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ISSN 2347-1921

Investigation of Laplace-Stieltjes transform for the ergodic distribution of the semi-markov random process with positive tendency, negative jumps and delaying boundary at zero

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Abstract: One of the important problems of stochastic processes theory is to define the Laplace-Stieltjes transform for the ergodic distribution of semi-markov random process. With this purpose, we will investigate the semi-markov random processes with positive tendency, negative jumps and delaying boundary at zero in this article. The Laplace transform on time, Laplace-Stieltjes transform on phase of the conditional and unconditional distributions and Laplace-Stieltjes transform of the ergodic distribution are defined. The characteristics of the ergodic distribution will be calculated on the basis of the final results.

Keywords: Laplace-Stieltjes transform; semi-markov random process; ergodic distribution; process with positive tendency and negative jumps.

2010 Mathematics Subject Classification: **60G05, 60J10, 60B05**

Council for Innovative Research
Peer Review Research Publishing System

Journal: Journal of Advances in Mathematics

Vol 6, No. 3
editor@cirworld.com
www.cirworld.com, member.cirworld.com

ISSN 2347-1921

Introduction

There are number of works devoted to definition of the distribution of the semi-markov processes and its main boundary functionals. Some authors are used the asymptotic, factorization and etc. methods ([2],[4],[5],[6],[9][12]) But other authors narrowing the class of distributions of walking are found the evident form for Laplace transforms for distributions and its main characteristics. In [7] The Laplace transformation for the distribution of the time of the system sojourn within a given band and its first and second moments are found .In [8] a model of inventory control is considered. It is described by a semi-markov random walk with a negative drift at an angle of 0 < α < 90°, with positive random jumps, a delay, an absorbing screen at zero, and a reflecting screen for a > 0 at an angle α. The Laplace transformation is found for the distribution of the first moment storehouse exhaustion, and the first and the second moments are explicitly obtained. In [9] The Laplace-Stieltjes transform with respect to phase, the Laplace transform with respect to time, the conditional distribution, the unconditional distribution, and the Laplace-Stieltjes transform of the ergodic distribution of the process of semi-markov random walk with negative drift, nonnegative jumps, delays, and boundary screen at zero are obtained. In [10] The first passage of the zero level of the semi-markov process with positive tendency and negative jumps will be included as a random variable. The Laplace transform for the distribution of this random variable is defined. In [11] for the step process of semi-markov random walk with delaying boundary in $a > 0$ the evident form of Laplace transform by time was found.

The presented work explicitly defines the Laplace transform on time, Laplace-Stieltjes transform on phase of the conditional and unconditional distributions and Laplace-Stieltjes transform of the ergodic distribution for the semi-markov random processes with positive tendency, negative jumps and delaying boundary at zero.

1. Problem

Let's assume that in probability space $\{\Omega, F, P(\cdot)\}$ is given the sequence of independent , equally distributed and independent themselves positive random variables ξ_k and ζ_k, $k = \overline{1,\infty}$. Using these random variables we will derive the following semi-markov random process:

$$X_1(t) = z + t - \sum_{i=1}^{k-1} \zeta_i , \qquad \text{if} \qquad \sum_{i=1}^{k-1} \xi_i \le t < \sum_{i=1}^{k} \xi_i, \quad k = \overrightarrow{1,\infty}$$

$X_1(t)$ is called semi-markov random processes with positive tendency and negative jumps.

General form of process semi-markov random walk with delaying boundary is given by A.A. Borovkov[1]

If process $X_1(t)$ is some process without boundary, then process $X(t)$ with delaying boundary at zero is defined following :

$$X(t) = X_1(t) - \inf_{0 \le S \le t} (0, X_1(s)) \qquad \text{or} \qquad X(t) = \max(0, \inf_{0 \le S \le t} (0, X_1(s)))$$

Idea of construction of the process semi-markov random walk is following :

Let $X_1(0) = z \ge 0$. Process $X(t)$ is equally to process $X_1(t)$ until, the process $X_1(t)$ is positive.

