

Uniqueness for entropy solutions to fully nonlinear equations

Chiraz KOURAICHI and Abdelmajid SIAI

Ecole Nationale d'Ingénieurs de Monastir, Tunisie

chiraz.kouraichi@yahoo.fr

Institut Préparatoire aux Etudes d'Ingénieurs de Nabeul, Tunisie

absiai@yahoo.fr

ABSTRACT

Let X be a metric space, $\mathcal{M}(X, \mathcal{B}, \mu)$ the space of μ -measurable functions, $\Omega \subset \mathbb{R}^N$ be a domain with boundary $\partial\Omega$ and $a(x, \xi)$ be an operator of Leray-Lions type. If β and γ are nondecreasing continuous function on \mathbb{R} such that $\beta(0) = \gamma(0) = 0$ and $(f, g) \in L^1(X, \mathcal{B}, \mu)$, then, there exists a unique entropy solution u in $\mathcal{M}(X, \mathcal{B}, \mu)$ to the problem $-\operatorname{div}[a(\cdot, Du)] + \beta(u) = f$ in Ω and $a(\cdot, Du)v + \gamma(\tau u) = g$ on $\partial\Omega$.

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Entropy solution; Renormalized solution; Elliptic problem; Measure space.

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Introduction

Let $\Omega \subset \mathbb{R}^N$, $N \geq 3$, be a domain (not necessarily bounded) with boundary $\partial\Omega$. For example, to prove existence and uniqueness for a solution to an elliptic or parabolic problem in Ω related with an operator of Leray-Lions type: $Au = -\text{div}[a(\cdot, Du)]$ with non linear Neumann condition on the boundary $\partial\Omega$:

$$a(\cdot, Du) \cdot \nu + \gamma(\tau u) \ni g, g \in L^1(\partial\Omega), \quad (1.1)$$

where γ is a maximal monotone graph in \mathbb{R}^2 , ν is the vector field of exterior normal to the boundary $\partial\Omega$, when $g = 0$, many authors even in the linear case $Au = -\Delta u$, (see [1], for example) include the boundary condition (1.1) in the definition of the domain $D(A)$ of the operator A . In the case where $g \neq 0$, [2] apply the same process to a family of operators B^g in the elliptic case (resp: $B^{g(t)}$ in the parabolic case). Besides supplementary technical difficulties, even in case where A is linear some notion of multivalued linear operator is needed. This is no more necessary when applying theorems 4.1 of [8] and [9]. More precisely if $X = \Omega \cup \partial\Omega$, $L^1(\Omega) \times L^1(\partial\Omega)$ is identified to $L^1(X)$. Then, let A be The operator defined as follows: $(u, \tau u) \in D(A)$ if there exist $(f, g) \in L^1(X)$ such u is an entropy solution to the next problems:

$$\begin{cases} -\text{div}[a(\cdot, Du)] = f \text{ in } \Omega, \\ a(\cdot, Du) \cdot \nu = g \text{ on } \partial\Omega \end{cases} \quad (1.2)$$

In the sense that, if $T_k(r) = \max\{-k, \min(r, k)\}$, $k > 0$, $r \in \Omega$, $\forall \varphi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$,

$$\int_{\Omega} a(x, Du) DT_k(u - \varphi) \leq \int_{\Omega} f T_k(u - \varphi) + \int_{\partial\Omega} g T_k(\tau u - \tau \varphi).$$

If A_1 is the restriction of A to $L^1(X)$, then A_1 is said to be accretive in $L^1(X)$, if then the next inequality holds

$$\int_{\Omega} (f_1 - f_2) \varphi_0 + \int_{\{u_1 = u_2\}} |f_1 - f_2| + \int_{\partial\Omega} (g_1 - g_2) \psi_0 + \int_{\{\tau u_1 = \tau u_2\}} |g_1 - g_2| \geq 0, \quad (1.3)$$

