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On Totally (p,k) Quasiposinormal Operator

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ABSTRACT

In this paper we study some properties of totally (p,k) - quasiposinormal operator. And also we show that Weyl's theorem and algebraically Weyl's theorem holds for totally (p,k) -quasiposinormal operator.

Keywords:

(p,k) - quasiposinormal operator; totally (p,k) quasiposinormal; algebraically totally (p,k) –quasiposinormal.

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1. INTRODUCTION AND PRELIMINARIES

Let $B(H)$ denote the algebra of all bounded linear operators acting on an infinite dimensional separable Hilbert space H. Recall that an operator $T \in B(H)$ is positive, $T \geq 0$, if $(Tx, x) \geq 0$ for all $x \in H$, and posinormal if there exists a positive $\lambda \in B(H)$ such that $TT^* = T^* \lambda T$. Here λ is called interrupter of T. In other words, an operator T is called posinormal if $TT^* \leq c^2 T^* T$, where T^* is the adjoint of T and $c \succ 0$ [4]. An operator T is said to be p - posinormal if $(TT^*)^p \leq c^2 (T^* T)^p$ for some $c \succ 0$. It is clear that 1 - posinormal is posinormal. The conditionally totally posinormal was introduced by Bhagawati prashad and Carlos Kubrusly [1]. Salah Mecheri [9] and D.Senthilkumar [14] et al studied Generalized Weyl's theorem and Weyls theorem for posinormal and p - posinormal operators. Mi young Lee and Sang Hun Lee was introduced "on (p,k) - quasiposinormal operator [13]. In this paper we want to focus that Weyl's theorem holds for totally (p,k) - quasiposinormal operators.

1.1 Definition : An operator T is said to be (p,k) - quasiposinormal if

$$T^{*k}(c^2 (T^*T)^p - (TT^*)^p)T^k \geq 0,$$

where k is a positive integer, $0 < p \leq 1$ and $c \succ 0$.

1.2 Definition: An operator T is called totally (p,k) - quasiposinormal, if the translate $T-\lambda$ is (p,k) - quasiposinormal operator for all $\lambda \in C$.

Let $B(H)$ and $K(H)$ denote, respectively, the algebra of bounded linear operators and the ideal of compact operators acting on an infinite dimensional separable Hilbert space H. If $T \in B(H)$ we shall write N(A) and R(T) for the null space and the range of T, respectively. Also, let $\alpha(T) := \dim N(T)$, $\beta(T) := \dim N(T^*)$ and let $\sigma(T)$, $\sigma_a(T)$, and $\prod_0(T)$, denote the spectrum, approximate point spectrum and the set of all Riesz points of T, respectively. An operator $T \in B(H)$ is called Fredholm if it has a closed range, a finite dimensional null space, and its range has finite co - dimension. The index of a Fredholm operator is given by

$$I(A) := \alpha(T) - \beta(T),$$

T is called Weyl if it is of index zero, and Browder if it is Fredholm of finite ascent and decent, equivalently [8, Theorem 7.9.3] if T is Fredholm and $T-\lambda$ is invertible for sufficiently small $|\lambda| > 0$, $\lambda \in C$. The essential spectrum $\sigma_e(T)$, the Weyl spectrum $\sigma_w(T)$ and the Browder spectrum $\sigma_b(T)$ of T are defined by [7, 8]

$$\sigma_e(T) = \{\lambda \in C : T - \lambda \text{ is not Fredholm}\}$$

$$\sigma_w(T) = \{\lambda \in C : T - \lambda \text{ is not Weyl}\}$$

$$\sigma_b(T) = \{\lambda \in C : T - \lambda \text{ is not Browder}\}$$

respectively. Evidently

$$\sigma_e(T) \subseteq \sigma_w(T) \subseteq \sigma_b(T) = \sigma_e(T) \cup acc\, \sigma(T),$$

where we write **acc K** for the accumulation points of $K \subseteq C$. If we write $iso K = K \setminus acc K$, then we let

$$\prod_{00}(T) := \{\lambda \in iso\, \sigma(T) : 0 < \alpha(T-\lambda) < \infty\}, \quad p_{00}(T) := \sigma(T) \setminus \sigma_b(T).$$

