(x)=\sum_{i,j=1}^{N}\frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial y}{\partial x_j}(x)\right), x\in\Omega$$

where the coefficients $a_{ij}=a_{ji}, 1\le i,j\le N$ satisfy the following conditions:

(i) (H_1): the coefficients $a_{ij}\in\mathcal{C}^1(\overline{\Omega})$

(ii) (H_2): there exists a constant $\beta>0$ such that

$$\sum_{i,j=1}^{N}a_{ij}(x)\xi_i\xi_j \ge \beta\sum_{i=1}^{N}\xi_i^2, x\in\overline{\Omega}, \xi\in\mathbb{R}^N.$$

Fractional diffusion equations describe anomalous diffusion on fractals (physical objects of fractional dimension, like some amorphous semiconductors or strongly porous materials; see [1, 2] and references therein). Fractional diffusion equations have been studied by several authors. For instance, in [3], Oldham and Spanier discuss the relation between a regular diffusion equation and a fractional diffusion equation that contains a first order derivative in space and half order derivative in time. Mainardi [4] and Mainardi *et al.* [5, 6] generalized the diffusion equation by replacing the first time derivative with a fractional derivative of order α. These authors proved that the process changes from slow diffusion to classical diffusion, then to diffusion-wave and finally to classical wave when α increases from 0 to 2. The fundamental solutions of the Cauchy problems associated to these generalized diffusion equation ($0<\alpha\le 2$) are studied in [6, 7]. By means of Fourier-Laplace transforms, the authors expressed these solutions in term of Wright-type functions that can be interpreted as spatial probability density functions evolving in time with similarity properties. Wyss in [11] used Mellin transform theory to obtain a closed form solution of the fractional diffusion equation in terms of Fox's H-function. In [12], Metzler and Klafter used the method of images and the Fourier-Laplace transform technique to solve fractional diffusion equation for different boundary value problems. We also refer to [8, 9, 10] where Agrawal *et al.* and Agrawal studied the solutions for a fractional diffusion wave equation.

Concerning the calculus of variations and optimal control of fractional differential equation, the filed is in full expansion. In [13], Agrawal presented a general formulation and solution scheme for fractional optimal control problem. That is an optimal control problem in which either the performance index or the differential equations governing the dynamics of the system or both contain at least one fractional derivative term. In that paper, the fractional derivative was defined in the Riemann-Liouville sense and the formulation was obtained by means of fractional variation principle [15] and the Lagrange multiplier technique. Following the same technique, Frederico *et al.* [16] obtained a Noether-like theorem for fractional optimal control problem in the sense of Caputo. Recently, Agrawal [14] presented an eigenfunction expansion approach for a class of distributed system whose dynamics are defined in Caputo sense. Following the same approach as Agrawal, in [17] Ozdemir investigated fractional optimal control problem of a distributed system in cylindrical coordinates whose dynamics are defined in Riemann-Liouville sense. For a time fractional diffusion equation with source term, Yamamoto *et al.* [29]

discuss an inverse problem of determining a spatially varying function of the source by final over-determining data. We also refer to [30] where for initial value/boundary value problems for fractional diffusion equation, Yamamoto *et al.* used the eigenfunction expansions to prove stability in the backward problem in time.

Various fractional optimal control problem are also studied by Mophou *et al.* when the fractional time derivative is expressed in the Riemann-Liouville sense. For instance, we refer to the boundary optimal control [19, 21], optimal control of a fractional diffusion equation with state constraints [20]. Following these works, we want to control, in this paper, the fractional diffusion equation with delay (1). Actually, this kind of equations can help to model the phenomenon of diffusion in the soil. In this case, the delay can be understood through cultural practices which by increasing soil aggregates and encouraging vegetation growth and the density of the vegetation cover, could curb the penetration of water into the soil. This can slow down soil saturation before surface flow or check runoff. This delay in the diffusion can also be considered as the result of the presence of obstacles, arrangements cropping for instance or as hedgerows and other cross fencing as in Bamiléké villages in western Cameroon [22, 23, 24].

In this paper, we are concerned with the optimal control of the fractional diffusion equation with delay (1). To this end, we consider the following control system:

$$\begin{cases} \mathcal{D}_l^{\alpha} y - Ay + My &= f + v \quad in \quad Q, \\ y &= 0 \quad on \quad \Sigma, \\ y(0) &= 0 \quad in \quad \Omega. \end{cases} \tag{3}$$

where the control v belongs to $\mathcal{U}_{ad}$ which is closed subset of $L^2(Q)$. The linear and continuous operator $M : W(0,T) \to L^2((0,T); H^2(\Omega))$ is defined by

$$My(x,t) = \begin{cases} y(x,t-\tau) & if \quad (x,t) \in (\tau,T)\times\Omega, \\ 0 & if \quad (x,t) \in (0,\tau)\times\Omega. \end{cases} \tag{4}$$

The function f is given by

$$f(t) = \begin{cases} f_1(x,t) & if \quad (x,t) \in (\tau,T)\times\Omega, \\ \mathcal{D}_l^{\alpha} g(x,t) + Ag(x,t) & if \quad (x,t) \in (0,\tau)\times\Omega. \end{cases} \tag{5}$$

Then, we consider the optimal control problem:

$$\min_{u\in\mathcal{U}_{ad}} J(v) = \|y(v) - z_d\|_{L^2(Q)} + N\|v\|_{L^2(Q)}$$

where $N > 0$ and $z_d \in L^2(Q)$.