For any $F_i = (f_i, g_i) \in L^1(X)$, $U_i = (u_i, \tau u_i) \in D(A_1)$, $i=1,2$ so that $AU_i = F_i$ and $\varphi_0 = \text{sign}_0(u_1 - u_2)$, $\psi_0 = \text{sign}_0(\tau u_1 - \tau u_2)$, where: $\text{sign}(r) = \text{sign}_0(r) = \frac{r}{|r|}$, $r \in \mathbb{R}$, $r \neq 0$, $\text{sign}(0) = [-1, 1]$ and $\text{sign}_0(0) = 0$. Inequalities (i) in ([8] theorem 4.1) extends (1.3) to the case where $(u_i, \tau u_i) \in D(A)$, and inequality (ii) states that if in addition $(u_i, \tau u_i) \in D(A_1)$ $i=1,2$, then for every $\varphi \in \text{sign}(u_1 - u_2)$ and $\psi \in \text{sign}(\tau u_1 - \tau u_2)$, we have:

$$\int (f_1 - f_2) \varphi + \int_{\partial\Omega} (g_1 - g_2) \psi \geq 0. \quad (1.4)$$

Inequalities (1.3) and (1.4) where applied in [8], and similarly in [9], to prove existence and uniqueness of a solution to the problem: $-\text{div}[a(\cdot, Du)] + \beta(u) \ni f$ and $a(\cdot, Du) \cdot \nu + \gamma(\tau u) \ni g$ on $\partial\Omega$.

It is well known from [7] that uniqueness of weak solutions for degenerate problems, is not guaranteed. However, if two kind of solutions u and v in a class of uniqueness, such as entropy solutions or renormalized solutions, are obtained as a limit of a sequence (u_n) of some regular solutions related to some sequence data (f_n) , from uniqueness of u_n related to some f_n , the fact that entropy solutions, renormalized solutions etc are particular weak solutions, then by applying (1.3), formally we shall be able to prove that $u = v$. This seems to be new, since, even that studied separately existence and uniqueness of entropy and renormalized solutions have not proved that it is the same solution (see [6]).

Theorems 4.1 [8] may be extended to general measure spaces (X, \mathcal{B}, μ) and the closure in $\mathcal{M}(X, \mathcal{B}, \mu) \times L^1(X, \mathcal{B}, \mu)$ of the operator A_1 and thus, a larger class of measurable functions that are weak solutions obtained as limit in $\mathcal{M}(X, \mathcal{B}, \mu)$ with data in $f \in L^1(X)$ of a sequence of entropy solutions, or renormalized solutions that are in $L^1(X)$. It is proved first, that for this particular case, entropy and renormalized solution is the same one. This will be extended, next, for general entropy and renormalized solutions in $\mathcal{M}(X, \mathcal{B}, \mu)$, that are not necessary integrable, but satisfy some specified conditions of regularity that is required in the definition of this kind of solutions.

Order Preserving Inequalities

Let be given a metric space X and a complete measure space (X, \mathcal{B}, μ) such that X is σ -infinite and μ is regular, (see [4]). The space of μ -measurable real valued functions $\mathcal{M}(X, \mathcal{B}, \mu)$, equipped with some distance as in [8] and [9] is a Frchet space and its topology is equivalent to the local convergence in measure. In the sequel, the spaces $\mathcal{M}(X, \mathcal{B}, \mu)$ and $L^1(X, \mathcal{B}, \mu)$ are noted simply \mathcal{M} and L^1 .

Definition 2.1.A: $X \rightarrow 2^X$ an operator, possibly multivalued in X , is said to be **accretive** in X , if one of the following equivalent properties is satisfied,

- (i) $(x_1 - x_2, y_1 - y_2) \geq 0$, if $x_1, x_2 \in D(A)$, $y_1 \in Ax_1$, $y_2 \in Ax_2$.
- (ii) The resolvent $J_\lambda^A = (I + \lambda A)^{-1}$ is a contraction from $R(I + \lambda A)$ to X , for every $\lambda > 0$.



