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Homotopy Continuation Method for Solving a Class of Slightly Perturbed Equations

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Abstract. In the present paper, an iterative method of second order convergence for solving a class of slightly perturbed equations is established using homotopy continuation technique. The accuracy of the method is very high and verified through two applications, in the first application we fix all the input parameters of our algorithm and allow the homotopy function Q to vary. For this application we considered as a typical example, Kepler equation of elliptical motion. In the second applications, the function Q is fixed while the other input parameters of the algorithm are varied, and we used for these applications seven highly transcendental slightly perturbed equations. The most important characteristic of the method is that it does not need any prior knowledge of the initial guess, a property which avoids the critical situations between divergent to very slow convergent solutions, that may exist in the application of other numerical methods which depending on initial guess. Moreover, the method allows freedom in choosing the Q function in order to secure any required accuracy.

Keywords: Homotopy continuation method; perturbation techniques; nonlinear Equations.

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1. Introduction

Usually, equations resulting in most problems of applied mathematics are highly transcendental and could be solved by iterative methods which in turn need: (a) initial guess, (b) an iterative scheme. In fact, these two points are not separated from each other, but there is a full agreement that, even accurate iterative schemes are extremely sensitive to initial guess. Moreover, in many cases the initial guess may lead to drastic situation between divergent and very slow convergent solutions.

In the field of numerical analysis, powerful techniques have been devoted [1] to solve transcendental equations without any priori knowledge of the initial guess. These techniques are known as *homotopy continuation methods.* The method was first applied to the universal initial value problem of space dynamics [2], in stellar statistics in reference [3], and for non linear algebraic equation in many papers of these are for example reference **[**4, 5,6].

In the present paper, an iterative method of second order convergence for solving a class of slightly perturbed equations is established using homotopy continuation technique. This class is of the form

$$\eta = \xi + \alpha\Phi(\eta), \quad |\alpha| < 1. \tag{1}$$

The accuracy of the method is very high and verified through two applications, in the first application we fix all the input parameters of our algorithm and allow the homotopy function Q to vary. For this application we considered as a typical example, Kepler's equation of elliptical motion. In the second applications, the function Q is fixed while the other input parameters of the algorithm are varied, and we used for these applications seven highly transcendental slightly perturbed equations. The most important characteristic of the method is that it does not need any priori knowledge of the initial guess, a property which avoids the critical situations between divergent to very slow convergent solutions, that may exist in the application of other numerical methods which depending on initial guess. Moreover, the method allows some freedom in choosing the Q function in order to secure any required accuracy.

2. Homotopy continuation method for solving $Y(x)=0$

Suppose one wishes to obtain a solution of a single non-linear equation in one variable x (say)

$$Y(x) = 0, \tag{2}$$

where, $\mathrm{Y}: R \to R$ is a mapping which , for our application assumed to be smooth that is, a map has as many continuous derivatives as requires. Let us consider the situation in which no priori knowledge concerning the zero point of Y is available. Since we assume that such a priori knowledge is not available, then any of the iterative methods will often fail to calculate the zero $\bar{x}$, because poor starting value is likely to be chosen. As a possible remedy, one defines a homotopy or deformation $\mathrm{H}: R \times R \to R$ such that

$$\mathrm{H}(x,1) = Q(x) \qquad ; \qquad \mathrm{H}(x,0) = \mathrm{Y}(x) \ ,$$

where $Q: R \to R$ is a (trivial) smooth map having known zero point and H is also smooth . Typically, one may choose a convex

$$\mathrm{H}(x,\lambda) = \lambda Q(x) + (1-\lambda)\mathrm{Y}(x) \tag{3}$$

and attempt to trace an implicitly defined curve $\Phi(z) \in \mathrm{H}^{-1}(0)$ from a starting point $(x_1, 1)$ to a solution point $(\bar{x}, 0)$. If this succeeds, then a zero point $\bar{x}$ of Y is obtained.

2.1. Computational Algorithm

Purpose: To solve $Y(x) = 0$ by homotopy continuation method.

Input: (1) The function $Q(x)$ with defined root x_1 such that $H(x_1, 1) = 0.$

(2) Positive integer *m*.