To solve this problem, we use the classical optimal control theory developed by J.L. Lions [31]. So, we prove that optimal control problem has a unique solution. Interpreting the Euler-Lagrange first order optimality condition with an adjoint problem defined by means of a right fractional Caputo derivative, we obtain an optimality system for the optimal control. As far as we know, the result presented here is new in fractional optimal control.

The paper is organized as follows. Section 2 is devoted to some definitions and preliminary results. In Section 3, we prove the existence of the optimal control for system (12).

2 Preliminaries

Definition 2.1 [25] *Let* $f : \mathbb{R}_+ \to \mathbb{R}$ *be a continuous function on* $\mathbb{R}^+$ *and* $\alpha > 0$. *Then the expression*

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds, t > 0$$

is called the Riemann-Liouville integral of order α.

Definition 2.2 [26] *Let* $\alpha \in (0,1)$ *and let* $f : \mathbb{R}_+ \to \mathbb{R}$. *The left Caputo fractional derivative of order* α *of* f *is defined by*

$$\mathcal{D}_l^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f'(s)ds, t > 0,$$

provided that the integral is defined.

Definition 2.3 [27]. *Let* $\alpha \in (0,1)$ *and let* $f : \mathbb{R}_+ \to \mathbb{R},\ 0 < \alpha < 1$ *and* $T > 0$. *The right Caputo fractional derivative of order* α *of* f *is defined by*

$$\mathcal{D}_r^{\alpha} f(t) = \frac{-1}{\Gamma(1-\alpha)} \int_t^T (s-t)^{-\alpha} f'(s)ds, \tag{6}$$

provided that the integral is defined.

Remark 2.4 *The right fractional Caputo derivative represents the future state of* $f(t)$. *For more details on this derivative we refer to* [27].

We need the following Lemmas which give the integration by parts for a fractional diffusion equation with Caputo derivatives for the resolution of the optimal control problem associate to (11).

Lemma 2.5 *Let* $0 < \alpha < 1$. *Then for any* $\varphi \in \mathcal{C}^{\infty}(\overline{Q})$, *we have*

$$\begin{aligned}
&\int_0^T \int_{\Omega} \left(\mathcal{D}_l^{\alpha} y(x,t) - Ay(x,t) \right) \varphi(x,t) dx dt = \\
&\int_{\Omega} \varphi(x,T) I^{1-\alpha} y(x,T) dx - \frac{1}{\Gamma(1-\alpha)} \int_{\Omega} y(x,0) \left(\int_0^T t^{-\alpha} \varphi(x,t) dt \right) dx + \\
&\int_0^T \int_{\partial\Omega} y \frac{\partial \varphi}{\partial \nu_A} d\sigma dt - \int_0^T \int_{\partial\Omega} \frac{\partial y}{\partial \nu_A} \varphi d\sigma dt + \\
&\int_{\Omega} \int_0^T y(x,t) \left(\mathcal{D}_r^{\alpha} \varphi(x,t) - A\varphi(x,t) \right) dx dt.
\end{aligned}$$

Proof. See annex in Section 4

Lemma 2.6 *[18] Let* $0 < \beta < 1$, $\mathbb{X}$ *be a Banach space and* $f \in \mathcal{C}([0,T], \mathbb{X})$. *Then for all* $t_1, t_2 \in [0,T]$,

$$\left\| I^{\beta} f(t_1) - I^{\beta} f(t_2) \right\|_{\mathbb{X}} \le \frac{\| f \|_{L^{\infty}((0,T);\mathbb{X})}}{\Gamma(\beta+1)} | t_1 - t_2 |^{\beta}$$

Remark 2.7 *Since* $\mathcal{C}([0,T], \mathbb{X}) \subset L^{\infty}((0,T); \mathbb{X}) \subset L^2((0,T); \mathbb{X})$ *because* $[0,T]$ *is a bounded subset of* $\mathbb{R}$, *Lemma 2.6 holds for* $f \in L^2((0,T); \mathbb{X})$ *and we have that* $I^{\beta} f \in \mathcal{C}([0,T], \mathbb{X}) \subset L^2((0,T); \mathbb{X})$.

Consider the following fractional diffusion equation with the Caputo fractional time derivative:

$$\begin{cases} \mathcal{D}_l^{\alpha} y - Ay + c(x)y & = & f & in & Q, \\ y & = & 0 & on & \Sigma, \\ y(0) & = & y^0 & in & \Omega \end{cases} \tag{7}$$

where the function $c \in \mathcal{C}(\overline{\Omega})$ and satisfies $c(x) \ge 0$ for all $x \in \overline{\Omega}$. The operator A satisfies assumptions (H_1) and (H_2) given in Page 1018. We have the following results.