Definition 2.2. A is **ism-accretive** in X , if the resolvent J_λ^A is a contraction everywhere defined in X , for every $\lambda > 0$.

Definition 2.3. The operator A is **m-completely accretive** in X , if A is m-accretive and

$$\int_X (AU_1 - AU_2)p(U_1 - U_2) \geq 0, U_1, U_2 \in D(A), p \in \mathcal{P}_0. \tag{2.1}$$

$\mathcal{P}_0 = \{p: \mathbb{R} \rightarrow \mathbb{R}, p \text{ Lipschitz, odd, non decreasing and } p' \text{ has a compact support}\}$.

The function a of Leray-Lions type is defined as follows,

(H.1) $\left\{ \begin{array}{l} a: \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}^N \\ (x, \xi) \mapsto a(x, \xi) \end{array} \right.$ is a Carathéodory function in the sense that, a is continuous in ξ ,

for almost every $x \in \Omega$, and measurable in x for any $\xi \in \mathbb{R}^N$.

(H.2) there exist $p, C_1 \in \mathbb{R}, 1 < p < N$, and $C_1 > 0$, so that, $\langle a(x, \xi), \xi \rangle \geq C_1 |\xi|^p$, for a. e. $x \in \Omega$ and every $\xi \in \mathbb{R}^N$.

(H.3) $\langle a(x, \xi_1) - a(x, \xi_2), \xi_1 - \xi_2 \rangle > 0$, if $\xi_1 \neq \xi_2$, for a. e. $x \in \Omega$.

(H.4) There exists some $h_0 \in L^{p'}(\Omega)$, $p' = \frac{p}{p-1}$ and a positive constant C_2 such that $|a(x, \xi)| \leq C_2 (h_0(x) + |\xi|^{p-1})$, for a. e. $x \in \Omega$ and every $\xi \in \mathbb{R}^N$.

If A_1 is an m-accretive operator in $L^1(\Omega)$, its closure $A = \overline{A_1}$ is defined as follows: For any $(u, f) \in \mathcal{M} \times L^1$, then $Au = f$, if there exist $(u_n, f_n) \in (L^1)^2$ such that, $(u_n, f_n) \rightarrow (u, f)$ in $\mathcal{M} \times L^1$ and $Au_n = f_n$.

(\mathcal{A}_1): A_1 is Kato accretive in L^1 , in the sense that for any pair $(u_1, f_1), (u_2, f_2) \in D(A_1) \times L^1$, we have:

$$\lim_{t \rightarrow 0} \left(\int_X |(f - g) + t(u - v)| - |f - g| \right) = \int_X (f - g) \text{sign}_0(u - v) + \int_{\{u=v\}} |f - g| \geq 0,$$

where $\text{sign}(r) = \text{sign}_0(r) = \frac{r}{|r|}$, if $r \in \mathbb{R}, r \neq 0$, $\text{sign}(0) = [-1, 1]$ and $\text{sign}_0(0) = 0$.

Theorem 2.1. A_1 and A are defined as previously, let be given $f, g \in L^1$, then we have the following:

(i) If $u, v \in D(A)$ are solutions to $Au = f$ and $Av = g$, then they satisfy:

$$\int_X (f - g) \text{sign}_0(u - v) + \int_{\{u=v\}} |f - g| \geq 0. \tag{2.2}$$

(ii) If in addition $u, v \in D(A_1)$, then: $\int_X (f - g) \varphi \geq 0$, for every $\varphi \in \text{sign}(u - v)$ (2.3)

Proof. (i) If $(u_n, f_n), (v_n, g_n) \in (L^1)^2, A_1 u_n = f_n$ and $A_1 v_n = g_n, u_n \rightarrow u, v_n \rightarrow v$ in \mathcal{M} and $f_n \rightarrow f, g_n \rightarrow g$ in L^1 . Since A_1 is accretive in L^1 , if setting $h_n = f_n - g_n, \omega_n = u_n - v_n, h = f - g, \omega = u - v, \varphi_n = \text{sign}_0(\omega_n)$ and $\varphi = \text{sign}_0(\omega)$, this leads to:

$$\int_X h_n \varphi_n + \int_{\{\omega=0\}} |h_n| \geq 0, \forall n \in \mathbb{N}. \tag{2.4}$$

