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Argument estimates of certain classes of P-Valent meromorphic functions involving certain operator

A. O. Mostafa[1],M. K. Aouf [2] and S. M. Madian[3]

[1,2]Department of Mathematics, Faculty of Science,
Mansoura University, Mansoura 35516, Egypt.
3 Basic Sciences Department, Higher Institute of
Engineering and Technology, New Damietta , Egypt
[1]adelaeg254@yahoo.com, [2]mkaouf127@yahoo.com
[3]samar_math@yahoo.com

ABSTRACT

In this paper, by making use of subordination , we investigate some inclusion relations and argument properties of certain classes of p-valent meromorphic functions involving certain operator.

Indexing terms/Keywords

Argument estimates, Hadamard product, certain operator, meromorphic functions.

SUBJECT CLASSIFICATION

2010 Mathematics Subject Classification: 30C45.

1. Introduction

For any integer $m > -p$, let $\sum_{p,m}$ denote the class of meromorphic functions $f(z)$ of the form:

$$f(z) = z^{-p} + \sum_{k=m}^{\infty} a_k z^k \ (m > -p;\ p \in \mathbb{N} = \{1, 2, \cdots\}), \tag{1.1}$$

which are analytic and p -valent in the punctured open unit disc $\mathbb{U}^* = \{z \in \mathbb{C} : 0 < |z| < 1\} = \mathbb{U} \setminus \{0\}$. For convenience, we write $\sum_{p,1-p} = \sum_p$ and $\sum_{1,0} = \sum$. If $f(z)$ and $g(z)$ are analytic in $\mathbb{U}$, we say that $f(z)$ is subordinate to $g(z)$, written $f \prec g$ or $f(z) \prec g(z)$ $(z \in \mathbb{U})$, if there exists a Schwarz function $w(z)$ in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ $(z \in \mathbb{U})$, such that $f(z) = g(w(z))$ $(z \in \mathbb{U})$. Furthermore, if $g(z)$ is univalent in $\mathbb{U}$, then the following equivalence relationship holds true (see [7] and [18]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \ \text{and} \ f(\mathbb{U}) \subset g(\mathbb{U}).$$

For functions $f(z) \in \sum_{p,m}$, given by (1.1) and $g(z) \in \sum_{p,m}$ defined by

$$g(z) = z^{-p} + \sum_{k=m}^{\infty} b_k z^k \ \ (m > -p; p \in \mathbb{N}),$$

the Hadamard product (or convolution) of $f(z)$ and $g(z)$ is given by

$$(f*g)(z)=z^{-p}+\sum_{k=m}^{\infty}a_k b_k z^k=(g*f)(z).$$

For complex parameters $\alpha_1,\alpha_2,\ldots,\alpha_q$ and

$\beta_1,\beta_2,\ldots,\beta_s$ $(\alpha_i,\beta_j\notin \mathrm{Z}_0^-=\{0,-1,-2,\ldots\}, i=1,2,\ldots,q,\ j=1,2,\ldots,s)$, the generalized hypergeometric function ${}_qF_s\ (\alpha_1,\alpha_2,\ldots,\alpha_q;\beta_1,\beta_2,\ldots,\ \beta_s;\ z\ )$ is defined by (see [22])

$${}_qF_s(\alpha_1,\alpha_2,\ldots,\alpha_q;\ \beta_1,\beta_2,\ldots,\beta_s;\ z)=\sum_{k=0}^{\infty}\frac{(\alpha_1)_k\ldots(\alpha_q)_k}{(\beta_1)_k\ldots(\beta_s)_k(1)_k}z^k$$

$$(q\le s+1;\ s,q\in \mathrm{N}_0=\mathrm{N}\cup\{0\};\ z\in \mathrm{U}),$$

where $(\theta)_k$, is the Pochhammer symbol defined in terms of the Gamma function Γ, by

$$(\theta)_v=\frac{\Gamma(\theta+v)}{\Gamma(\theta)}=\begin{cases}1 & (v=0;\ \theta\in \mathrm{C}^*=\mathrm{C}\setminus\{0\})\\ \theta(\theta+1)\ldots.(\theta+v-1) & (v\in \mathrm{N};\ \theta\in \mathrm{C}).\end{cases}$$

Liu and Srivastava [16] and Aouf [4] investigated recently the operator $\Upsilon_{p,q.s}(\alpha_1,\alpha_2,\ldots,\alpha_q;$ $\beta_1,\beta_2,\ldots,\beta_s):\ \sum_{p,m}\to\sum_{p,m}$, defined as follows:

$$\Upsilon_{p,q.s}(\alpha_1)=\Upsilon_{p,q.s}(\alpha_1,\alpha_2,\ldots,\alpha_q;\ \beta_1,\beta_2,\ldots,\beta_s;\ z)=z^{-p}\ {}_qF_s\ (\alpha_1,\alpha_2,\ldots,\alpha_q;\ \beta_1,\beta_2,\ldots,\beta_s;\ z),$$

$$=z^{-p}+\sum_{k=1-p}^{\infty}\frac{(\alpha_1)_{k+p}\ldots(\alpha_q)_{k+p}}{(\beta_1)_{k+p}\ldots(\beta_s)_{k+p}(1)_{k+p}}a_k z^k. \tag{1.2}$$

With aid of the function $\Upsilon_{p,q.s}(\alpha_1)$ given by (1.2), consider the function $\Upsilon^*_{p,q.s}(\alpha_1)$ defined by:

$$\Upsilon_{p,q.s}(\alpha_1)*\Upsilon^*_{p,q.s}(\alpha_1)=\frac{1}{z^p(1-z)^{\lambda+p}}(\lambda>-p;p\in \mathrm{N};z\in \mathrm{U}^*). \tag{1.3}$$