Theorem 2.8 [Theorem 4.1[28]] *Let* $f \equiv 0$ *and* $y^0 \in H_0^1(\Omega)$. *Then problem (7) has a unique solution* $y \in C([0,T]; L^2(\Omega)) \cap C([0,T]; H^2(\Omega) \cap H_0^1(\Omega))$ *such that* $D_C^{\alpha} y \in C([0,T]; L^2(\Omega))$. *Moreover, there exists a constant* $C > 0$ *such that*

$$\|y\|_{L^2((0,T);H^2(\Omega))} + \|D_l^\alpha y\|_{L^2(Q)} \leq C\|y^0\|_{H^1(\Omega)}. \quad (8)$$

Theorem 2.9 [Theorem 4.2[28]] *Let* $f \in L^2(Q)$ *and* $y^0 \equiv 0$. *Then problem (7) has a unique solution* $y \in L^2((0,T);H^2(\Omega) \cap H_0^1(\Omega))$. *Moreover, there exists a constant* $C > 0$ *such that*

$$\|y\|_{L^2((0,T);H^2(\Omega))} + \|D_l^\alpha y\|_{L^2(Q)} \leq C\|f\|_{L^2(Q)}. \quad (9)$$

Lemma 2.10 *Let* $f \in L^2(Q)$ *and* $y \in L^2((0,T);H^2(\Omega))$ *be such that* $\mathcal{D}_l^\alpha \in L^2(Q)$ *and* $\mathcal{D}_l^\alpha y - Ay = f$. *Then* $y|_\Sigma$ *exists and belongs* $L^2((0,T);H^{-1/2}(\partial\Omega))$. [(i)]

(i) $y|_\Sigma$ exists and belongs $L^2((0,T);H^{-1/2}(\partial\Omega))$.

(ii) $y(0)$ belongs to $L^2(\Omega))$.

Proof. Since $a_{ij} \in C^1(\overline{\Omega})$ for $1 \leq i, j \leq n$, proceeding as in [19, 20], we have (i).

On the other hand, in view of Lemma 2.6, $I^\alpha(\mathcal{D}_l^\alpha y(t)) \in L^2(\Omega)$ because $\mathcal{D}_l^\alpha y \in L^2(Q)$. Hence, $y(0)$ exists and belongs to $L^2(\Omega)$ since $I^\alpha(\mathcal{D}_l^\alpha y(t)) = y(t) - y(0)$ and $y(t) \in L^2(\Omega)$.

3 Optimal control

Before going further, let us justify equation (12). So, define M and f as in (4) and (5) respectively. Then $f \in L^2(Q)$. Indeed, observing in the one hand that $f_1 \in L^2((\tau,T)\times\Omega)$, and on the other hand that, g belongs $W(0,\tau)$, we obtain that $Ag \in L^2((0,\tau)\times\Omega)$ which combining with the fact that $\mathcal{D}_l^\alpha g \in L^2((0,\tau)\times\Omega)$ implies that $\mathcal{D}_l^\alpha g + Ag \in L^2((0,\tau)\times\Omega)$. Set

$$g(x,0) = 0, \quad (10)$$

we have that (1) can be rewritten as

$$\begin{cases} \mathcal{D}_l^\alpha y - Ay + My = f & in \quad Q, \\ y = 0 & on \quad \Sigma, \\ y(0) = 0 & in \quad \Omega. \end{cases} \quad (11)$$

We thus consider the following system:

$$\begin{cases} \mathcal{D}_l^\alpha y - Ay + My = f + v & in \quad Q, \\ y = 0 & on \quad \Sigma, \\ y(0) = 0 & in \quad \Omega. \end{cases} \quad (12)$$

where $f \in L^2(Q)$ and the control v belongs to $\mathcal{U}_{ad}$ which is closed subset of $L^2(Q)$. In view of the Theorem 2.9, we know that the solution $y = y(v)$ of (12) belongs to $L^2((0,T);H^2(\Omega))$. Thus we can define the functional

$$J(v) = \|y - z_d\|^2_{L^2(\Omega)} + N\|v\|^2_{L^2(Q)} \quad (13)$$

where $z_d \in L^2(Q)$ and $N > 0$. We are interested in the optimal control problem: Find $u \in \mathcal{U}_{ad}$ such that

$$J(u) = \inf_{v \in \mathcal{U}_{ad}} J(v). \tag{14}$$

Proposition 3.1 *There exists a unique optimal control* u *such that (14) holds.*