Next, if $T_{\frac{1}{k}}(r) = \max\left\{-\frac{1}{k}, \min\left(\frac{1}{k}, r\right)\right\}, k \in \mathbb{N}^*, r \in \mathbb{R}$, then

$$\begin{aligned} \left| \int_X h \varphi - \int_X h_n \varphi_n \right| &\leq \int_X |h| \left| \varphi - k T_{\frac{1}{k}}(\omega) \right| + \int_X |h| \left| k T_{\frac{1}{k}}(\omega) - k T_{\frac{1}{k}}(\omega_n) \right| + \int_X |h - h_n| \left| k T_{\frac{1}{k}}(\omega_n) \right| + \int_X |h_n| \left| k T_{\frac{1}{k}}(\omega_n) - \varphi_n \right| \\ &= I_1 + I_2 + I_3 + I_4 \end{aligned}$$

Then, by applying the Lebesgue theorem successively to I_1, I_2, I_3 , we may assume that:

$\forall \varepsilon > 0, \exists k_0, n_0 = n_0(k_0), n_1 \in \mathbb{N}^*,$ s. t. $I_1 \leq \varepsilon$, if $k \geq k_0, I_2 \leq \varepsilon$, if $n \geq n_0$ and $I_3 \leq \varepsilon$, if $n \geq n_1$. Thus if n and k are large enough, then: $\left| \int_X h \varphi - \int_X h_n \varphi_n \right| \leq 3\varepsilon + \int_X |h_n| \left| k T_{\frac{1}{k}}(\omega_n) - \varphi_n \right|$, thus $\lim_{n \rightarrow +\infty} \left| \int_X h \varphi - \int_X h_n \varphi_n \right| \leq 3\varepsilon + \int_X |h| \left| k T_{\frac{1}{k}}(\omega) - \varphi \right|$.

Since the left term in the last inequality do not depend to k and $\lim_{k \rightarrow +\infty} \int_X |h| \left| k T_{\frac{1}{k}}(\omega) - \varphi \right| = 0$, then

$\lim_{n \rightarrow +\infty} \int_X h_n \varphi_n = \int_X h \varphi$, hence

$$\forall \varepsilon > 0, \exists n_\varepsilon \in \mathbb{N}, \text{ such that } \int_X h \varphi \geq \int_X h_n \varphi_n - \varepsilon \text{ if } n \geq n_\varepsilon. \tag{2.5}$$

Now, for the right term in (2.4), denote $E_n = \{\omega_n = 0\}$, if C is a compact in X and $\eta > 0$, then by Egorov theorem: $\forall k \in \mathbb{N}^*, \exists N_k \in \mathbb{B}, N_k \subset C$ and $n_k \in \mathbb{N}$ so that $n \geq n_k$, then $\mu(N_k) \leq \eta, |u_n - u| \leq \frac{1}{k}$ and $|v_n - v| \leq \frac{1}{k}$ uniformly on $C \setminus N_k$, therefore, after possibly



replacing N_k with $\cup_{1 \leq i \leq k} N_i$, we assume that $(N_k)_k$ is increasing and while setting $F_k = \cup_{n \geq n_k} E_n$ and $F'_k = (F_k \cap (C \setminus N_k))_k$, it ensuer that $F'_k \downarrow F_{k \geq 1} = \cap F'_k$, if $k \rightarrow +\infty$.