This function leads us to the following family of linear operators $\mathbf{M}^{\lambda,m}_{p,q,s}(\alpha_1):\ \sum^{\lambda}_{p,m}\to\sum^{\lambda}_{p,m}$, which are given by:

$$\mathbf{M}^{\lambda,m}_{p,q,s}(\alpha_1)=\Upsilon^*_{p,q.s}(\alpha_1)*f(z)(f\in\sum\nolimits_{p,m}). \tag{1.4}$$

The linear operator $\mathbf{M}^{\lambda,m}_{p,q,s}(\alpha_1)$ was defined by Patel and Patil [20] and Mostafa [17]. If $f(z)$ is given by (1.1), then from (1.4), we deduce that

$$\mathbf{M}^{\lambda,m}_{p,q,s}(\alpha_1)f(z)=z^{-p}+\sum_{k=m}^{\infty}\frac{(\beta_1)_{k+p}\ldots(\beta_s)_{k+p}(\lambda+p)_{k+p}}{(\alpha_1)_{k+p}\ldots(\alpha_q)_{k+p}}a_k z^k$$

$$(f \in \sum_{p,m}; \lambda, m > -p; p \in \mathbb{N}; z \in \mathbb{U}^*). \qquad (1.5)$$

It is easily verified from (1.5) that (see [20] and [17])

$$z\left(\mathbf{M}^{\lambda,m}_{p,q,s}(\alpha_1) f(z)\right)' = (\lambda + p)\mathbf{M}^{\lambda+1,m}_{p,q,s}(\alpha_1) f(z) - (\lambda + 2p)\mathbf{M}^{\lambda,m}_{p,q,s}(\alpha_1) f(z) \qquad (1.6)$$

and

$$z\left(\mathbf{M}^{\lambda,m}_{p,q,s}(\alpha_1 + 1) f(z)\right)' = \alpha_1 \mathbf{M}^{\lambda,m}_{p,q,s}(\alpha_1) f(z) - (\alpha_1 + p)\mathbf{M}^{\lambda,m}_{p,q,s}(\alpha_1 + 1) f(z). \qquad (1.7)$$

For a function $f \in \sum_p$ and $\mu > 0$, let $F_{\mu,p} : \sum_p \to \sum_p$ be the integral operator defined by (see [13]):

$$F_{\mu,p}(f)(z) = \frac{\mu}{z^{\mu+p}} \int_0^z t^{\mu+p-1} f(t)dt = z^{-p} + \sum_{k=1-p}^{\infty} \frac{\mu}{\mu + k + p} a_k z^k \quad (1.8)$$

$$(f \in \sum_p; \mu > 0; p \in \mathbb{N}; z \in \mathbb{U}^*).$$

It follows from (1.8) that:

$$z(\mathbf{M}^{\lambda,m}_{p,q,s}(\alpha_1) F_{\mu,p}(f)(z))' = \mu \mathbf{M}^{\lambda,m}_{p,q,s}(\alpha_1) f(z) - (\mu + p)\mathbf{M}^{\lambda,m}_{p,q,s}(\alpha_1) F_{\mu,p}(f)(z). \quad (1.9)$$

We note that:

Putting $\lambda = 1 - p \ (p \in \mathbb{N})$ in (1.5), then the operator $\mathbf{M}^{1-p,m}_{p,q,s}(\alpha_1)$ reduces to the operator $\mathbf{M}^{m}_{p,q,s}(\alpha_1)$, defined by:

$$\mathbf{M}^{m}_{p,q,s}(\alpha_1) f(z) = z^{-p} + \sum_{k=m}^{\infty} \frac{(\beta_1)_{k+p} \dots (\beta_s)_{k+p}}{(\alpha_1)_{k+p} \dots (\alpha_q)_{k+p}} a_k z^k \quad (f \in \sum_{p,m}; m > -p; p \in \mathbb{N}; z \in \mathbb{U}^*)$$

Also, by specializing the parameters $\lambda,\ m,\ p,\ \alpha_i\ (i = 1,2,\dots,q),\ \beta_j\ (j = 1,2,\dots,s),\ q$ and s, we have:

(i) $\mathbf{M}^{0,m}_{p,2,1}(p, p; p) f(z) = \mathbf{M}^{1}_{p,2,1}(p + 1, p; p) f(z) = f(z)(p \in \mathbb{N})$;

(ii) $\mathbf{M}^{1,m}_{p,2,1}(p, p; p) f(z) = \frac{2pf(z) + zf'(z)}{p}\ (p \in \mathbb{N})$;

(iii) $\mathbf{M}^{2,m}_{p,2,1}(p + 1, p; p) f(z) = \frac{(2p+1) f(z) + zf'(z)}{p+1}\ (p \in \mathbb{N})$;

(iv) $\mathbf{M}^{n,0}_{p,2,1}(a,1; a) f(z) = D^{n+p-1} f(z)\ (n > -p, a > 0, p \in \mathbb{N})$ (see Yang [23] and Aouf ([2] and [3])), which for $p = 1$ reduces to the operator $D^n f(z) \quad (n > -1)$ (see Cho [8]);

(v) $\mathbf{M}^{0,m}_{p,2,1}(p + 1, p; p) f(z) = \frac{p}{z^{2p}} \int_0^z t^{2p-1} f(t)dt\ (p \in \mathbb{N})$;

(vi) $\mathbf{M}^{1-p,m}_{p,2,1}(\mu + 1,1; \mu) f(z) = F_{\mu,p}(f)(z) \quad (p \in \mathbb{N}, \mu > 0)$, this integral operator is defined by (1.8);