Let $X_1(t) \le 0$; then $X(t)$ is equally to zero until, the process $X_1(t)$ will not have positive jump. In moment of jump of the process $X_1(t)$, process $X(t)$ will be have jump, such is equally to jump of the process $X_1(t)$.

The obtained process is called a process of a semi-markov random walk with positive tendency, negative jumps and delaying boundary at zero.

The aim of the present study is to find an evident form of the Laplace-Stieltjes transform of the ergodic distribution for X(t).

2. Definition of Laplace transform on time for the distribution of the process X(t)

In accordance with formula of total probability for $x \ge 0$ we have

$$P\{X(t) < x \mid X(0) = z\} = P\{X(t) < x; \xi_1 > t \mid X(0) = z\} + P\{X(t) < x; \xi_1 < t \mid X(0) = z\} =$$

$$= P\{z + t < x; \xi_1 > t\} + \int_{s=0}^{\infty} \int_{y=0}^{\infty} P\{\xi_1 \in ds; X(s) \in dy \mid X(0) = z\} \cdot P\{X(t-s) < x \mid X(0) = y\}$$

(1)

We denote

$$\begin{cases} R(t,x\,|\,z) = P\{X(t) < x\,|\,X(0) = z\}, & x > 0 \\ \tilde{R}(\theta,x\,|\,z) = \int\limits_{t=0}^{\infty} e^{-\theta t} R(t,x\,|\,z), & \theta > 0 \\ \approx \\ R(\theta,\alpha\,|\,z) = \int\limits_{0}^{\infty} e^{-\alpha x} d_x \bar{R}(t,x\,|\,z), & \alpha > 0. \end{cases} \quad ,$$

(2)

In this case equation (1) will be as follows :

$$R(t,x\,|\,z) = P\{z + t - x < 0\}P\{\xi_1 > t\} + \int\limits_{y=0}^{\infty}\int\limits_{s=0}^{t} P\{\xi_1 \in ds\}\, d_y P\{\max(0, z + s - \zeta_1) < y\}\,\mathrm{R(t - s, x\,|\,y)}$$

Both sides of this equation we applied Laplace transform by “t”

$$\int\limits_{0}^{\infty} e^{-\theta t} R(t,x\,|\,z)dt = \int\limits_{0}^{\infty} e^{-\theta t}\, \varepsilon(x - z - t)P\{\xi_1 > t\}dt + \int\limits_{y=0}^{\infty} \tilde{R}(\theta,x\,|\,y) \int\limits_{t=0}^{\infty} e^{-\theta t}\, d_y\, \mathrm{P}\{\max(0, z + t - \zeta_1) < y\}\,\mathrm{dP}\{\xi_1 < t\}$$

where, $\theta > 0$.

Further

$$\tilde{R}(\theta,x\,|\,z) = \varepsilon(x - z)\int\limits_{0}^{x-z} e^{-\theta t} P\{\xi_1 > t\}dt + \int\limits_{y=0}^{\infty} \tilde{R}(\theta,x\,|\,y) \int\limits_{t=0}^{\infty} e^{-\theta t}\, d_y\, \mathrm{P}\{0 < y\}\mathrm{P}\{\mathrm{z + t -}\zeta_1 < y\}\,\mathrm{d\,P}\{\xi_1 < t\}$$

After some simplifications we will get:

$$\tilde{R}(\theta,x\,|\,z) = \varepsilon(x - z)\int\limits_{0}^{x-z} e^{-\theta t} P\{\xi_1 > t\}dt + \int\limits_{y=0}^{\infty} \tilde{R}(\theta,x\,|\,y) \int\limits_{t=0}^{\infty} e^{-\theta t} d_y \varepsilon(y)[1 - \mathrm{P}\{\zeta_1 < z + t - y\}]\,\mathrm{dP}\{\xi_1 < t\} =$$