Next, for μ almost any $x \in F'_k$, there exist n such that

$$|u(x) - v(x)| \leq |u(x) - u_n(x)| + |u_n(x) - v_n(x)| + |v_n(x) - v(x)| \leq \frac{2}{k}.$$

Thus $u = v$, μ .a.e on F ,

Therefore $F'_k \downarrow \cap_{k \geq 1} F'_k \subset E = \{\omega = 0\}$ and $\int_E |h| \geq \int_{F'_k} |h| - \varepsilon$, of sufficiently large k . (2.6)

Now, with the help of (2.5) and (2.4), if n is large enough, then $\int_X h\varphi + \int_{F_k} |h_n| \geq \int_X h_n \varphi_n - \varepsilon + \int_{E_n} |h_n| \geq -\varepsilon$.

As $\lim_{n \rightarrow +\infty} \int_{F_k} |h_n| = \int_{F_k} |h|$, it arises that $\int_X h\varphi + \int_{F_k} |h| \geq -\varepsilon$,

For every $\varepsilon > 0$. Therefore $\int_X h\varphi + \int_{F_k} |h| \geq 0$, if $k \geq k_0$. (2.7)

Next, $F_k = F'_k \cup (F_k \cap N_k) \cup (F_k \cap C^c)$. Since $h \in L^1$, then we may suppose that the compact C is sufficiently large and $\mu(N_k)$ is sufficiently small so that $\int_{C^c} |h| \leq \varepsilon$ and $\int_{N_k} |h| \leq \varepsilon$, therefore $\int_E |h| \geq \int_{F'_k} |h| - \varepsilon \geq \int_{F_k} |h| - 3\varepsilon$ and then, in view of (2.6) and (2.7).

$$\int_X (f - g) \text{sign}_0(u - v) + \int_{\{u=v\}} |f - g| = \int_X h\varphi + \int_E |h| \geq -4\varepsilon, \text{ for any } \varepsilon > 0.$$

This completes the proof of (2.2).

(ii) Its proof is the same as in [8] and [9], we give an outline for this. For $\alpha > 0$, consider

$$W_{i,\alpha} = (\omega_{i,\alpha}, \tau\omega_{i,\alpha}) = J_\alpha^{A_1} u_i = (I + \alpha A_1)^{-1} u_i \in D(A_1), i=1, 2,$$

$$Y_{i,\alpha} = (y_{i,\alpha}, z_{i,\alpha}) = A_1 W_{i,\alpha} = A_{1,\alpha} u_i = \frac{1}{\alpha} (u_i - W_{i,\alpha})$$

Consider $p_n(r) = n T_{\frac{1}{n}}(r)$ and $j_n(r) = \int_0^r p_n(s) ds$, then by definition of the subdifferential $\partial j_n = j'_n$, we have

$$j_n(u_1 - u_2) - j_n(\omega_{1,\alpha} - \omega_{2,\alpha}) \geq \alpha p_n(\omega_{1,\alpha} - \omega_{2,\alpha})(y_{1,\alpha} - y_{2,\alpha}) \geq 0, \mu.a.e. \text{ on } X. \text{ Then}$$

$$\int_X j_n(u_1 - u_2) - \int_X j_n(\omega_{1,\alpha} - \omega_{2,\alpha}) \geq \alpha \int_X p_n(\omega_{1,\alpha} - \omega_{2,\alpha})(y_{1,\alpha} - y_{2,\alpha}).$$

Since $|j_n(r)| \uparrow j(r) = |r|$, if $r \rightarrow +\infty$, then applying the Lebesgue convergence theorem in $L^1(X, \mathcal{B}, \mu)$, we obtain:

$$\int_X (|u_1 - u_2| - |\omega_{1,\alpha} - \omega_{2,\alpha}|) \geq 0$$

Next, $\partial j(r) = \text{sign}(r)$, then for every $\varphi \in \text{sign}(u_1 - u_2)$, $\mu.a.e.$ on X we have:

$$|u_1 - u_2| - |\omega_{1,\alpha} - \omega_{2,\alpha}| \leq [(u_1 - \omega_{1,\alpha}) - (u_2 - \omega_{2,\alpha})] \varphi = \alpha (y_{1,\alpha} - y_{2,\alpha}) \varphi, \text{ on } \mathbb{R}^N, \text{ then}$$

$$\int_X (y_{1,\alpha} - y_{2,\alpha}) \varphi \geq \int_X (|u_1 - u_2| - |\omega_{1,\alpha} - \omega_{2,\alpha}|) \geq 0$$

Since, $Y_{i,\alpha} \rightarrow A_1 u_i$ in $L^1(X, \mathcal{B}, \mu)$, if $\alpha \rightarrow 0$, then (2.3) is proved.