(vii) $\mathbf{M}^{\lambda,m}_{p,2,1}(c, p + \lambda; a) f(z) = L_p(a; c) f(z)\ \left(p \in \mathbb{N}, a \in \mathbb{R}, c \in \mathbb{R} \setminus \mathbb{Z}_0^-\right)$ (see Liu [15]);

(viii) $\mathbf{M}_{p,2,1}^{\lambda,1-p}(a,1;c)f(z)=L_p^{\lambda}(a;c)f(z)\ \left(\lambda>-p,\ p\in\mathrm{N},a,c\in\mathrm{R}\setminus\mathrm{Z}_0^-\right)$ (see Aouf et al. [5]), which for $p=1$ reduces to $L^{\lambda}(a;c)$ (see Aghalary [1]);

(ix) $\mathbf{M}_{p,2,1}^{\eta-p,1-p}(n+p,\eta;\eta)f(z)=I_{n+p-1,\eta}f(z)\ (\eta>0,n>-p,\ p\in\mathrm{N})$ (see Aouf and Xu [6]), which $p=1$ reduces to $I_{n,\eta}$ (see Yuan et al. [24]);

(x) $\mathbf{M}_{1,q,s}^{\sigma-1,0}(\alpha_1)f(z)=H_{\sigma,q,s}(\alpha_1)f(z)\ (\sigma>0)$ (see Cho and Kim [9]);

(xi) $\mathbf{M}_{1,2,1}^{x-1,0}(a,1;c)f(z)=I_x(a,c)f(z)\quad (x>0,a,c\in\mathrm{R}\setminus\mathrm{Z}_0^-)$ (Cho and Noor [10]).

Let M be the class of functions $h(z)$ which are analytic and univalent in U and for which $h(\mathrm{U})$ is convex with $h(0)=1$ and Re $\{h(z)\}>0,\ z\in\mathrm{U}$.

Now, by using the linear operator $\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1)$, we define a subclass of $\sum_{p,m}$ by

$$\sum\nolimits_{p,q,s}^{\lambda,m}(\alpha_1;h)=\left\{f\ :\ f\in\sum\nolimits_{p,m} and\ -\frac{z(\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1)f(z))'}{p\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1)f(z)}\prec h(z)(h\in M;z\in\mathrm{U})\right\}\ (1.10)$$

We also set

$$\sum\nolimits_{p,q,s}^{\lambda,m}(\alpha_1;\tfrac{1+Az}{1+Bz})=\sum\nolimits_{p,q,s}^{\lambda,m}(\alpha_1;A,B)\ \ (-1<B<A\le 1;z\in\mathrm{U}).\quad (1.11)$$

From (1.10) and (1.11) and by using the result of Silverman and Silvia [21], we observe that a function $f(z)$ is in the class $\sum_{p,q,s}^{\lambda,m}(\alpha_1;A,B)$ if and only if

$$\left|\frac{z\left(\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1)f(z)\right)'}{\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1)f(z)}+\frac{p(1-AB)}{1-B^2}\right|<\frac{p(A-B)}{1-B^2}\ \ (-1<B<A\le 1;z\in\mathrm{U}).\qquad (1.12)$$

In the present paper, we investigate some inclusion relationships and argument properties of certain meromorphically p -valent functions in U^* in connection with the linear operator $\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1)$.

2. Preliminaries

In order to prove our main results, we need to the following lemmas.

Lemma 1 [12]. *Let β and υ be complex constants and let h(z) be convex (univalent) in U with $h(0)=1$ and $\mathrm{Re}\{\beta h(z)+\upsilon\}>0$. If $q(z)=1+q_1z+\ldots$ is analytic in U , then*

$$q(z)+\frac{zq'(z)}{\beta q(z)+\upsilon}\prec h(z),$$

implies

$$q(z) \prec h(z) .$$

Lemma 2 [18]. *Let* $h(z)$ *be convex (univalent) in* U *and* $\psi(z)$ *be analytic in* U *with* $\mathrm{Re}\{\psi(z)\} \geq 0.$ *If* q *is analytic in* U *and* $q(0) = h(0)$ *, then*

$$q(z) + \psi(z) z q'(z) \prec h(z) ,$$

implies

$$q(z) \prec h(z) .$$

Lemma 3 [19]. *Let* $q(z)$ *be analytic in* U *, with* $q(0) = 1$ *and* $q(z) \neq 0 \quad (z \in \mathrm{U}).$ *If there exists a point* $z_0 \in \mathrm{U},$ *such that*

$$|\arg q(z)| < \frac{\pi}{2}\tau \quad \textit{for} \quad |z| < |z_0| \tag{2.1}$$

and

$$|\arg q(z_0)| = \frac{\pi}{2}\tau \quad (0 < \tau \leq 1). \tag{2.2}$$

Then we have

$$\frac{z_0 q'(z_0)}{q(z_0)} = i x \tau, \tag{2.3}$$

where

$$x \geq \frac{1}{2}(b + \frac{1}{b}) \quad \textit{when} \quad \arg q(z_0) = \frac{\pi}{2}\tau, \tag{2.4}$$

$$x \geq -\frac{1}{2}(b + \frac{1}{b}) \quad \textit{when} \quad \arg q(z_0) = -\frac{\pi}{2}\tau \tag{2.5}$$

and

$$q(z_0)^{\frac{1}{\tau}} = \pm i b \quad (b > 0). \tag{2.6}$$

3. Some inclusion relationships

By using Lemma 1, we obtain the following results:

Theorem 1. *Let* $h(z) \in M$ *with* $\max\limits_{z\in \mathrm{U}} \operatorname{Re}\{h(z)\} < \min\{\frac{\lambda+2p}{p}, \frac{\alpha_1+p}{p}\}$ *(* ✞☢✍p, $p \in \mathrm{N})$ *. Then*

$$\sum_{p,q,s}^{\lambda+1,m}(\alpha_1;h) \subset \sum_{p,q,s}^{\lambda,m}(\alpha_1;h) \subset \sum_{p,q,s}^{\lambda,m}(\alpha_1+1;h).$$

Proof. To prove the first part, we show that $\sum_{p,q,s}^{\lambda+1,m}(\alpha_1;h) \subset \sum_{p,q,s}^{\lambda,m}(\alpha_1;h)$. Let $f \in \sum_{p,q,s}^{\lambda+1,m}(\alpha_1;h)$ and set

$$R(z) = -\frac{z\left(\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1)f(z)\right)'}{p\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1)f(z)} \quad (z \in \mathrm{U}), \tag{3.1}$$

where $R(z)$ is analytic with $R(0)=1$. Using (1.6) in (3.1), we obtain

$$pR(z) - (\lambda+2p) = -(\lambda+p)\frac{\mathbf{M}_{p,q,s}^{\lambda+1,m}(\alpha_1)f(z)}{\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1)f(z)}. \tag{3.2}$$

Differentiating (3.2) logarithmically with respect to z and multiplying by z , we have

$$R(z) + \frac{zR'(z)}{-pR(z)+\lambda+2p} = -\frac{z\left(\mathbf{M}_{p,q,s}^{\lambda+1,m}(\alpha_1)f(z)\right)'}{p\mathbf{M}_{p,q,s}^{\lambda+1,m}(\alpha_1)f(z)} \prec h(z), \tag{3.3}$$

from Lemma 1, it follows that $R(z) \prec h(z)$ in U, that is, that $f \in \sum_{p,q,s}^{\lambda,m}(\alpha_1;h)$.

To prove the second part, let $f \in \sum_{p,q,s}^{\lambda,m}(\alpha_1;h)$ and put

$$s(z) = -\frac{z\left(\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1+1)f(z)\right)'}{p\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1+1)f(z)} \quad (z \in \mathrm{U}),$$

then, by using the arguments similar to those detailed above and using (1.7) instead of (1.6), it follows that $s(z) \prec h(z)$ in U, which implies $f \in \sum_{p,q,s}^{\lambda,m}(\alpha_1+1;h)$. Therefore we compelet the proof of Theorem 1.

Taking $h(z) = \frac{1+Az}{1+Bz}$ $(-1 < B < A \le 1)$ in Theorem 1, we have

Corollary 1. *Let* $\frac{1+A}{1+B} < \min\{\frac{\lambda+2p}{p}, \frac{\alpha_1+p}{p}\}$ *and* $-1 < B < A \le 1$. *Then*

$$\sum_{p,q,s}^{\lambda+1,m}(\alpha_1;A,B) \subset \sum_{p,q,s}^{\lambda,m}(\alpha_1;A,B) \subset \sum_{p,q,s}^{\lambda,m}(\alpha_1+1;A,B).$$

Theorem 2. *Let* $h(z) \in M$ *with Re* $\{h(z)\} < \frac{\mu+p}{p}$ $(\mu > 0)$, *if* $f \in \sum_{p,q,s}^{\lambda,m}(\alpha_1;h)$, *then* $F_{\mu,p}(f) \in \sum_{p,q,s}^{\lambda,m}(\alpha_1;h)$, where $F_{\mu,p}(f)$ is defined by (1.8) .

Proof. Let $f \in \sum_{p,q,s}^{\lambda,m}(\alpha_1;h)$ and set

$$L(z) = -\frac{z\left(\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1)F_{\mu,p}(f)(z)\right)'}{p\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1)F_{\mu,p}(f)(z)} \quad (z \in \mathrm{U}), \tag{3.4}$$

where $L(z)$ is analytic with $L(0)=1$. Applying (1.9) to (3.4), we get

$$pL(z)-(\mu+p)=-\mu\frac{\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1)f(z)}{\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1)F_{\mu,p}(f)(z)}. \tag{3.5}$$

Differentiating (3.5) logarithmically with respect to z and multiplying by z, we have

$$L(z)+\frac{zL'(z)}{-pL(z)+\mu+p}=-\frac{z\left(\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1)f(z)\right)'}{p\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1)f(z)}\prec h(z).$$

Hence, by virtue of Lemma 1, we conclude that $L(z)\prec h(z)$ in U, which implies $F_{\mu,p}(f)\in\sum_{p,q,s}^{\lambda,m}(\alpha_1;h)$. This compelets the proof of Theorem 2.

Taking $h(z)=\frac{1+Az}{1+Bz}$ $(-1<B<A\le 1)$ in Theorem 2, we have

Corollary 2. *Let* $\frac{1+A}{1+B}<\frac{\mu+p}{p}\ (\mu>0)$ *and* $-1<B<A\le 1$, *if* $f\in\sum_{p,q,s}^{\lambda,m}(\alpha_1;A,B)$, *then* $F_{\mu,p}(f)\in\sum_{p,q,s}^{\lambda,m}(\alpha_1;A,B)$.