$$= \varepsilon(x - z)\int\limits_{0}^{x-z} e^{-\theta t} P\{\xi_1 > t\}dt + \tilde{R}(\theta,x\,|\,0)\int\limits_{0}^{\infty} e^{-\theta t}\,\mathrm{dP}\{\xi_1 < t\} - \int\limits_{y=0}^{\infty} \tilde{R}(\theta,x\,|\,y) \int\limits_{t=0}^{\infty} e^{-\theta t} d_y \mathrm{P}\{\zeta_1 < z + t - y\}\,\mathrm{dP}\{\xi_1 < t\} =$$

$$= \varepsilon(x - z)\int\limits_{0}^{x-z} e^{-\theta t} P\{\xi_1 > t\}dt + \tilde{R}(\theta,x\,|\,0)\int\limits_{0}^{\infty} e^{-\theta t}\,\mathrm{dP}\{\xi_1 < t\} - \int\limits_{y=0}^{\infty} \tilde{R}(\theta,x\,|\,y) \int\limits_{t=\max(0,y-z)}^{\infty} e^{-\theta t} d_y \mathrm{P}\{\zeta_1 < z + t - y\}\,\mathrm{dP}\{\xi_1 < t\}$$

If take into account

$$\max\{0, y - z\} = \begin{cases} 0, & \text{if } \ y < z \\ \mathrm{y - z}, & \text{if } \ y > z \end{cases}$$

that is, why we get

$$\tilde{R}(\theta,x\,|\,z) = \varepsilon(x - z)\int\limits_{0}^{x-z} e^{-\theta t} P\{\xi_1 > t\}dt +$$

$$+ \tilde{R}(\theta,x\,|\,0)\int\limits_{0}^{\infty} e^{-\theta t}\,\mathrm{dP}\{\xi_1 < t\} -$$

ISSN 2347-1921

$$-\int_{y=0}^{z}\widetilde{R}(\theta,x\,|\,y)\int_{t=0}^{\infty}e^{-\theta\,t}d_y\mathrm{P}\{\zeta_1<z+t-y\}\mathrm{dP}\{\xi_1<t\}-$$

$$-\int_{y=z}^{\infty}\widetilde{R}(\theta,x\,|\,y)\int_{t=y-z}^{\infty}e^{-\theta\,t}d_y\mathrm{P}\{\zeta_1<z+t-y\}\mathrm{dP}\{\xi_1<t\} \qquad \textbf{(3)}$$

Both sides of this equation we applied Laplace transform by “x ” [see (2)]

$$\overset{\approx}{R}(\theta,\alpha\,|\,\mathrm{z})=\int_{x=0}^{\infty}e^{-\alpha\,x}d_x\varepsilon(x-z)\int_{t=0}^{x-z}e^{-\theta\,t}P\{\xi_1>t\}dt+$$

$$+\overset{\approx}{R}(\theta,\alpha\,|\,0)\int_{0}^{\infty}e^{-\theta\,t}\,\mathrm{dP}\{\xi_1<t\}-$$

$$-\int_{y=0}^{z}\overset{\approx}{R}(\theta,\alpha\,|\,\mathrm{y})\int_{t=0}^{\infty}e^{-\theta\,t}d_y\mathrm{P}\{\zeta_1<z+t-y\}\mathrm{dP}\{\xi_1<t\}-$$

$$-\int_{y=z}^{\infty}\overset{\approx}{R}(\theta,\alpha\,|\,\mathrm{y})\int_{t=y-z}^{\infty}e^{-\theta\,t}d_y\mathrm{P}\{\zeta_1<z+t-y\}\mathrm{dP}\{\xi_1<t\}$$

Take into consideration

$$\int_{x=0}^{\infty}e^{-\alpha\,x}d_x\varepsilon(x-z)\int_{t=0}^{x-z}e^{-\theta\,t}P\{\xi_1>t\}dt=\int_{x=0}^{\infty}e^{-\alpha\,x}\varepsilon(x-z)\,d_x\int_{t=0}^{x-z}e^{-\theta\,t}P\{\xi_1>t\}dt+\int_{x=0}^{\infty}e^{-\alpha\,x}\int_{t=0}^{x-z}e^{-\theta\,t}P\{\xi_1>t\}dt\,d_x\varepsilon(x-z)$$

$$=\int_{x=z}^{\infty}e^{-\alpha\,x}d_x\int_{t=0}^{x-z}e^{-\theta\,t}P\{\xi_1>t\}dt=\int_{z}^{\infty}e^{-\alpha\,x}e^{-\theta(x-z)}P\{\xi_1>x-z\}dx$$