Applications

We consider the problem

$$\begin{cases} -\text{div}[a(\cdot, Du)] = f \text{ in } \Omega \\ a(\cdot, Du)v = g \text{ on } \partial\Omega \end{cases} \tag{3.1}$$

Where $f, g \in L^1$. Let $T_k(r) = \max\{-k, \min\{r, k\}\}$, $k > 0$ and $r \in \Omega$. $\mathcal{M}(\Omega) = \{u: \Omega \rightarrow \mathbb{R}, u \text{ is measurable}\}$.

$\mathcal{L}_0(\Omega) = \{u \in \mathcal{M}(\Omega), \text{ such that } \text{meas}\{|u| > k\} < +\infty, \text{ for every } k > 0\}$.

Definition 3.1. u is an entropy solution for the problem (3.1), if $u \in \mathcal{L}_0(\Omega)$, $DT_k(u) \in L^p(\Omega)$, $\forall k > 0$ and $\forall \varphi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$,



$$\int_{\Omega} a(x, Du)DT_k(u - \varphi) \leq \int_{\Omega} fT_k(u - \varphi) + \int_{\partial\Omega} gT_k(\tau u - \tau\varphi).$$

Definition 3.2. u is a renormalized solution for the problem (3.1), if $u \in L^0(\Omega)$, $DT_k(u) \in L^p(\Omega)$, $\forall k > 0$,

$$\lim_{h \rightarrow +\infty} \int_{h \leq |u| \leq h+h} |Du|^p = 0 \text{ and } \forall \varphi \in W^{1,p}(\Omega) \cap L^\infty(\Omega), \int_{\Omega} S(u)a(x, Du)D\varphi + \int_{\Omega} S'(u)\varphi a(x, Du)Du = \int_{\Omega} f\varphi S(u) + \int_{\partial\Omega} g\tau\varphi S(\tau u)$$

For all regular function S such that has a compact support.

Lemma3.1. *A renormalized solution in L^1 is an entropy solution.*

Proof. If u is a renormalized solution of (3.1), $\forall \psi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$

$$\int_{\Omega} S(u)a(x, Du)\nabla\psi + \int_{\Omega} S'(u)\psi a(x, Du)Du = \int_{\Omega} f\psi S(u) + \int_{\partial\Omega} g\tau\psi S(\tau u).$$

Let $\psi = T_k(u - \varphi)$, $\varphi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$, and $S = S_n$ with S_n regular, $0 \leq S_n \leq 1$, $S(x) = 0$ if $|x| \geq n + 1$, $S(x) = 1$ if $|x| \leq n$ and radial piecewise linear. Then

$$\int_{\Omega} S_n(u)a(x, Du)DT_k(u - \varphi) + \int_{\Omega} S'_n(u)T_k(u - \varphi)a(x, Du)Du \leq \int_{\Omega} fT_k(u - \varphi)S_n(u) + \int_{\partial\Omega} gT_k(\tau u - \tau\varphi)S_n(\tau u).$$

If $n \rightarrow \infty$, $S_n \rightarrow 1$, then $|fS_n T_k(u - \varphi)| \leq |fT_k(u - \varphi)|$ and $|gS_n T_k(\tau u - \tau\varphi)| \leq |gT_k(\tau u - \tau\varphi)|$.