4. Some argument properties

Theorem 3. Let $f(z)\in\sum_{p,m}$, $0<\delta\le 1$, $0\le\ell<p$ and

$$\lambda\ge\frac{p(A-B)}{1+B}-p\ \ (-1<B<A\le 1;\ p\in\mathrm{N}).$$

If

$$\left|\arg\left(-\frac{z\left(\mathbf{M}_{p,q,s}^{\lambda+1,m}(\alpha_1)f(z)\right)'}{\mathbf{M}_{p,q,s}^{\lambda+1,m}(\alpha_1)g(z)}-\ell\right)\right|<\frac{\pi}{2}\delta,$$

for some $g(z)\in\sum_{p,q,s}^{\lambda+1,m}(\alpha_1;A,B)$ *, then*

$$\left|\arg\left(-\frac{z\left(\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1)f(z)\right)'}{\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1)g(z)}-\ell\right)\right|<\frac{\pi}{2}\tau,$$

where $\tau(0<\tau\le 1)$ *is the solution of the equation*

$$\delta=\tau+\frac{2}{\pi}\tan^{-1}\left(\frac{\tau\cos\frac{\pi}{2}t(A,B)}{\frac{(\lambda+p)(1-B)+p(A-B)}{1-B}+\tau\sin\frac{\pi}{2}t(A,B)}\right) \tag{4.1}$$

and

$$t(A,B)=\frac{2}{\pi}\sin^{-1}\left(\frac{p(A-B)}{(\lambda+2p)(1-B^2)-p(1-AB)}\right). \tag{4.2}$$

Proof. Let

$$q(z)=\frac{1}{p-\ell}\left(-\frac{z\left(\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1)f(z)\right)'}{\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1)g(z)}-\ell\right), \tag{4.3}$$

where $q(z)$ is analytic with $q(0)=1$. Applying the identity (1.6), we have

$$[-(p-\ell)q(z)-\ell]\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1)g(z)=(\lambda+p)\mathbf{M}_{p,q,s}^{\lambda+1,m}(\alpha_1)f(z)-(\lambda+2p)\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1)f(z). \tag{4.4}$$

Differentiating (4.4) with respect to z and multiplying by z, we obtain

$$-(p-\ell)zq'(z)\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1)g(z)+[-(p-\ell)q(z)-\ell]z(\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1)g(z))'$$

$$=(\lambda+p)z(\mathbf{M}_{p,q,s}^{\lambda+1,m}(\alpha_1)f(z))'-(\lambda+2p)z(\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1)f(z))'. \tag{4.5}$$

Then, by using (4.3), (4.4) and (4.5), we have

$$\frac{1}{p-\ell}\left(-\frac{z\left(\mathbf{M}_{p,q,s}^{\lambda+1,m}(\alpha_1)f(z)\right)'}{\mathbf{M}_{p,q,s}^{\lambda+1,m}(\alpha_1)g(z)}-\ell\right)=q(z)+\frac{zq'(z)}{-r(z)+\lambda+2p},$$

where

$$r(z)=-\frac{z\left(\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1)g(z)\right)'}{\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1)g(z)}.$$

From Corollary 1, since $g(z)\in\sum_{p,q,s}^{\lambda+1,m}(\alpha_1;A,B)$, then $g(z)\in\sum_{p,q,s}^{\lambda,m}(\alpha_1;A,B)$, which from (1.12) leads to

$$r(z)\prec p\frac{1+Az}{1+Bz}.$$

Letting

$$-r(z)+\lambda+2p=\rho e^{i\frac{\pi}{2}\phi}\quad (z\in\mathrm{U}),$$

then from (1.12) we have

$$\frac{(\lambda+p)(1+B)-p(A-B)}{1+B}<\rho<\frac{(\lambda+p)(1-B)+p(A-B)}{1-B}$$

and

$$-t(A,B)<\phi<t(A,B),$$

where t is defined by (4.2).

Let h be a function which maps U onto the angular domain $\{\omega : |\arg\omega| < \frac{\pi}{2}\delta\}$ with $h(0)=1$. Applying Lemma 2, for this h with $\psi(z)=\frac{1}{-r(z)+\lambda+2p}$, we see that $\mathrm{Re}\{q(z)\}>0$ in U and hence $q(z)\neq 0$ in U. If there exists a point $z_0 \in \mathrm{U}$ such that the conditions (2.1) and (2.2) are satisfied, then by Lemma 3, we have (2.3) under the restrictions (2.4) and (2.5).

At first, suppose that $q(z_0)^{\frac{1}{\tau}} = ib \quad (b>0)$. Then

$$\arg\left[-\frac{1}{p-\ell}\left(\frac{z_0\left(\mathbf{M}_{p,q,s}^{\lambda+1,m}(\alpha_1)f(z_0)\right)'}{\mathbf{M}_{p,q,s}^{\lambda+1,m}(\alpha_1)g(z_0)}+\ell\right)\right]$$

$$=\arg\left(q(z_0)+\frac{z_0q'(z_0)}{-r(z_0)+\lambda+2p}\right)=\frac{\pi}{2}\tau+\arg\left(1+i\ell\tau\left(\rho e^{i\frac{\pi}{2}\phi}\right)^{-1}\right)$$

$$=\frac{\pi}{2}\tau+\tan^{-1}\left(\frac{\ell\tau\sin\frac{\pi}{2}(1-\phi)}{\rho+\ell\tau\cos\frac{\pi}{2}(1-\phi)}\right)$$

$$\geq\frac{\pi}{2}\tau+\tan^{-1}\left(\frac{\tau\cos\frac{\pi}{2}t(A,B)}{\frac{(\lambda+p)(1-B)+p(A-B)}{1-B}+\tau\sin\frac{\pi}{2}t(A,B)}\right)=\frac{\pi}{2}\delta,$$

where ☠ and $t(A,B)$ are given by (4.1) and (4.2), respectively. This contradicts to the assumption of the theorem.