Output: Solution x of $Y(x) = 0$.

Computational sequence:

1. Set $x = x_1, \lambda = (m-1)/m, \ \Delta\lambda = 1/m.$
2. For *i* = 1 to m do

begin

Solve $\mathrm{H}(y,\lambda)=\lambda Q(x)+(1-\lambda)\mathrm{Y}(y)$ iteratively for y using x as starting value.

$x=y.$

$\lambda=\lambda-\Delta\lambda.$

end.

3. Applications of the homotopy continuation method

We divide the applications of the homotopy continuation method for solving a class of slightly perturbed equations into two parts: in the first part we study how the selection of the function Q affects the accuracy of the solution. To do so we fix all the input parameters in our algorithm of Subsection 2.1 and change Q. This study is the subject of Subsection 3.1. In the second part of the applications, we study the accuracy of the solution when Q is fixed and the other input parameters of the algorithm are varied. This study is the subject of Subsection 3.2.

3.1. The effect of the function Q

To study the effect of the choice of the function Q on the accuracy of solution on slightly perturbed equations, we consider a typical example of Kepler's equation of elliptic motion, this equation is given as

$$M=E-e\sin E, \tag{4}$$

where

$$M=\sqrt{\frac{\mu}{a^3}}(t-\tau)=n(t-\tau). \tag{5}$$

The angle M ,called the *mean anomaly* by Kepler, E the *eccentric anomaly, e* the *eccentricity* of the orbit $(e<1)$, μ the *gravitational parameter*, *n* the *mean motion*, a the *semi-major axis* of the orbit, *t* the time, and τ is the time of *passage through pericenter* (the closet distance to the focus).

When $M=j\pi$, where *j* is any integer, the value $E=j\pi$ uniquely satisfies Kepler's equation. It is to be noted the *M* and *E* always lie in the same semi ellipse. When *M* has some given value between $j\pi$ and $(j+1)\pi$,where *j* is any integer, then [7], there is but one real value of *E* satisfying Kepler's equation and it lies between $j\pi$ and $(j+1)\pi$. If *M* does not already lie in the closed interval [0, 2π], then we have reduce it to that range and hence E, lies in [0, 2π]. The solution of Kepler's equation is to find E when M and e are given.

Now, writing Equation (3) in the form $E=M+e\sin E$, it becomes of the same form as Equation (1). As an example, let $M=30^o$ and $e=0.1$ then solve Kepler's equation by the homotopy method. There are many choices of the function $Q(E)$ of these are listed in what follows with the corresponding values of x_1

a. $Q(E)=E-M \Rightarrow x_1=M$,

b. $Q(E)=E-M-e\sin M \Rightarrow x_1=M+e\sin M$,

c. $Q(E)=E-M-e\sin M-0.5\,e^2\sin(2M) \Rightarrow x_1=M+e\sin M+0.5\,e^2\sin(2M),$

d. $Q(E)=E-M-\dfrac{e\sin M}{1-e\sin(M+e)+\sin M} \Rightarrow x_1=M+e\sin M+\dfrac{e\sin M}{1-e\sin(M+e)+\sin M}.$

Apply the above algorithm of Subsection 2.2 for each case of $Q(E)$ and x_1 with the same value of $m=10$.The solution E and the corresponding $Y(E)$ are listed for each case in Table I.

Cleary, the choice of the Q function is an important factor in the homotopy method. On other words, the method is flexible in allowing freedom in selecting the homotopy function Q in order to secure the required accuracy.

Table I: The effect of the Q function on the solution of Keler's equation

Cases	E^{ra}	$Y(E)$
Case 1 of $Q(E)$	0.5782568453295369171	$-1.5676166280135\times10^{-6}$
Case 2 of $Q(E)$	0.5782551467642060850	$-1.129206819023\times10^{-8}$
Case 3 of $Q(E)$	0.5782551345005499431	$-5.53922463\times10^{-11}$
Case 4 of $Q(E)$	0.5782553617109310595	$-2.082387774160\times10^{-7}$

3.2. Applications for solving transcended equations

In the present subsection, we give some applications of the homotopy method for solving other slightly perturbed equations.