At last we received the following integral equation for $\overset{\approx}{R}(\theta,\alpha\,|\,\mathrm{z})$ when ξ_k and ζ_k, $k=\overline{1,\infty}$ equally distributed and independent themselves positive random variables

$$\overset{\approx}{R}(\theta,\alpha\,|\,\mathrm{z})=e^{\theta\,z}\int_{z}^{\infty}e^{-(\alpha+\theta)\,x}P\{\xi_1>x-z\}dx+$$

$$+\overset{\approx}{R}(\theta,\alpha\,|\,0)\int_{0}^{\infty}e^{-\theta\,t}\mathrm{P}\{\zeta_1>z+t\}\mathrm{dP}\{\xi_1<t\}-$$

$$-\int_{y=0}^{z}\overset{\approx}{R}(\theta,\alpha\,|\,y)\int_{t=0}^{\infty}e^{-\theta\,t}d_y\mathrm{P}\{\zeta_1<z+t-y\}\mathrm{dP}\{\xi_1<t\}-$$

$$-\int_{y=z}^{\infty}\overset{\approx}{R}(\theta,\alpha\,|\,y)\int_{t=y-z}^{\infty}e^{-\theta\,t}d_y\mathrm{P}\{\zeta_1<z+t-y\}\mathrm{dP}\{\xi_1<t\}$$

(4)

We will solve this integral equation in special case.

Let’s assume that ξ_1 random variable has the Erlangian distribution of n order, while ζ_1 random variable has the single order Erlangian distribution:

ISSN 2347-1921

$$P\{\xi_1(\omega)<t\}=\left\{1-\left[1+\mu t+\frac{(\mu t)^2}{2!}+\cdots+\frac{(\mu t)^{n-1}}{(n-1)!}\right]e^{-\mu t}\right\}\varepsilon(t)\ , \qquad \mu>0$$

$$\mathrm{P}\{\zeta_1(\omega)<t\}=\left[1-e^{-\lambda t}\right]\varepsilon(t)\,, \qquad \lambda>0$$

where

$$\varepsilon(t)=\begin{cases}0, & \mathrm{t}<0\\ 1, & \mathrm{t}>0\end{cases}$$

In this case equation (4) will be as follows:

$$\begin{aligned}\overset{\approx}{R}(\theta,\alpha\,|\,\mathrm{z}) &= \frac{(\alpha+\mu+\theta)^{\mathrm{n}}-\mu^n}{(\alpha+\mu+\theta)^{\mathrm{n}}\ (\alpha+\theta)}\,\mathrm{e}^{-\alpha\,\mathrm{z}}+\left[\frac{\mu}{\lambda+\mu+\theta}\right]^n e^{-\lambda z}\,\overset{\approx}{R}(\theta,\alpha\,|\,0)\,- \\ &-\lambda\left[\frac{\mu}{\lambda+\mu+\theta}\right]^n \mathrm{e}^{-\lambda \mathrm{z}}\int_0^{\mathrm{z}} e^{\lambda\,\mathrm{y}}\,\overset{\approx}{R}(\theta,\alpha\,|\,\mathrm{y})\mathrm{dy}\,- \\ &-\frac{\lambda\mu^{\mathrm{n}}}{(n-1)!}\,\mathrm{e}^{-\lambda \mathrm{z}}\int_{\mathrm{y=z}}^{\infty} e^{\lambda\,\mathrm{y}}\,\overset{\approx}{R}(\theta,\alpha\,|\,\mathrm{y})\int_{\mathrm{t=y-z}}^{\infty}\mathrm{e}^{-(\lambda+\mu+\theta)\mathrm{t}}\ \mathrm{t}^{\mathrm{n-1}}\mathrm{dtdy}\end{aligned}$$