We say that Weyl's theorem holds for T if

$$\sigma(T) \setminus \sigma_W(T) = \prod_{00}(T).$$

More generally, Berkani in [2] says that the generalized Weyl's theorem holds for T provided

$$\sigma(T) \setminus \sigma_{BW}(T) = E(T),$$

where $E(T)$ and $\sigma_{Bw}(T)$ denote the isolated point spectrum which are eigenvalues (with no restriction on multiplicity) and the set of complex numbers λ for which $T-\lambda I$ fails to be Weyl, respectively. Let X be a Banach space. An operator $T \in B(X)$ is called B-Fredholm by Berkani [2] if there exists $n \in N$ for which the induced operator

$$T_n : T^n(X) \to T^n(X)$$

is Fredholm in the usual sense, and B-Weyl if in addition T_n has index zero. Note that if the generalized Weyl's theorem holds for T, then so does Weyl's theorem [2].

2. MAIN RESULTS

In this section we study some properties of totally (p,k) - quasiposinormal operator. The following lemma summarizes the basic properties of such operators.

2.1 Lemma: If T is totally (p,k) - quasiposinormal operator, then $\ker T \subset \ker T^*$, $\ker T \subset \ker T^2$, $r(T) = \|T\|$ and $T|_M$ is a totally (p,k) quasiposinormal operator, where $r(T)$ denotes the spectral radius of T and M is any invariant subspace for T.

2.2 Lemma: Every totally (p,k) - quasiposinormal operator has the single valued extension property.

Proof: It is easy to prove that, by Lemma 2.1, $T-\lambda$ has finite ascent for each λ. Hence T has the single valued extension property by [10].

Recall that an operator $X \in L(H,K)$ is called a quasiaffinity if it has trivial kernal and dense range. An operator $S \in L(H)$ is said to be quasiaffine transform of an operator $T \in L(K)$ if there is a quasiaffinity $X \in L(H,K)$ such that $XS = TX$

2.3 Corollary: Let T be any totally (p,k) - quasiposinormal operator. If S is any quasiaffine transform of T, then S has the single valued extension property.

Proof: Since $\ker(S-\lambda) \subset \ker(S-\lambda)^2$, it suffices to show that $\ker(S-\lambda)^2 \subset \ker(S-\lambda)$. If $x \in \ker(S-\lambda)^2$, then $(S-\lambda)^2 x = 0.$ Let X be a quasiaffinity such that $XS = TX$. Then $X(S-\lambda)^2 x = 0.$ Hence $(T-\lambda)^2 Xx = 0.$ Thus $Xx \in \ker(T-\lambda)^2.$ Since $\ker(T-\lambda) = \ker(T-\lambda)^2$ by the proof of Lemma 2.2, $Xx \in \ker(T-\lambda)$ Therefore, $X(S-\lambda)x = (T-\lambda)Xx = 0.$ Since X is one-to-one, $(S-\lambda)x = 0.$ Thus $x \in \ker(S-\lambda).$

If T has the single valued extension property, then for any $x \in H$ there exists a unique maximal open set $\rho T(x)(\supset \rho(T))$ and a unique H - valued analytic function f defined in $\rho T(x)$ such that $(T-\lambda) f(\lambda) = x$, $\lambda \in \rho T(x)$. Moreover, if F is a closed set in C and $\sigma_T(x) = C\, \rho T(x)$, then

$$H_T(F) = \{x \in H : \sigma_T(x) \subset F\}$$

is a linear subspace of H [3].