By dominated convergence

$$\int_{\Omega} fS_n T_k(u - \varphi) \rightarrow \int_{\Omega} fT_k(u - \varphi) \text{ and } \int_{\partial\Omega} gS_n T_k(\tau u - \tau\varphi) \rightarrow \int_{\partial\Omega} gT_k(\tau u - \tau\varphi).$$

Since $DT_k(u - \varphi) \in L^p(\Omega)$ and $a(x, Du) \in L^p(\Omega)$ then

$$\int_{\Omega} S_n(u)a(x, Du)DT_k(u - \varphi) \rightarrow \int_{\Omega} a(x, Du)DT_k(u - \varphi) \text{ and } \left| \int_{\Omega} S'_n(u)a(x, Du)T_k(u - \varphi)Du \right| = \left| \int_{n \leq |u| \leq n+1} a(x, Du)T_k(u - \varphi)Du \right|.$$

By (H4) $\left| \int_{n \leq |u| \leq n+1} a(x, Du)T_k(u - \varphi)Du \right| \leq c \|T_k(u - \varphi)\|_{\infty} \int_{n \leq |u| \leq n+1} |Du|^p$ for $k=1$ by definition of renormalized solution:

$$\lim_{h \rightarrow \infty} \int_{h \leq |u| \leq h+1} |Du|^p = 0 \text{ then}$$

$$\int_{\Omega} S'_n(u)a(x, Du)T_k(u - \varphi)Du \rightarrow 0 \text{ finally}$$

$$\int_{\Omega} a(x, Du)DT_k(u - \varphi) \leq \int_{\Omega} fT_k(u - \varphi) + \int_{\partial\Omega} gT_k(\tau u - \tau\varphi).$$

Then the renormalized solution is an entropy solution.

Lemma3.2. *An entropy solution is a renormalized solution.*

Proof. From the uniqueness of entropy and renormalized solutions and by Lemma 3.1 we can conclude that an entropy solution is a renormalized solution.

Theorem 3.1. If $u, v \in \mathcal{M}$ are two entropy solutions to (3.1) then $u=v$.

Proof. If f and $g \in L^1$, $u \in \mathcal{M}$ is an entropy solution to (3.1) and $v \in \mathcal{M}$ is a renormalized solutions to (3.1) (entropy solution).

There exists $v_n \in L^1$ is a renormalized solution to (3.1) with $v_n \rightarrow v$ in \mathcal{M} and $(f_n, g_n) \rightarrow (f, g) \in L^1$. Consider then, for a fixed k ,

$$S_1(h) = \{|u - v_n| < k\} \cap [\{|u| < h\} \cup \{|v_n| < h\}]$$

$$S_2(h) = \{|u - v_n| < k\} \cap [\{|u| \geq h\} \cup \{|v_n| < h\}]$$

$$S'_2(h) = \{|u - v_n| < k\} \cap [\{|v_n| \geq h\} \cup \{|u| < h\}],$$

We select $\varphi = T_h v_n$ in the equation related to u . Then, taking into account that

$$\int_{S'_2} \langle a(\cdot, Du), Du \rangle \geq 0, \text{ and } \int_{S_2} \langle a(\cdot, Du), -Dv_n \rangle \leq \int_{S_2} \langle a(\cdot, Du), Du - Dv_n \rangle, \text{ we have}$$



$$-\int_{S_2} \langle a(\cdot, Du), -Dv_n \rangle + \int_{S_1} \langle a(\cdot, Du), Du - Dv_n \rangle \leq \int_{\Omega} fT_k(u - T_h v_n) + \int_{\partial\Omega} gT_k(\tau u - T_h \tau v_n).$$

On the other hand by (H4),

$$\left| \int_{S_2} \langle a(\cdot, Du), -Dv_n \rangle \right| \leq C \|Dv_n\|_{L^p(\{h-k \leq |v| < h\})} \times (\|h_0\|_{L^{p'}(\{h \leq |u| < h+k\})} + \|Du\|_{L^{p'}(\{h \leq |u| < h+k\})})$$

$$\text{or } \lim_{h \rightarrow +\infty} \int_{h \leq |v_n| < h+k} |Dv_n|^p = 0 \text{ then } \lim_{h \rightarrow +\infty} \int_{S_2} \langle a(\cdot, Du), -Dv_n \rangle = 0$$