Proof. Let $v_n \in \mathcal{U}_{ad}$ be a minimizing sequence such that

$$J(v_n) \to \inf_{v \in \mathcal{U}_{ad}} J(v). \tag{15}$$

Then $y_n = y(v_n)$ is solution of (12). This means that y_n satisfies:

$$\mathcal{D}_l y_n - Ay_n + My_n = f + v_n \quad in \qquad Q, \tag{16a}$$

$$y_n = 0 \quad on \Sigma, \tag{16b}$$

$$y_n(x,0) = 0 \quad in \Omega \tag{16c}$$

Moreover, in view of (15), there exists $C > 0$ independent of n such that

$$\|v_n\|_{L^2(Q)} \leq C, \tag{17}$$

$$\|y_n\|_{L^2(Q)} \leq C. \tag{18}$$

Hence, we deduce from (16) that

$$\|\mathcal{D}_l y_n - Ay_n + My_n\|_{L^2(Q)} \leq C,, \tag{19}$$

and from Theorem 2.9 that

$$\|\mathcal{D}_l^{\alpha} y_n\|_{L^2(Q)} \leq C, \triangle ABC \tag{20}$$

$$\|y_n\|_{L^2((0,T);H^2(\Omega))} \leq C. \tag{21}$$

Hence there exists (u, δ, y) in $L^2(Q) \times L^2(Q) \times L^2((0,T);H^2(\Omega))$ and a subsequence extracted from (v_n), $(\mathcal{D}_l^{\alpha} y_n)$ and (y_n) (still called (v_n), $(\mathcal{D}_l^{\alpha} y_n)$ and (y_n)) such that

$$v_n \rightharpoonup u \quad \text{weakly in} \quad L^2(Q), \tag{22}$$

$$y_n \rightharpoonup y \quad \text{weakly in} \quad L^2((0,T);H^2(\Omega)), \tag{23}$$

$$\mathcal{D}_l y_n - Ay_n + My_n \rightharpoonup \beta \quad \text{weakly in} \quad L^2(Q), \tag{24}$$

$$\mathcal{D}_l^{\alpha} y_n \rightharpoonup \delta \quad \text{weakly in} \quad L^2(Q). \tag{25}$$

Since v_n is in $\mathcal{U}_{ad}$ which is a closed subset of $L^2(Q)$, we have that

$$u \in \mathcal{U}_{ad}. \tag{26}$$

Let $M \in \mathcal{L}(W(0,T), L^2((0,T);H^2(\Omega)))$ be the linear and continuous operator defined in Page 3. Then, $M \in \mathcal{L}(W(0,T), L^2(Q))$ and we can define the adjoint M^* of M in $\mathcal{L}(L^2(Q), W(0,T))$ by:

$$M * \varphi(x,t) = \begin{cases} y(x,t+\tau) & if \quad (x,t) \in (0, T-\tau) \times \Omega, \\ 0 & if \quad (x,t) \in (T-\tau, T) \times \Omega. \end{cases} \tag{27}$$

Set

$$\mathbb{D}(Q) = \{\varphi \in \mathcal{C}^{\infty}(Q) \ such \ that \ \varphi|_{\partial\Omega} = 0, \varphi(x,0) = \varphi(x,T) = 0 \ in \Omega\}$$

and denote by $\mathbb{D}'(Q)$ the dual of $\mathbb{D}(Q)$.

Using Lemma 2.5, we have

$$\int_0^T \int_\Omega \left(\mathcal{D}_l^{\alpha} y_n(x,t) - Ay_n(x,t) + My_n(x,t)\right)\varphi(x,t)dxdt$$
$$= \int_0^T \int_\Omega y_n(x,t)\left(\mathcal{D}_r^{\alpha}\varphi(x,t) - A\varphi(x,t) + M^*\varphi(x,t)\right)dxdt, \forall \varphi \in \mathbb{D}(Q).$$

Therefore, it follows from (23) and Lemma 2.5 that

$$\lim_{n\to\infty} \int_0^T \int_\Omega \left(\mathcal{D}_l^{\alpha} y_n(x,t) - Ay_n(x,t) + My_n(x,t)\right)\varphi(x,t)dxdt =$$
$$\int_0^T \int_\Omega y(x,t)\left(\mathcal{D}_r^{\alpha}\varphi(x,t) - A\varphi(x,t) + M^*\varphi(x,t)\right)dxdt =$$
$$\int_0^T \int_\Omega \left(\mathcal{D}_l^{\alpha} y(x,t) - Ay(x,t) + My(x,t)\right)\varphi(x,t)dxdt, \quad \forall \varphi \in \mathbb{D}(Q).$$

This implies that

$$\mathcal{D}_l^{\alpha} y_n - Ay_n + My_n \qquad \mathcal{D}_l^{\alpha} y - Ay + My \quad weakly \ in \quad \mathbb{D}'(Q).$$