2.4 Corollary: If T is totally (p,k) - quasiposinormal operator, then

$$H_T(F) = \{x \in H : \lim_{n\to\infty} \|(T-\lambda)^n x\|^{\frac{1}{n}} = 0\}$$

Proof: Since T has the single valued extension property by Lemma 2.2, the proof follows from [10].

2.5 Lemma: If T is totally (p,k) - quasiposinormal operator, then it is isoloid.

Proof: Since T has the translation invariance property, it suffices to show that if $0 \in iso\,\sigma(T)$, then $0 \in \sigma_p(T)$. Choose $\rho > 0$ sufficiently small that 0 is the only point of $\sigma(T)$ contained in or on the circle $|\lambda| = \rho$. Define

$$E = \int_{|\lambda|=\rho} (\lambda I - T)^{-1} d\lambda$$

Then E is the Riesz idempotent corresponding to 0. So $M = E(H)$ is an invariant subspace for T, $M \neq \{0\}$, and $\sigma(T|_M) = 0$. Since $(T|_M)$ is also totally (p,k) - quasiposinormal operator, $T|_M = 0$. Therefore, T is not one-to-one. Thus $0 \in \sigma_p(T)$.

2.6 Theorem: Weyl's theorem holds for any totally (p,k) - quasiposinormal operator.

Proof: If T is totally (p,k) - quasiposinormal operator, then it has the single valued extension property from Lemma 2.2. By [5, Theorem 2], it suffices to show that $H_T(\lambda)$ is finite dimensional for $\lambda \in \pi_{00}(T)$. If $\lambda \in \pi_{00}(T)$, then $\lambda \in iso\,\sigma(T)$ and $0 < \dim\ker(T-\lambda) < \infty$. Since $\ker(T-\lambda)$ is a reducing subspace for $T-\lambda$, write $T-\lambda = 0 \oplus (T_1 - \lambda)$, where 0 denotes the zero operator on $\ker(T-\lambda)$, and $T_1 - \lambda = (T_1 - \lambda)|_{(\ker(T-\lambda))^\perp}$ is injective. Therefore,

$$\sigma(T-\lambda) = \{0\} \cup \sigma(T_1 - \lambda)$$

If $T_1 - \lambda$ is not invertible, $0 \in \sigma(T_1 - \lambda)$. Since $\sigma(T-\lambda) = \{0\} \cup \sigma(T_1 - \lambda)$, $\sigma(T-\lambda) = \sigma(T_1 - \lambda)$. Since $\lambda \in \pi_{00}(T)$, $\lambda \in iso\,\sigma(T_1)$. Since T is totally (p,k) - quasiposinormal, it is easy to show that T_1 is totally (p,k) quasiposinormal operator. Since T_1 is isoloid by Lemma 2.5, $\lambda \in \sigma_p(T_1)$. Therefore, $\ker(T_1 - \lambda) \neq \{0\}$ So we have a contradiction. Thus $T_1 - \lambda$ is invertible. Therefore, $(T-\lambda)((\ker(T-\lambda))^\perp) = (\ker(T-\lambda))^\perp$. Thus $(\ker(T-\lambda))^\perp) \subset ran(T-\lambda)$ Since $\ker(T-\lambda) \subset \ker(T-\lambda)^* = (ran(T-\lambda))^\perp$. Therefore, $ran(T-\lambda) = (\ker(T-\lambda))^\perp$. Thus $ran(T-\lambda)$ is closed. Since $\dim\ker(T-\lambda) < \infty$, $T-\lambda$ is semi-Fredholm. By [11, Lemma 1], $H_T(\{\lambda\})$ is finite dimensional.

2.7 Definition: An operator $T \in B(H)$ is called algebraically totally (p,k) - quasiposinormal operator if there exists a nonconstant complex polynomial p such that $p(T)$ is totally (p,k) - quasiposinormal operator.

The following facts follow from the above definition and the well known facts of totally (p,k) - quasiposinormal operator.