(5)

We will get differential equation from this integral equation. For this purpose, we will multiply both sides of equation (5) by $e^{\lambda z}$ and derive on z . Then we will multiply both sides of last equation by $e^{-(\lambda+\mu+\theta)z}$ and derive on z . If repeat this process (n-1) time we have following differential equation**:**

$$\sum_{k=o}^{n} C_n^k\left[\lambda\,\overset{\approx}{R}^{(\mathrm{k})}(\theta,\alpha\,|\,\mathrm{z})+\overset{\approx}{R}^{(\mathrm{k+1})}(\theta,\alpha\,|\,\mathrm{z})\right](-1)^{\mathrm{n-k}}\ (\mu+\theta)^{\mathrm{n-k}}+(-1)^{n-1}\lambda\mu^n\,\overset{\approx}{R}(\theta,\alpha\,|\,\mathrm{z})=$$

$$=(-1)^{\mathrm{n-1}}\,\frac{[(\alpha+\mu+\theta)^{\mathrm{n}}-\mu^{n}](\lambda-\alpha)}{(\alpha+\theta)}\,\mathrm{e}^{-\alpha\,\mathrm{z}}$$

(6)

3. The general solution of the differential equation (6)

The general solution of this differential equation will be

$$\overset{\approx}{R}(\theta,\alpha\,|\,\mathrm{z})=\mathrm{C}_1(\theta,\alpha)e^{k_1(\theta)z}+\mathrm{C}_2(\theta,\alpha)e^{k_2(\theta)z}+\cdots+\mathrm{C}_{\mathrm{n}}(\theta,\alpha)e^{k_n(\theta)z}+\overset{\approx}{R}_{\mathrm{sp}}(\theta,\alpha\,|\,\mathrm{z}) \qquad \textbf{(7)}$$

where

$k_i(\theta)\,,\quad \mathrm{i}=1,2,...\mathrm{n},$ -are the roots of characteristic equation of (6)

$\overset{\approx}{R}_{\mathrm{sp}}(\theta,\alpha\,|\,\mathrm{z})$ - is the special solution of the equation (5)

$$\overset{\approx}{R}_{\mathrm{sp}}(\theta,\alpha\,|\,\mathrm{z})=\mathrm{A\,e}^{-\alpha\,\mathrm{z}}$$

where

ISSN 2347-1921

$$A = \frac{(\lambda - \alpha)[(\alpha+\mu+\theta)^{n} - \mu^{n}]}{(\alpha+\theta)\prod_{i=1}^{n}[\alpha + k_i(\theta)]}$$

$$\begin{cases} \overset{\approx}{R}(\theta,\alpha\,|\,0) = \frac{(\alpha+\mu+\theta)^{n} - \mu^{n}}{(\alpha+\mu+\theta)^{n}\,(\alpha+\theta)} + \left[\frac{\mu}{\lambda+\mu+\theta}\right]^{n} \overset{\approx}{R}(\theta,\alpha\,|\,0) - \frac{\lambda\mu^{n}}{(n-1)!}\int_{y=0}^{\infty} e^{\lambda y}\,\overset{\approx}{R}(\theta,\alpha\,|\,y)\int_{t=y}^{\infty} e^{-(\lambda+\mu+\theta)t} t^{n-1}\,dtdy \\ \overset{\approx}{R}'(\theta,\alpha\,|\,0) = -\alpha\frac{(\alpha+\mu+\theta)^{n} - \mu^{n}}{(\alpha+\mu+\theta)^{n}\,(\alpha+\theta)} - \lambda\left[\frac{\mu}{\lambda+\mu+\theta}\right]^{n} \overset{\approx}{R}(\theta,\alpha\,|\,0) + \frac{\lambda^{2}\mu^{n}}{(n-1)!}\int_{y=0}^{\infty} e^{\lambda y}\,\overset{\approx}{R}(\theta,\alpha\,|\,y)\int_{t=y}^{\infty} e^{-(\lambda+\mu+\theta)t} t^{n-1}\,dtdy - \\ \qquad - \frac{\lambda\mu^{n}}{(n-1)!}\int_{0}^{\infty} e^{-(\mu+\theta)y}\,\overset{\approx}{R}(\theta,\alpha\,|\,y)\,y^{n-1}dy \\ \cdots\cdots\cdots \\ \sum_{k=0}^{n-1} C_n^k\left[\lambda \overset{\approx}{R}{}^{(k)}(\theta,\alpha\,|\,0) + \overset{\approx}{R}{}^{(k+1)}(\theta,\alpha\,|\,0)\right] = (-1)^{n-k}\frac{[(\alpha+\mu+\theta)^{n} - \mu^{n}](\lambda-\alpha)}{(\alpha+\theta)} - (-1)^{n-1}\lambda\mu^{n}\int_{z}^{\infty} e^{-(\mu+\theta)y}\,\overset{\approx}{R}(\theta,\alpha\,|\,y)dy \end{cases} \quad (8)$$