Next, suppose that $q(z_0)^{\frac{1}{\alpha}} = -ib \quad (b>0)$. Applying the same method as the above, we have

$$\arg\left[-\frac{1}{p-\ell}\left(\frac{z_0\left(\mathbf{M}_{p,q,s}^{\lambda+1,m}(\alpha_1)f(z_0)\right)'}{\mathbf{M}_{p,q,s}^{\lambda+1,m}(\alpha_1)g(z_0)}+\ell\right)\right]$$

$$\leq-\frac{\pi}{2}\tau-\tan^{-1}\left(\frac{\tau\cos\frac{\pi}{2}t(A,B)}{\frac{(\lambda+p)(1-B)+p(A-B)}{1-B}+\tau\sin\frac{\pi}{2}t(A,B)}\right)=-\frac{\pi}{2}\delta,$$

where ☠ and $t(A,B)$ are given by (4.1) and (4.2), respectively, which contradicts the assumption. This completes the proof of Theorem 3.

Taking $q=2, s=\lambda=\delta=A=1, B=0$ and $\alpha_1=\alpha_2=\beta_1=p\ (p\in\mathrm{N})$ in Theorem 3, we have the following corollary:

Corollary 3. *Let* $f(z)\in\sum_{p,m}$. *If*

$$-\mathrm{Re}\left\{\frac{z(2(p+1)(2p+1)f'(z)+4(1+p)zf''(z)+z^2f'''(z))}{2p(2p+1)g(z)+2(2p+1)zg'(z)+z^2g''(z)}\right\}>\ell\ (0\leq\ell<p),$$

for some $g(z)\in\sum_{p,m}$ *satisfying the condition*

$$\left|\frac{z(2(p+1)(2p+1)g'(z)+4(p+1)zg''(z)+z^2g'''(z))}{2p(2p+1)g(z)+2(2p+1)zg'(z)+z^2g''(z)}+p\right|<p,$$

then

$$-\operatorname{Re}\left\{\frac{z((2p+1)f'(z)+zf''(z))}{zf'(z)+2pf(z)}\right\}>\ell.$$

Taking $q=2, s=\delta=A=1, \lambda=2, B=0, \alpha_1=p+1$ and $\alpha_2=\beta_1=p \quad (p\in \mathbb{N})$ in Theorem 3, we have the following corollary:

Corollary 4. *Let* $f(z)\in\sum_{p,m}$. *If*

$$-\operatorname{Re}\left\{\frac{z(2((p+1))(2p+3)f'(z)+2(2p+3)zf''(z)+z^2f'''(z))}{2(p+1)(2p+1)g(z)+4(p+1)zg'(z)+z^2g''(z)}\right\}>\ell \quad (0\le\ell<p),$$

for some $g(z)\in\sum_{p,m}$ *satisfying the condition*

$$\left|\frac{z(2((p+1))(2p+3)g'(z)+2(2p+3)zg''(z)+z^2g'''(z))}{2(p+1)(2p+1)g(z)+4(p+1)zg'(z)+z^2g''(z)}+p\right|<p,$$

then

$$-\operatorname{Re}\left\{\frac{z(2(p+1)f'(z)+zf''(z))}{(2p+1)g(z)+zg'(z)}\right\}>\ell.$$

Taking $q=2, s=1, \lambda=\eta-p, \quad m=1-p, \alpha_1=n+p$ and $\alpha_2=\beta_1=\eta \quad (\eta>0, n>-p, p\in\mathbb{N})$ in Theorem 3, we have the following corollary:

Corollary 5. Let $f(z) \in\sum_p, \quad 0<\delta\le1, \quad 0\le\ell<p$ and

$$\eta\ge\frac{p(A-B)}{1+B} \quad (-1<B<A\le1;\, p\in\mathbb{N}).$$

If

$$\left|\arg\left(-\frac{z\left(I_{n+p-1,\eta+1}f(z)\right)'}{I_{n+p-1,\eta+1}g(z)}-\ell\right)\right|<\frac{\pi}{2}\delta,$$

for some $g(z)\in\sum_{p,m}$ *satisfying the conditon*

$$-\frac{z\left(I_{n+p-1,\eta}g(z)\right)'}{I_{n+p-1,\eta}g(z)}\prec p\frac{1+Az}{1+Bz}, \tag{4.6}$$

then

$$\left|\arg\left(-\frac{z\left(I_{n+p-1,\eta}f(z)\right)'}{I_{n+p-1,\eta}g(z)}-\ell\right)\right|<\frac{\pi}{2}\tau,$$

where $\tau(0<\tau\leq 1)$ *is the solution of the equation* (4.1) with $\lambda=\eta-p\,(\eta>0)$.

Taking $p=1$ in Corollary 5, we have the following corollary:

Corollary 6. Let $f(z)\in\sum$, $0<\delta\leq 1$, $0\leq\ell<1$ and

$$\eta\geq\frac{A-B}{1+B}\quad(-1<B<A\leq 1).$$

If

$$\left|\arg\left(-\frac{z\left(I_{n,\eta+1}f(z)\right)'}{I_{n,\eta+1}g(z)}-\ell\right)\right|<\frac{\pi}{2}\delta,$$

for some $g(z)\in\sum$ *satisfying the conditon*

$$-\frac{z\left(I_{n,\eta}g(z)\right)'}{I_{n,\eta}g(z)}\prec\frac{1+Az}{1+Bz},\tag{4.7}$$

then

$$\left|\arg\left(-\frac{z\left(I_{n,\eta}f(z)\right)'}{I_{n,\eta}g(z)}-\ell\right)\right|<\frac{\pi}{2}\tau,$$

where [illegible] 1 [illegible] *is the solution of the equation* (4.1) with $\lambda=\eta-1\,(\eta>0)$.

The proof of the next theorem is akin to that of Theorem 3 and so, we omit it.