Hence, in view of (24) and (25), we obtain that

$$\mathcal{D}_l^{\alpha} y - Ay + My = \beta \in L^2(Q), \tag{28}$$

$$\mathcal{D}_l^{\alpha} y = \delta \in L^2(Q). \tag{29}$$

So, passing to the limit in (16a) while using (24), (22) and (28), we deduce that

$$\mathcal{D}_l^{\alpha} y - Ay + My = f + u \quad in \ Q. \tag{30}$$

Since $y \in L^2((0,T); H^2(\Omega))$, Lemma 2.10 allows us to say that $y|_{\partial\Omega}$ and $y(0)$ exist and belong respectively to $H^{-1/2}(\partial\Omega)$ and to $L^2(\Omega)$. Consequently, multiplying (16a) by $\varphi \in \mathcal{C}^{\infty}(\overline{Q})$ with $\varphi|_{\partial\Omega} = 0$ and $\varphi(T,x) = 0$ on Ω, and integrating by parts over Q, we obtain by using Lemma 2.5,

$$\int_0^T \int_\Omega \left(\mathcal{D}_l^{\alpha} y_n(x,t) - Ay_n(x,t) + My_n(x,t)\right)\varphi(x,t)dxdt =$$
$$+\int_0^T \int_\Omega y_n(x,t)\left(\mathcal{D}_r^{\alpha}\varphi(x,t) - A\varphi(x,t) + M^*\varphi(x,t)\right)dxdt$$

Passing this latter identity to the limit when $n \to \infty$ while using (24), (28) and (23), we get

$$\begin{aligned}&\int_0^T \int_\Omega \left(\mathcal{D}_l^{\alpha} y(x,t) - Ay(x,t) + My(x,t)\right)\varphi(x,t)dxdt = \\ &\int_0^T \int_\Omega y(x,t)\left(\mathcal{D}_r^{\alpha}\varphi(x,t) - A\varphi(x,t) + M^*\varphi(x,t)\right)dxdt.\end{aligned} \tag{31}$$

Integrating by part the right side of (31) while using Lemma 2.5, we obtain

$$\int_0^T\int_\Omega\left(\mathcal{D}_l^\alpha y(x,t)-Ay(x,t)+My(x,t)\right)\varphi(x,t)dxdt =$$
$$\int_\Omega y(x,0)\left(\frac{1}{\Gamma(1-\alpha)}\int_0^T t^{-\alpha}\varphi(x,t)dt\right)dx -$$
$$\int_0^T\langle y,\frac{\partial\varphi}{\partial\nu_A}\rangle_{H^{-1/2}(\partial\Omega),H^{1/2}(\partial\Omega)}dt + \qquad (32)$$
$$\int_0^T\int_\Omega\left(\mathcal{D}_l^\alpha y(x,t)-Ay(x,t)+My(x,t)\right)\varphi(x,t)dxdt,$$
$$for\ all\ \varphi\in\mathcal{D}^\infty(\overline{Q})\ with\ \varphi|_{\partial\Omega}=0\ and\ \varphi(x,T)=0\ on\ \Omega.$$

where $\langle .,. \rangle_{Y,Y'}$ represents the duality bracket between the spaces Y and Y'.

Hence, (32) yields

$$0 = \int_\Omega y(x,0)\left(\frac{1}{\Gamma(1-\alpha)}\int_0^T t^{-\alpha}\phi(x,t)dt\right)dx$$
$$- \int_0^T\langle y,\frac{\partial\varphi}{\partial\nu_A}\rangle_{H^{-1/2}(\partial\Omega),H^{1/2}(\partial\Omega)}dt$$
$$for\ all\ \varphi\in\mathcal{C}^\infty(\overline{Q})\ with\ \varphi|_{\partial\Omega}=0\ and\ \varphi(x,T)=0\ on\ \Omega.$$

Therefore taking in this latter identity φ such $\frac{\partial\varphi}{\partial\nu}=0$ on $\partial\Omega$, we obtain

$$y(x,0)=0 \quad in\ \Omega \qquad (33)$$

and then,

$$y=0 \quad on\ \partial\Omega. \qquad (34)$$

In view of (30), (33) and (34), we deduce that $y=y(u)$ is a solution of (12) with $u\in\mathcal{U}_{ad}$ because of (26). From weak lower semi-continuity of the function $v\longrightarrow J(v)$ we deduce

$$\liminf_{n\to\infty} J(v_n)\geq J(u).$$

Hence according to (15), we deduce that

$$J(u)\leq \inf_{v\in\mathcal{U}_{ad}} J(v)$$

which implies that

$$J(u)\leq \inf_{v\in\mathcal{U}_{ad}} J(v)$$

The uniqueness of u is straightforward from the strict convexity of J.

Theorem 3.2 *If u is solution of (14), then there exist $p\in L^2((0,T);H^2(\Omega))$ such that (u,y,p) satisfies the following optimality system:*

$$\begin{cases} \mathcal{D}_l^{\alpha} y(x,t) - Ay(x,t) + My(x,t) & = & f + u & in & Q, \\ y(x,t) & = & 0, & on & \Sigma, \\ y(x,0) & = & 0 & in & \Omega, \end{cases} \tag{35}$$

$$\begin{cases} \mathcal{D}_r^{\alpha} p(x,t) - Ap(x,t) + p(x,t+\tau) & = & y - z_d & in & (0,T-\tau)\times\Omega, \\ \mathcal{D}_r^{\alpha} p(x,t) - Ap(x,t) & = & y - z_d & in & (T-\tau,T)\times\Omega, \\ p(x,t) & = & 0 & on & \Sigma, \\ p(x,T) & = & 0 & in & \Omega, \end{cases} \tag{36}$$

and

$$\int_0^T \int_\Omega (Nu + p)\varphi dxdt \geq 0 \quad \forall \varphi \in \mathcal{U}_{ad}. \tag{37}$$

Proof. Relations (30), (33) and (34) give (35).