By finding $C_1(\theta,\alpha)$ ….. $C_n(\theta,\alpha)$ from equation (5) we will get the following system of algebraic equations:

By

exploitation of equation (7), equation (8) becomes

$$\begin{cases} \sum_{i=1}^{n} C_i(\theta,\alpha) + A = \frac{(\alpha+\mu+\theta)^{n} - \mu^{n}}{(\alpha+\mu+\theta)^{n}\,(\alpha+\theta)} + \left[\frac{\mu}{\lambda+\mu+\theta}\right]^{n}\left[\sum_{i=1}^{n} C_i(\theta,\alpha) + A\right] - \\ \qquad - \frac{\lambda\mu^{n}}{(n-1)!}\int_{y=0}^{\infty} e^{\lambda y}\left[\sum_{i=1}^{n} C_i(\theta,\alpha)e^{k_i(\theta)y} + Ae^{-\alpha y}\right]\int_{t=y}^{\infty} e^{-(\lambda+\mu+\theta)t} t^{n-1}\,dtdy \\ \sum_{i=1}^{n} C_i(\theta,\alpha)k_i(\theta) - \alpha A = -\alpha\frac{(\alpha+\mu+\theta)^{n} - \mu^{n}}{(\alpha+\mu+\theta)^{n}\,(\alpha+\theta)} - \lambda\left[\frac{\mu}{\lambda+\mu+\theta}\right]^{n}\left[\sum_{i=1}^{n} C_i(\theta,\alpha) + A\right] + \\ \qquad + \frac{\lambda^{2}\mu^{n}}{(n-1)!}\int_{y=0}^{\infty} e^{\lambda y}\left[\sum_{i=1}^{n} C_i(\theta,\alpha)e^{k_i(\theta)y} + Ae^{-\alpha y}\right]\int_{t=y}^{\infty} e^{-(\lambda+\mu+\theta)t} t^{n-1}\,dtdy - \\ \qquad - \frac{\lambda\mu^{n}}{(n-1)!}\int_{0}^{\infty} e^{-(\mu+\theta)y}\left[\sum_{i=1}^{n} C_i(\theta,\alpha)e^{k_i(\theta)y} + Ae^{-\alpha y}\right] y^{n-1}dy \\ \cdots\cdots\cdots \\ \sum_{k=0}^{n} C_n^k\left\{\lambda\left[\sum_{i=1}^{n} k^{n}C_i(\theta,\alpha) + \alpha^{n}A\right] + \left[\sum_{i=1}^{n} k^{n+1}C_i(\theta,\alpha) + \alpha^{n+1}A\right]\right\} = (-1)^{n-1}\,\alpha^{n-1}\frac{[(\alpha+\mu+\theta)^{n} - \mu^{n}](\lambda-\alpha)}{(\alpha+\theta)} - \\ \qquad - (-1)^{n-1}\lambda\mu^{n}\int_{z}^{\infty} e^{-(\mu+\theta)y}\left[\sum_{i=1}^{n} C_i(\theta,\alpha)e^{k_i(\theta)y} + Ae^{-\alpha y}\right]dy \end{cases} \quad (9)$$