Theorem 4. *Let* $f(z)\in\sum_{p,m}$, $0<\delta\leq 1$, $\ell>p$ *and*

$$\lambda\geq\frac{p(A-B)}{1+B}-p\quad(-1<B<A\leq 1;\, p\in\mathrm{N}).$$

If

$$\left|\arg\left(\frac{z\left(\mathbf{M}_{p,q,s}^{\lambda+1,m}(\alpha_1)f(z)\right)'}{\mathbf{M}_{p,q,s}^{\lambda+1,m}(\alpha_1)g(z)}+\ell\right)\right|<\frac{\pi}{2}\delta$$

for some $g(z)\in\sum_{p,q,s}^{\lambda+1,m}(\alpha_1;A,B)$ *, then*

$$\left|\arg\left(\frac{z\left(\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1)f(z)\right)'}{\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1)g(z)}+\ell\right)\right|<\frac{\pi}{2}\tau,$$

where $\tau (0<\tau \le 1)$ *is the solution of equation (4.1).*

Theorem 5. *Let* $f(z)\in \sum_{p,m}$, $0<\delta \le 1,\ 0\le \ell <p$ *and*

$$\alpha_1 \ge \frac{p(A-B)}{1+B}\quad (-1<B<A\le 1;\ p\in \mathrm{N}).$$

If

$$\left|\arg\left(-\frac{z\left(\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1)f(z)\right)'}{\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1)g(z)}-\ell\right)\right|<\frac{\pi}{2}\delta,$$

for some $g(z)\in \sum_{p,q,s}^{\lambda,m}(\alpha_1;A,B)$ *, then*

$$\left|\arg\left(-\frac{z\left(\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1+1)f(z)\right)'}{\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1+1)g(z)}-\ell\right)\right|<\frac{\pi}{2}\tau,$$

where $\tau (0<\tau \le 1)$ *is the solution of equation (4.1)* .

Proof. Let

$$X(z)=\frac{1}{p-\ell}\left(-\frac{z\left(\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1+1)f(z)\right)'}{\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1+1)g(z)}-\ell\right), \tag{4.8}$$

where $X(z)$ is analytic with $X(0)=1$. Using (1.7), we have

$$[-(p-\ell)X(z)-\ell]\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1+1)g(z)=\alpha_1\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1)f(z)-(\alpha_1+p)\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1)f(z). \tag{4.9}$$

Differentiating (4.9) with respect to z and multiplying by z, we obtain

$$\begin{aligned}&-(p-\ell)zX'(z)\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1+1)g(z)+[-(p-\ell)X(z)-\ell]z(\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1+1)g(z))'\\&=\alpha_1 z(\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1)f(z))'-(\alpha_1+p)z(\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1)f(z))'.\end{aligned} \tag{4.10}$$

Then, by using (4.8), (4.9) and (4.10), we have

$$\frac{1}{p-?}\left(-\frac{z\left(\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1)f(z)\right)'}{\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1)g(z)}-\ell\right)=X(z)+\frac{zX'(z)}{-j(z)+\alpha_1+p},$$

where

$$j(z)=-\frac{z\left(\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1+1)g(z)\right)'}{\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1+1)g(z)}.$$

The remaining part of the proof is similar to that of Theorem 3 and so we omit it.

Taking $q=2, s=1, \lambda=\eta-p, \quad m=1-p, \alpha_1=n+p$ and $\alpha_2=\beta_1=\eta \quad (\eta>0, n>-p, p\in \mathrm{N})$ in Theorem 5, we have the following corollary:

Corollary 7. *Let* $f(z)\in \sum_{p,m}, \quad 0<\delta\le 1, 0\le \ell<p$ *and*

$$n\ge \frac{p(A-B)}{1+B}-p \ (-1<B<A\le 1; p\in \mathrm{N}).$$

If

$$\left|\arg\left(-\frac{z\left(I_{n+p-1,\eta}f(z)\right)'}{I_{n+p-1,\eta}g(z)}-\ell\right)\right|<\frac{\pi}{2}\delta,$$

for some $g(z)\in \sum_{p,m}$ *satisfying (4.6), then*

$$\left|\arg\left(-\frac{z\left(I_{n+p,\eta}f(z)\right)'}{I_{n+p,\eta}g(z)}-\ell\right)\right|<\frac{\pi}{2}\tau,$$

where $\tau(0<\tau\le 1)$ *is the solution of equation (4.1)* .

Taking $p=1$ in Corollary 7, we have the following corollary:

Corollary 8. *Let* $f(z)\in \sum, \quad 0<\delta\le 1, 0\le \ell<1$ *and*

$$n\ge \frac{(A-B)}{1+B}-1 \ (-1<B<A\le 1).$$

If

$$\left|\arg\left(-\frac{z\left(I_{n,\eta}f(z)\right)'}{I_{n,\eta}g(z)}-\ell\right)\right|<\frac{\pi}{2}\delta,$$

for some $g(z)\in \sum$ *satisfying (4.7), then*

$$\left|\arg\left(-\frac{z\left(I_{n+1,\eta}f(z)\right)'}{I_{n+1,\eta}g(z)}-\ell\right)\right|<\frac{\pi}{2}\tau,$$

where $\tau(0<\tau\le 1)$ *is the solution of equation (4.1)* .

The proof of the next theorem is akin to that of Theorem 5 and so, we omit it.