If $T \in B(H)$ is algebraically totally (p,k) - quasiposinormal operator and $M \subseteq H$ is invariant under T, then $T|_M$ is algebraically totally (p,k) - quasiposinormal operator.

2.8 Lemma: If $T \in B(H)$ is algebraically totally (p,k) quasiposinormal operator and quasinilpotent, then T is nilpotent.

Proof: Suppose $p(T)$ is totally (p,k) quasiposinormal operator for some nonconstant polynomial p. Since totally (p,k) quasiposinormal is translation-invariant, we may assume $p(0) = 0$ .Thus we can write $p(\lambda) \equiv a_0 \lambda^m (\lambda - \lambda_1) \cdots (\lambda - \lambda_n)(m \neq 0, \lambda_i \neq 0)$ for every $1 \leq i \leq n$. if T is quasinilpotent, then $\sigma(p(T)) = p(\sigma(T)) = p(\{0\}) = 0$, so that $p(T)$ is also quasinilpotent. Since the only (p,k) - quasiposinormal quasinilpotent operator is zero, it follows that

$$a_0 T^m (T - \lambda_1 I) \cdots (T - \lambda_n I) = 0$$

since $T - \lambda_i I$ is invertible for every $1 \leq i \leq n$., we have $T^m = 0$.

2.9 Lemma: If $T \in B(H)$ is algebraically totally (p,k) - quasiposinormal operator, then T is isoloid.

Proof: Suppose $p(T)$ is totally (p,k) - quasiposinormal for some nonconstant polynomial p. Let $\lambda \in \sigma(T)$. Then using the spectral decomposition, we can represent T as the direct sum $T = T_1 \oplus T_2$ where $\sigma(T_1) = \lambda$ and $\sigma(T_2) = \sigma(T) \setminus \lambda$. Note that $T_1 - \lambda I$ is also algebraically totally (p,k) quasiposinormal. Since $T_1 - \lambda I$ is quasinilpotent, by Lemma 2.8, $T_1 - \lambda I$ is nilpotent. Therefore $\lambda \in \pi_0(T)$. This shows that T is isoloid.

2.10 Theorem: Let T be an algebraically totally (p,k) - quasiposinormal operator. Then T is polaroid.

Proof: Let T be an algebraically totally (p,k) - quasiposinormal operator. Then $p(T)$ is totally (p,k) - quasiposinormal for some non constant polynomial p. Let $\lambda \in iso\,\sigma(T)$. Using the spectral projection $P = \frac{1}{2\pi i}\int_{\partial D}(\mu - T)^{-1}d\mu$, where D is a closed disc centered at μ which contains no other point of $\sigma(T)$. We can represent T as the direct sum, $T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}$ where $\sigma(T_1) = \{\lambda\}$ and $\sigma(T_2) = \sigma(T)\setminus\{\lambda\}$. Since T_1 is algebraically (p,k) - quasiposinormal operator and $\sigma(T_1) = \lambda$. But $\sigma(T_1) - \lambda I = 0$ it follows from Lemma 2.8, that $T_1 - \lambda I$ is nilpotent. Therefore $T_1 - \lambda I$ has finite ascent and descent. On the other hand, since $T_1 - \lambda I$ is invertible, clearly it has finite ascent and descent. Therefore $T - \lambda I$ has finite ascent and descent. Therefore λ is a pole of the resolvent of T. Thus if $\lambda \in iso(\sigma(T))$ implies $\lambda \in \pi(T)$, and so $iso(\sigma(T)) \subset \pi(T)$. Hence T is Polaroid.

2.11 Theorem: Let $T^* \in B(H)$ be an algebraically totally (p,k) - quasiposinormal operator. Then T is a - isoloid.

Proof: Suppose T^* is algebraically totally (p,k) - quasiposinormal operator. Since T^* has SVEP, then $\sigma(T) = \sigma_a(T)$. Let $\lambda \in \sigma_a(T) = \sigma(T)$. But T^* is polaroid, hence T is also polaroid. Therefore it is isoloid, and hence $\lambda \in \sigma_p(T)$. Thus T is a - isoloid.