Equation (1) is written as

$$F(\eta)=\xi-\eta+\alpha\Phi(\eta)=0\,. \tag{7}$$

Now the problem reduces to: find the root of Equation (7) by using homotopy continuation method for given functions (Φ,Q) and the numerical values ξ and $|\alpha|<1.$ Note that, for all applications listed in Table II, we used for the function Q the same expression $Q(\eta)=\eta-\xi$. In this table the first three columns are ξ,α and Φ (input), while the forth column is *m* (output) which is the number of iterations used to reach the value of the root η (output) of the fifth column, finally in the six column $F(\eta)\times10^{-7}$ (output) which is the accuracy check of the computed root η .

Table II: Homotopy continuation method for solving Equation (7) for some α,ξ , Φ and fixed $Q(\eta)=\eta-\xi$.

α	ξ	$\Phi(\eta)$	m	η	$F(\eta)\times10^{-7}$
0.1	0.3	ηe^{η}	4	0.313441	1.77804
0.01	0.3	$\cos\eta\sin\eta$	3	0.302847	-0.479795
0.04	0.3	$2/3-\eta+\eta^3/3$	4	0.314501	0.385445
0.04	0.3	$\tan\eta/\eta$	7	0.341631	12.5622
0.06	1.8	$J_2(\eta)$	7	1.81427	0.261322
0.06	0.08	$P_4(\eta)$	7	0.100268	-31.1064
0.1	0.6	$\sin^5\eta$	7	0.605995	1.75784

Where, $J_n(\eta)$ and $P_n(\eta)$ are respectively, the Bessel function of the first kind and the Legendre polynomial.

4. Conclusion

In the present paper, an iterative method of second order convergence for solving a class of slightly perturbed equations is established using homotopy continuation technique. The accuracy is very high and verified through two applications, in the first application we fix all the input parameters of our algorithm and allow the homotopy function Q to vary. For this application we considered as a typical example, Kepler equation of elliptical motion. In the second applications, the function Q is fixed while the other input parameters of the algorithm are varied, and we used for these applications seven highly transcendental slightly perturbed equations. The most important characteristic of the method is that it does not need any priori knowledge of the initial guess, a property which avoids the critical situations between divergent to very slow convergent solutions, that may exist in the application of other numerical methods which depending on initial guess. Moreover, the method allows freedom in choosing the Q function in order to secure the required accuracy

References

[1] Allgower, E.L. and George, K., 1990: *Numerical Continuation Methods*, Springer – Verlag, Berlin.

[2] Sharaf, M. A. and Sharaf, A.A.:2003, Celestial Mechanics and Dynamical Astronomy, 86,351

[3] Sharaf, M. A, 2006: Computations of the Cosmic Distance Equation, Appl.Math.Comput.174, 1269-1278.

[4] Yuen, C. F., Hoong, L.K., Seng, C. F., 2010: Solving Nonlinear Algebraic Equation by Homotopy Analysis Method, Mathematics, Proceedings of the 6th IMT-GT Conference on Mathematics, Statistics and its applications, Universiti Tunku Abdul Rahman, Kuala Lumpur, Malaysia.

[5] H.Vázquez-Leal, U. Filobello-Niño, R. Castañeda-Sheissa, L. Hernández-Martínez, and A. Sarmiento-Reyes, “Modified HPMs inspired by homotopy continuation methods,” Mathematical Problems in Engineering, vol. 2012, Article ID 309123, 19 pages, 2012.

[6] HectorVazquez-Leal, Yasir Khan, Guillermo Fernández-Anaya, Agustín Herrera-May, Arturo Sarmiento-Reyes, Uriel Filobello-Nino, Víctor-M. Jimenez-Fernández, and Domitilo Pereyra-Díaz, “A General Solution for Troesch's Problem**,**” Mathematical Problems in Engineering, vol. 2012, Article ID 208375, 14 pages, 2012.

[7] Fitzpatrick, M. P., 1970:”*Principles of Celestial Mechanics”*, Academic Press, New York and London.