Now we proof linear dependence of this algebraic system

If to consider the following substitutions:

$$\prod_{i=1}^{n}\left[\mu+\theta-k_i(\theta)\right]=(-1)^{n+1}\lambda\mu^n$$

$$\prod_{i=1}^{n}\left[\lambda+k_i(\theta)\right]=(-1)^{n+1}\lambda\mu^n$$

$$\prod_{i=1}^{n}\left[\mu+\theta-k_i(\theta)\right]=(-1)^{n+1}\left[\lambda+k_j(\theta)\right]\left[\mu+\theta-k_j(\theta)\right]^n=(-1)^{n+1}\lambda\mu^n$$

$$\prod_{i=1}^{n}\left[\alpha+k_i(\theta)\right]=(\lambda-\alpha)(\mu+\theta+\alpha)^n-\lambda\mu^n$$

$$\int_y^{\infty}t^{n-1}e^{-(\lambda+\mu+\theta)t}dt=\frac{y^{n-1}}{\lambda+\mu+\theta}+\sum_{k=2}^{n}\frac{(n-1)!}{(n-k)!}\frac{y^{n-k}}{(\lambda+\mu+\theta)^k}e^{-(\lambda+\mu+\theta)y}$$

(10)

equation (9) becomes:

$$\begin{cases}\sum_{i=1}^{n}\left\{[\mu+\theta-k_i(\theta)]^n-\mu^n\right\}C_i(\theta,\alpha)=\dfrac{\alpha\mu^n[(\alpha+\mu+\theta)^n-\mu^n]}{(\alpha+\theta)\left[\lambda\mu^n-(\lambda-\alpha)(\mu+\theta+\alpha)^n\right]}=\alpha\mu^n A\\ \sum_{i=1}^{n}\left\{[\mu+\theta-k_i(\theta)]^n-\mu^n\right\}C_i(\theta,\alpha)=\dfrac{\alpha\mu^n[(\alpha+\mu+\theta)^n-\mu^n]}{(\alpha+\theta)\left[\lambda\mu^n-(\lambda-\alpha)(\mu+\theta+\alpha)^n\right]}=\alpha\mu^n A\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ \sum_{i=1}^{n}\left\{[\mu+\theta-k_i(\theta)]^n-\mu^n\right\}C_i(\theta,\alpha)=\dfrac{\alpha\mu^n[(\alpha+\mu+\theta)^n-\mu^n]}{(\alpha+\theta)\left[\lambda\mu^n-(\lambda-\alpha)(\mu+\theta+\alpha)^n\right]}=\alpha\mu^n A\end{cases}$$

(11)

Thus, (11) is a linear dependence equations system, as

$$C_2(\theta,\alpha)=C_3(\theta,\alpha)=\cdots=C_n(\theta,\alpha)=0$$

Then we have

$$C_1(\theta,\alpha)=\frac{\alpha\mu^n\ [(\alpha+\mu+\theta)^n-\mu^n]}{(\alpha+\theta)(\left[\mu+\theta-k_1(\theta)\right]^n-\mu^n)\left[\lambda\mu^n-(\lambda-\alpha)(\mu+\theta+\alpha)^n\right]}\qquad(12)$$

Then the general solution of integral equation (5) will be as follows:

$$\overset{\approx}{R}(\theta,\alpha\,|\,z)=\frac{\alpha\mu^n\ [(\alpha+\mu+\theta)^n-\mu^n]}{(\alpha+\theta)(\left[\mu+\theta-k_1(\theta)\right]^n-\mu^n)\left[\lambda\mu^n-(\lambda-\alpha)(\mu+\theta+\alpha)^n\right]}e^{k_1(\theta)z}+$$

$$+\frac{(\lambda-\alpha)[(\alpha+\mu+\theta)^n-\mu^n]}{(\alpha+\theta)[\left[\lambda\mu^n-(\lambda-\alpha)(\mu+\theta+\alpha)^n\right]]}e^{-\alpha z}$$

(13)

This expression is the Laplace transform on time, Laplace-Stieltjes transform on phase for **conditional** distribution of the process $X(t)$

4. Ergodic distribution of the process.

ISSN 2347-1921

We will need to find Laplace transform on time, Laplace-Stieltjes transform on phase for **unconditional** distribution of the process $X(t)$.