2.12 Theorem: Let T be an algebraically totally (p,k) - quasiposinormal operator. Then T has SVEP.

Proof: First we show that if T is totally (p,k) - quasiposinormal operator, then T has SVEP. Suppose that T is totally (p,k) - quasiposinormal operator. If $\prod_0(T) = \phi$, then clearly T has SVEP. Suppose that $\prod_0(T) \neq \phi$. Let $\Delta(T) = \lambda \in \prod_0(T) : N(T - \lambda) \subseteq N(T^* - \bar{\lambda})$. Since T is totally (p,k) - quasiposinormal operator and $\prod_0(T) \neq \phi$, $\Delta(T) \neq \phi$. Let M be the closed linear span of the subspaces $N(T-\lambda)$ with $\lambda \in \Delta(T)$. Then M reduces T and we can write T as $T_1 \oplus T_2$ on $H = M \oplus M^{\perp}$. Clearly $\lambda \in \prod_{00}(T)$, is normal and $\prod_0(T_2) = \phi$,. Since T_1 and T_2 have both SVEP, T has SVEP. Suppose that T is algebraically totally (p,k) - quasiposinormal operator. Then $p(T)$ is totally (p,k) - quasiposinormal operator for some non constant polynomial p. Since $p(T)$ has SVEP, it follows from [12, Theorem 3.3.9] that T has SVEP.

2.13 Theorem: Weyl's theorem holds for algebraically totally (p,k) - quasiposinormal operator.

Proof: Suppose that $\lambda \in \sigma(T)\setminus w(T)$. Then $T - \lambda$ is Weyl and not invertible, we claim that $\lambda \in \partial\sigma(T)$. Assume that λ is an interior point of $\sigma(T)$. Then there exist a neighbourhood U of λ, such that $\dim N(T-\lambda) > 0$ for all $\lambda \in u$. It follows from [6, Theorem 10] that T doesnot have single valued extension property [SVEP]. On the other hand, since $p(T)$ is (p,k) - quasiposinormal operator for some non constant polynomial p, it follows from Lemma 2.12. That T has SVEP. It is a contradiction. Therefore $\lambda \in \partial\sigma(T)$.

Conversely suppose that $\lambda \in \prod_{00}(T)$,. Using the Riesz idempotent $E_\lambda = \frac{1}{2\pi i}\int_{\partial D\lambda}(\mu - T)^{-1}d\mu$ where D is the closed disk centered at λ which contains no other point of $\sigma(T)$.

We can represent T as the direct sum $T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}$ where $\sigma(T_1) = \{\lambda\}$ and $\sigma(T_2) = \sigma(T)\setminus\{\lambda\}$. Now we consider two cases

Case (i) $\lambda = 0$:

Here T_1 is algebraically (p,k) - quasiposinormal operator and quasinilpotent. It follows from Lemma 2.8, that T_1 is nilpotent. We claim that $\dim R(E) < \infty$. For if $N(T_1)$ is infinite dimensional, then $0 \notin \prod_{00}(T)$,. It is contradiction. Therefore T_1 is a finite dimensional operator. So it follows that T_1 is Weyl. But since T_2 is invertible, we can conclude that T is Weyl. Therefore $0 \in \sigma(T)\setminus w(T)$.

Case (ii) $\lambda \neq 0$.

Then by Lemma 2.9, that $T_1 - \lambda$ is nilpotent. Since $\lambda \in \prod_{00}(T)$, $T_1 - \lambda$ is a finite dimensional operator. So $T_1 - \lambda$ is Weyl. Since $T_2 - \lambda$ is invertible, $T - \lambda$ is Weyl.

By case (i) and case (ii) , Weyl's theorem holds for T .

This completes the proof.

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