Theorem 6. *Let* $f(z)\in \sum_{p,m}, \quad 0<\delta\le 1, \ \ell>p$ *and*

$$\alpha_1 \geq \frac{p(A-B)}{1+B} \quad (-1 < B < A \leq 1;\ p \in \mathrm{N}).$$

If

$$\left|\arg\left(\frac{z\left(\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1) f(z)\right)'}{\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1) g(z)} + \ell\right)\right| < \frac{\pi}{2}\delta,$$

for some $g(z) \in \sum_{p,q,s}^{\lambda,m}(\alpha_1; A, B)$ *, then*

$$\left|\arg\left(\frac{z\left(\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1+1) f(z)\right)'}{\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1+1) g(z)} + \ell\right)\right| < \frac{\pi}{2}\tau,$$

where $\tau(0 < \tau \leq 1)$ *is the solution of equation (4.1)* .

Theorem 7. *Let* $f(z) \in \sum_{p,m}$, $0 < \delta \leq 1,\ 0 \leq \ell < p$ *and*

$$\mu \geq \frac{p(A-B)}{1+B} \quad (-1 < B < A \leq 1;\ p \in \mathrm{N}).$$

If

$$\left|\arg\left(-\frac{z\left(\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1) f(z)\right)'}{\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1) g(z)} - \ell\right)\right| < \frac{\pi}{2}\delta,$$

for some $g(z) \in \sum_{p,q,s}^{\lambda,m}(\alpha_1; A, B)$ *, then*

$$\left|\arg\left(-\frac{z\left(\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1) F_{\mu,p}(f)(z)\right)'}{\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1) F_{\mu,p}(g)(z)} - \ell\right)\right| < \frac{\pi}{2}\tau,$$

where $\tau(0 < \tau \leq 1)$ *is the solution of equation (4.1)* .

Proof. Let

$$k(z) = \frac{1}{p-\ell}\left(-\frac{z\left(\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1) F_{\mu,p}(f)(z)\right)'}{\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1) F_{\mu,p}(g)(z)} - \ell\right), \tag{4.11}$$

where $k(z)$ is analytic with $k(0) = 1$. Using (1.9), we have

$$[-(p-\ell)k(z) - \ell]\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1) F_{\mu,p}(g)(z) = \mu\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1) f(z) - (\mu + p)\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1) F_{\mu,p}(f)(z). \tag{4.12}$$

Differentiating (4.12) with respect to z and multiplying by z, we obtain

$$-(p-\ell)zk'(z)\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1) F_{\mu,p}(g)(z) + [-(p-\ell)k(z) - \ell]z(\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1) F_{\mu,p}(g)(z))'$$

$$=\mu z(\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1)f(z))'-(\mu+p)z(\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1)F_{\mu,p}(f)(z))'. \qquad (4.13)$$

Then, by using (4.11), (4.12) and (4.13), we have

$$\frac{1}{p-\ell}\left(-\frac{z\left(\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1)f(z)\right)'}{\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1)g(z)}-\ell\right)=k(z)+\frac{zk'(z)}{-\rho(z)+\mu+p},$$

where

$$\rho(z)=-\frac{z\left(\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1)F_{\mu,p}(g)(z)\right)'}{\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1)F_{\mu,p}(g)(z)}.$$

The remaining part of the proof is similar to that of Theorem 3 and so we omit it.

The proof of the next theorem is akin to that of Theorem 7 and so, we omit it.

Theorem 8. *Let* $f(z)\in\sum_{p,m}$, $0<\delta\le 1,\ \ell>p$ *and*

$$\mu\ge\frac{p(A-B)}{1+B}\quad(-1<B<A\le 1;\ p\in\mathrm{N}).$$

If

$$\left|\arg\left(\frac{z\left(\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1)f(z)\right)'}{\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1)g(z)}+\ell\right)\right|<\frac{\pi}{2}\delta$$

for some $g(z)\in\sum_{p,q,s}^{\lambda,m}(\alpha_1;A,B)$, then

$$\left|\arg\left(\frac{z\left(\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1)F_{\mu,p}(f)(z)\right)'}{\mathbf{M}_{p,q,s}^{\lambda,m}(\alpha_1)F_{\mu,p}(g)(z)}+\ell\right)\right|<\frac{\pi}{2}\tau,$$

where $\tau(0<\tau\le 1)$ is the solution of equation (4.1) .

Remark 1. *Specializing the parameters* p,q,s,m *and* λ *in the above results, we obtain the results for the corresponding operators defined in the introduction.*

Remark 2. *(i) Putting* $q=2,s=1,m=1-p,\quad \alpha_1=\beta_1=a\,(a>0,p\in\mathrm{N})$ *and* $\alpha_2=1$ *in Theorems 3 and 4, respectively, we obtain the results obtained by Cho and Owa [11, Theorems 2.1 and 2.2, respectively]* ;

(ii) Putting $q=2,s=1,m=0,\quad \alpha_1=a,\alpha_2=p=1$ *and* $\beta_1=a\quad(a>0)$ *in Theorems 3 and 4, respectively, we obtain the results obtained by Cho [8, Theorems 2.1 and 2.2, respectively]* ;

(iii) Putting $q=2,s=1,m=1-p,\quad \alpha_1=a,\alpha_2=1$ *and* $\beta_1=c\quad\left(a,c\in\mathrm{R}\setminus\mathrm{Z}_0^-,p\in\mathrm{N}\right)$ *in the above results, we obtain the results obtained by Aouf et al. [5]* ;

(iv) Putting $q=2, s=1, m=1-p$ *,* $\alpha_1=c, \alpha_2=p+\lambda$ *and* $\beta_1=a$ *(* $a\in\mathbb{R}, c\in\mathbb{R}\setminus\mathbb{Z}_0^-, \lambda>-p, p\in\mathbb{N})$ *in Theorem 4, we obtain the results obtained by Lashin [14, Theorem 2.2]* .

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