From construction process $X(t)$ is seen that

$$X(0) = X_1(0) = \xi_1(\omega)$$

Then we will get

$$\overset{\approx}{R}(\theta,\alpha) = \int_0^\infty \overset{\approx}{R}(\theta,\alpha \mid z)\, dP\{X(0) < z\}$$

Therefore

$$\overset{\approx}{R}(\theta,\alpha) = \int_{z=0}^{\infty}[\frac{\alpha\mu^n\ [(\alpha+\mu+\theta)^n-\mu^n]}{(\alpha+\theta)\left([\mu+\theta-k_1(\theta)]^n-\mu^n\right)\left[\lambda\mu^n-(\lambda-\alpha)(\mu+\theta+\alpha)^n\right]}e^{k_1(\theta)z} + \\ + \frac{(\lambda-\alpha)[(\alpha+\mu+\theta)^n-\mu^n]}{(\alpha+\theta)[\left[\lambda\mu^n-(\lambda-\alpha)(\mu+\theta+\alpha)^n\right]]}e^{-\alpha z}]\, d[1-e^{-\mu z}]$$

or

$$\overset{\approx}{R}(\theta,\alpha) = \frac{\alpha\mu^n\ [(\alpha+\mu+\theta)^n-\mu^n]}{(\alpha+\theta)\left([\mu+\theta-k_1(\theta)]^n-\mu^n\right)\left[\lambda\mu^n-(\lambda-\alpha)(\mu+\theta+\alpha)^n\right]}\frac{\mu}{[\mu-k_1(\theta)]} + \\ + \frac{(\lambda-\alpha)[(\alpha+\mu+\theta)^n-\mu^n]}{(\alpha+\theta)\left[\lambda\mu^n-(\lambda-\alpha)(\mu+\theta+\alpha)^n\right]}\frac{\mu}{[\alpha+\mu]}$$

This expression is Laplace transform on time, Laplace-Stieltjes transform on phase for **unconditional** distribution of the process $X(t)$.

Now, we will find Laplace-Stieltjes transform for ergodic distribution of the process $X(t)$.

In [3] (see p.363) proved a general theorem on the ergodicity of the process semi-markov random walk. The process described in this article a special case of this process.

Process $X(t)$ will be ergodic, if $E\xi_1 < E\zeta_1$, or

$$\frac{1}{\mu} < \frac{n}{\lambda} \Rightarrow \lambda < n\mu$$

If process $X(t)$ ergodic,then we can use Tauber's theorem [4]

$$Ee^{-\alpha X(\omega)} = \tilde{R}(\alpha) = \lim_{\theta\to 0} \theta\, \overset{\approx}{R}(\theta,\alpha)$$

We obtined

$$\tilde{R}(\alpha) = \frac{1}{(n-1)!}\frac{[(\alpha+\mu)^n - \mu^n]\,\mu}{[\lambda\mu^n + (\alpha-\lambda)(\alpha+\mu)^n]} \qquad \textbf{(14)}$$

Expression (14) is Laplace-Stieltjes transform for ergodic distribution of the process $X(t)$.Respectively, we will get the following characteristics for $\lambda < n\mu$:

$$\widetilde{R}'(0) = -EX(\omega) = \frac{n}{\lambda - n\mu}, \qquad \lambda < n\mu$$

$$\widetilde{R}''(0) - [\widetilde{R}'(0)]^2 = DX(\omega) = \frac{2n}{(\lambda - n\mu)^n}, \qquad \lambda < n\mu$$

5. Conclusions

In this article we have defined Laplace transforms on time, Laplace-Stieltjes transforms on phase for conditional and unconditional distributions and Laplace-Stieltjes transform for the ergodic distribution**.**

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