To prove (36) and (37), we express the Euler-Lagrange optimality condition which characterizes the optimal control u :

$$\frac{d}{d\mu} J(u + \mu\phi)|_{\mu=0} = 0, \ for\ all\ \varphi \in L^2(Q). \tag{38}$$

The state $z(\varphi)$ associated to the control $\varphi \in L^2(Q)$ is solution of

$$\begin{array}{lcccc} \mathcal{D}_l^{\alpha} z - Az + Mz & = & \varphi & in & Q, \\ z & = & 0, & on & \Sigma, \\ z(x,0) & = & 0 & in & \Omega \end{array} \tag{39}$$

After calculations, (38) gives

$$\int_0^T \int_\Omega z(y(u) - z_d) dxdt + N \int_0^T \int_\Omega u\varphi dxdt \geq 0 \quad \forall \varphi \in \mathcal{U}_{ad} \tag{40}$$

To interpret (40), we consider the adjoint state system:

$$\begin{array}{lcccc} \mathcal{D}_r^{\alpha} p - Ap + M^* p & = & y - z_d & in & Q, \\ p & = & 0 & on & \Sigma, \\ p(T) & = & 0 & in & \Omega. \end{array} \tag{41}$$

Make as in [18] the change of variable $t \to T - t$ in (41), the system becomes

$$\begin{array}{lcccc} \mathcal{D}_l^{\alpha} \tilde{p} - A\tilde{p} + M^* \tilde{p} & = & \tilde{y} - z_d & in & Q, \\ \tilde{p} & = & 0 & on & \Sigma, \\ \tilde{p}(0) & = & 0 & in & \Omega \end{array}$$

where $\tilde{y} - z_d = y(T-t,x) - z_d \in L^2(Q)$ since $y - z - d \in L^2(Q)$. Hence , using Theorem 2.9, we deduce that problem (41) has a unique solution in $p \in L^2((0,T); H^2(\Omega))$. Thus, multiplying (39) by p solution of (41), we obtain by using Lemma 2.5,

$$\begin{aligned} \int_0^T \int_\Omega \left(\mathcal{D}_l^{\alpha} z - Az + Mz \right) pdxdt &= \int_0^T \int_\Omega \left(\mathcal{D}_r^{\alpha} p - Ap + M^* p \right) zdxdt \\ &= \int_0^T \int_\Omega (y - z_d) zdxdt \end{aligned}$$

Hence, in view of (39) and (40), we deduce that

$$\int_0^T\int_\Omega (p+Nu)\varphi dxdt \geq \quad \forall \varphi \in L^2(Q).$$

4 Annex

Lemma 4.1 *For any* $\phi \in \mathcal{C}^\infty(\overline{Q}),$ *we have*

$$\int_0^T\int_\Omega \left(\mathcal{D}_l\alpha y(x,t) - Ay(x,t)\right)\varphi(x,t)dxdt =$$
$$+\int_\Omega \varphi(x,T)I^{1-\alpha}y(x,T)dx -$$
$$\int_\Omega y(x,0)\mathcal{D}^{1-\alpha}\varphi(x,0)dx +$$
$$\int_0^T\int_{\partial\Omega} y\frac{\partial\varphi}{\partial\nu_A}d\sigma dt - \int_0^T\int_{\partial\Omega}\frac{\partial y}{\partial\nu_A}\varphi d\sigma dt +$$
$$\int_\Omega\int_0^T y(x,t)\left(\mathcal{D}_r{}^\alpha\varphi(x,t) - A\varphi(x,t)\right)dxdt.$$

Proof. Let $\varphi \in \mathcal{C}^\infty(\overline{Q}),$. We have

$$\int_0^T\int_\Omega \left(\mathcal{D}_l{}^\alpha y(x,t) - Ay(t,x)\right)\varphi(x,t)dxdt =$$
$$\int_0^T\int_\Omega \mathcal{D}_l{}^\alpha y(x,t)\varphi(x,t)dxdt - \int_0^T\int_\Omega Ay(x,t)\varphi(x,t)dxdt.$$

We set

$$\begin{aligned} M_1 &= \int_0^T\int_\Omega \mathcal{D}_l{}^\alpha y(x,t)\varphi(x,t)dxdt,\\ M_2 &= -\int_0^T\int_\Omega Ay(t,x)\varphi(x,t)dxdt.\end{aligned}$$