Next, we do the same for the equation related to v_n , with test function $\varphi = T_h u$ and add the two inequalities.

$$\int_{\Omega} \lim_{h \rightarrow +\infty} \langle a(\cdot, Du) - a(\cdot, Dv_n), Du - Dv_n \rangle \mathbf{1}_{S_1(h)} + \lim_{h \rightarrow +\infty} \int_{S_2(h)} \langle a(\cdot, Du), -Dv_n \rangle + \lim_{h \rightarrow +\infty} \int_{S_2(h)} \langle a(\cdot, Dv_n), -Du \rangle$$

$$\leq \lim_{h \rightarrow +\infty} \int_{\Omega} f(T_k(u - T_h v_n)) + f_n(T_k(v_n - T_h u)) + \lim_{h \rightarrow +\infty} \int_{\partial\Omega} g(T_k(\tau u - \tau T_h v)) + g_n(T_k(\tau v_n - \tau T_h u))$$

Then, by applying the Lebesgue dominated convergence on the right, and letting $n \rightarrow \infty$, we obtain

$$\int_{\{|u-v|<k\}} \langle a(\cdot, Du) - a(\cdot, Dv), Du - Dv \rangle = 0, k > 0.$$

It arises from H3 (Leary-Lions), that $Du = Dv$, a.e. in Ω (if $n \rightarrow \infty$) and therefore $u - v = 0$, a.e. in Ω . This leads to $\tau T_k u = \tau T_k v$ a.e. on $\partial\Omega$, for any $k > 0$. Thus $\tau u = \tau v$ a.e. on $\partial\Omega$.

Theorem 3.2. If β, γ are nondecreasing continuous function on \mathbb{R} such that $\beta(0) = \gamma(0) = 0$, $(f, g) \in L^1$, then, there exists a unique entropy solution u in \mathcal{M} to the problem:

$$\begin{cases} -\text{div}[a(\cdot, Du)] + \beta(u) = f \text{ in } \Omega \\ a(\cdot, Du)v + \gamma(\tau u) = g \text{ on } \partial\Omega \end{cases} \tag{3.2}$$

Proof. If u and v are two entropy solutions to (3.2) in \mathcal{M} with the same data $(f, g) \in L^1$, then applying (2.2), since $\beta(u) = \beta(v)$ a.e. on $\{u=v\}$ and $\gamma(\tau u) = \gamma(\tau v)$ a.e. on $\{u=v\}$, one obtain:

$$-\int_{\Omega} |\beta(u) - \beta(v)| - \int_{\partial\Omega} |\gamma(\tau u) - \gamma(\tau v)| \geq 0 \text{ thus } \beta(u) = \beta(v) \text{ a.e. on } \Omega \text{ and } \gamma(\tau u) = \gamma(\tau v) \text{ on } \partial\Omega.$$

If $\beta(u) = h$ and $\gamma(\tau u) = k$, then u and v are two entropy solutions in \mathcal{M} to the problem, $A(u, \tau u) = (f-h, g-k)$. Then, the uniqueness of the entropy solution u to (3.2) derives from Theorem 3.1.

Corollary 3.1. If λ and $\tilde{\lambda}$ are bounded measure on Ω and $\partial\Omega$, then $-\text{div}[a(\cdot, Du)] + \beta(u) \ni \lambda$ on Ω and $a(\cdot, Du)v + \gamma(\tau u) \ni \tilde{\lambda}$ on $\partial\Omega$ has at least a weak solution.

Proof. Set $AU = (-\text{div}[a(\cdot, Du)], a(\cdot, Du)v)$, if $U_n, F_n \in L^1(\Omega \cup \partial\Omega)$ is some approximative sequence of solutions to the equation $AU_n + H_n \ni F_n, H_n \in BU_n$ previous arguments is that (H_n) is a Cauchy sequence in L^1 , then the classical methods are applied.

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