Then

$$\int_0^T\int_\Omega \left(\mathcal{D}_l{}^\alpha y(x,t) - Ay(x,t)\right)\varphi(x,t)dxdt \quad = \quad M_1 + M_2.$$

We have

$$\begin{aligned} M_2 &= -\int_0^T\int_\Omega Ay(x,t)\varphi(x,t)dxdt\\ &= -\int_0^T\int_{\partial\Omega}\frac{\partial y}{\partial\nu_A}\varphi d\sigma dt + \int_0^T\int_{\partial\Omega} y\frac{\partial\varphi}{\partial\nu_A}d\sigma dt \qquad (42)\\ &- \int_0^T\int_\Omega y(x,t)A\varphi(x,t)dxdt.\end{aligned}$$

$$M_1 = \int_0^T \int_\Omega \mathcal{D}_l^\alpha y(x,t)\varphi(x,t)dxdt,$$

$$= \int_\Omega \left[\int_0^T \varphi(x,t,) \left(\frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} y'(x,s)ds \right) dt \right] dx$$

$$= \int_\Omega \left[\int_0^T y'(x,s) \left(\frac{1}{\Gamma(1-\alpha)} \int_s^T (t-s)^{-\alpha} \varphi(x,t)dt \right) ds \right] dx$$

$$= \int_\Omega \left[\int_0^T y'(x,s)A(x,s)ds \right] dx$$

$$= \int_\Omega B(x)dx$$

where

$$B(x) = \int_0^T y'(x,s)A(x,s)ds,$$

$$A(s,x) = \frac{1}{\Gamma(1-\alpha)} \int_s^T (t-s)^{-\alpha} \varphi(x,t)dt.$$

We have

$$\begin{aligned} A(x,s) &= \frac{1}{\Gamma(1-\alpha)} \int_s^T (t-s)^{-\alpha} \varphi(x,t)dt \\ &= \frac{1}{(1-\alpha)\Gamma(1-\alpha)} \left[(t-s)^{1-\alpha} \varphi(x,t) \right]_{t=s}^{t=T} \\ &- \frac{1}{(1-\alpha)\Gamma(1-\alpha)} \int_s^T (t-s)^{1-\alpha} \varphi'(x,t)dt \\ &= \frac{1}{\Gamma(2-\alpha)} (T-s)^{1-\alpha} \varphi(x,T) - \frac{1}{\Gamma(2-\alpha)} \int_s^T (t-s)^{1-\alpha} \varphi'(x,t)dt. \end{aligned}$$

$$\begin{aligned} B(x) &= \int_0^T y'(x,s)A(x,s)ds \\ &= \frac{1}{\Gamma(2-\alpha)} \varphi(x,T) \int_0^T (T-s)^{1-\alpha} y'(x,s)ds \\ &- \frac{1}{\Gamma(2-\alpha)} \int_0^T y'(x,s) \left(\int_s^T (t-s)^{1-\alpha} \varphi'(x,t)dt \right) ds. \end{aligned}$$

Since

$$\begin{aligned}
\frac{\varphi(x,T)}{\Gamma(2-\alpha)}\int_0^T (T-s)^{1-\alpha} y'(x,s)ds &= \frac{\varphi(x,T)}{\Gamma(2-\alpha)}\left[(T-s)^{1-\alpha} y(x,s)\right]_{s=0}^{s=T} \\
&+ \frac{1-\alpha}{\Gamma(2-\alpha)}\varphi(x,T)\int_0^T (T-s)^{-\alpha} y(x,s)ds \\
&= -\frac{\varphi(x,T)}{\Gamma(2-\alpha)} T^{1-\alpha} y(x,0) \\
&+ \frac{\varphi(x,T)}{\Gamma(1-\alpha)}\int_0^T (T-s)^{-\alpha} y(x,s)ds \\
&= \varphi(x,T)\left(-\frac{T^{1-\alpha}}{\Gamma(2-\alpha)} y(x,0) + I^{1-\alpha} y(x,T)\right)
\end{aligned}$$

and

$$\begin{aligned}
&-\frac{1}{\Gamma(2-\alpha)}\int_0^T y'(x,s)\left(\int_s^T (t-s)^{1-\alpha}\varphi'(x,t)dt\right)ds \\
&= -\frac{1}{\Gamma(2-\alpha)}\int_0^T \varphi'(x,t)\left(\int_0^t (t-s)^{1-\alpha} y'(x,s)ds\right)dt \\
&= -\frac{1}{\Gamma(2-\alpha)}\int_0^T \varphi'(x,t)\left[(t-s)^{1-\alpha} y(x,s)\right]_{s=0}^{s=t} dt \\
&- \frac{1-\alpha}{\Gamma(2-\alpha)}\int_0^T \varphi'(x,t)\left(\int_0^t (t-s)^{-\alpha} y(x,s)ds\right)dt \\
&= \frac{y(x,0)}{\Gamma(2-\alpha)}\int_0^T t^{1-\alpha}\varphi'(x,t)dt - \\
&\int_0^T y(x,s)\left(\frac{1}{\Gamma(1-\alpha)}\int_s^T (t-s)^{-\alpha}\varphi'(x,t)dt\right)ds \\
&= \frac{y(x,0)}{\Gamma(2-\alpha)}\left[t^{1-\alpha}\varphi(x,t)\right]_{t=0}^{t=T} - \frac{y(x,0)(1-\alpha)}{\Gamma(2-\alpha)}\int_0^T t^{-\alpha}\varphi(x,t)dt - \\
&\int_0^T y(x,s)\left(\frac{1}{\Gamma(1-\alpha)}\int_s^T (t-s)^{-\alpha}\varphi'(x,t)dt\right)ds \\
&= \frac{y(x,0)}{\Gamma(2-\alpha)} T^{1-\alpha}\varphi(x,T) - \frac{y(x,0)}{\Gamma(1-\alpha)}\int_0^T t^{-\alpha}\varphi(x,t)dt - \\
&\int_0^T y(x,s)\left(\frac{1}{\Gamma(1-\alpha)}\int_s^T (t-s)^{-\alpha}\varphi'(x,t)dt\right)ds \\
&= \frac{y(x,0)}{\Gamma(2-\alpha)} T^{1-\alpha}\varphi(x,T) - y(x,0)\mathcal{D}^{1-\alpha}\varphi(x,0) + \int_0^T y(x,s)\mathcal{D}_r^{\alpha}\varphi(x,s)ds \\
&= y(x,0)\left(\frac{T^{1-\alpha}}{\Gamma(2-\alpha)}\varphi(x,T) - \mathcal{D}^{1-\alpha}\varphi(x,0)\right) + \int_0^T y(x,s)\mathcal{D}_r^{\alpha}\varphi(x,s)ds
\end{aligned}$$

where

$$\mathcal{I}^{\alpha} f(s) = \frac{1}{\Gamma(\alpha)}\int_s^T (t-s)^{\alpha-1} f(t)dt.$$

Thus

$$B(x) = \varphi(x,T)\left(-\frac{T^{1-\alpha}}{\Gamma(2-\alpha)}y(x,0)+I^{1-\alpha}y(x,T)\right) + y(x,0)\left(\frac{T^{1-\alpha}}{\Gamma(2-\alpha)}\varphi(x,T)+\mathcal{D}^{1-\alpha}\varphi(x,0)\right)+\int_0^T y(x,s)\mathcal{D}_r^{\alpha}\varphi(x,s)ds$$

and

$$\begin{aligned} M_1 &= \int_\Omega \varphi(x,T)\left(-\frac{T^{1-\alpha}}{\Gamma(2-\alpha)}y(x,0)+I^{1-\alpha}y(x,T)\right)dx \\ &+ \int_\Omega y(x,0)\left(\frac{T^{1-\alpha}}{\Gamma(2-\alpha)}\varphi(x,T)-\mathcal{D}^{1-\alpha}\varphi(x,0)\right)dx \\ &+ \int_\Omega\int_0^T y(x,s)\mathcal{D}_r^{\alpha}\varphi(x,s)ds. \end{aligned} \tag{43}$$

Hence adding (43) to (42), we obtain

$$\begin{aligned} &\int_0^T\int_\Omega\left(D^\alpha y(x,t)-Ay(x,t)+c(x)y(x,t)\right)\varphi(x,t)dxdt \\ &= \int_\Omega \varphi(x,T)\left(-\frac{T^{1-\alpha}}{\Gamma(2-\alpha)}y(x,0)+I^{1-\alpha}y(x,T)\right)dx \\ &+\int_\Omega y(x,0)\left(\frac{T^{1-\alpha}}{\Gamma(2-\alpha)}\varphi(x,T)-\mathcal{D}^{1-\alpha}\varphi(x,0)\right)dx \\ &+\int_\Omega\int_0^T y(x,t)\mathcal{D}_r^{\alpha}\varphi(x,t)dt \\ &-\int_0^T\int_{\partial\Omega}\frac{\partial y}{\partial\nu_A}\varphi d\sigma dt+\int_0^T\int_{\partial\Omega}y\frac{\partial\varphi}{\partial\nu_A}d\sigma dt \\ &-\int_0^T\int_\Omega y(x,t)A\varphi(x,t)dxdt \\ &= \int_\Omega \varphi(x,T)I^{1-\alpha}y(x,T)dx \\ &-\int_\Omega y(x,0)\mathcal{D}^{1-\alpha}\varphi(x,0)dx \\ &-\int_0^T\int_{\partial\Omega}\frac{\partial y}{\partial\nu_A}\varphi d\sigma dt+\int_0^T\int_{\partial\Omega}y\frac{\partial\varphi}{\partial\nu_A}d\sigma dt \\ &+\int_\Omega\int_0^T y(x,t)\left(\mathcal{D}_r^{\alpha}\varphi(x,t)-A\varphi(x,t)\right)dxdt. \end{aligned}$$

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