ISSN 2347-1921
Volume 13 Number5
Journal of Advances in Mathematics

Common Fixed Point Results for Compatible Map in Digital Metric Spaces

Sumitra Dalal
Department of Mathematics, Jazan University, K.S.A
Email # mathsqueen_d@yahoo.com

ABSTRACT:

The aim of this paper is to define the concept of compatible maps and its variants in the setting of digital metric spaces and establish some common fixed point results for these maps. Also, an application of the proposed results is quoted in this note.

Keywords: Compatible maps, fixed point results, digital metric space, metric space.

Subject Classification: 47H10, 54H25

INTRODUCTION:

Digital topology is an emerging area based on general topology and functional analysis and focuses on studying digital topological properties of $n-$ dimensional digital spaces, where as Euclidean topology deals with topological properties of subspaces of the $n-$ dimensional real space, which has contributed to the study of some areas of computer sciences such as computer graphics, Image processing, approximation theory, mathematical morphology, optimization theory etc. Rosenfield [15] was the first to consider digital topology as a tool to study digital images. Boxer [4], then introduced the digital fundamental group of a discrete object and produced the digital versions of the topological concepts [2], also later studied digital continuous functions [3]. Ege and Karaca [7,8,9] established relative and reduced Lefschetz fixed point theorem for digital images and proposed the notion of a digital metric space and proved the famous Banach Contraction Principle for digital images.

Fixed point theory leads to lots of applications in mathematics , computer science, engineering, game theory, fuzzy theory, image processing and so forth [4, 7,14]. In metric spaces, this theory begins with the Banach fixed-point theorem which provides a constructive method of finding fixed points and an essential tool for solution of some problems in mathematics and engineering and consequently has been generalized in many ways. Up to now, several developments have occurred in this area [see 7,10,14,17,18]. A major shift in the arena of fixed point theory came in 1976 when Jungck [11], defined the concept of commutative maps and proved the common fixed point results for such maps. After which, Sessa [16] gave the concept of weakly compatible, and Jungck ([12,13])gave the concepts of compatibility and weak compatibility. Certain altercations of commutativity and compatibility can also be found in [1,6,14,19].

This paper is organized as follows. In the first part, we give the required background about the digital topology and fixed point theory. In the next section, we state and prove main results for compatible mappings and compatible mappings of types (A) and (P) in digital metric spaces. Our results improve and generalize many other results existing in literature. Finally, we give an important application of fixed point theorem to digital images. Lastly, we make some conclusions.

PRELIMINARIES

Let X be subset of Z^n for a positive integer n where Z^n is the set of lattice points in the n- dimensional Euclidean space and ρ represent an adjacency relation for the members of X. A digital

image consists of (X,ρ).

Definition 2.1 [4]: Let l,n be positive integers, $1 \le l \le n$ and two distinct points

$a=(a_1,a_2,...,a_n), b=(b_1,b_2,...,b_n) \in Z^n$, a and b are k_l- adjacent if there are at most l indices i such that $|a_i - b_i| = 1$ and for all other indices j such that $|a_j - b_j| \neq 1, a_j = b_j$.

A ρ-neighbour [4] of $a \in Z^n$ is a point of Z^n that is ρ- adjacent to a where $\rho \in \{2,4,6,8,18,26\}$ and $n \in \{1,2,3\}$. The set

$$N_\rho(a) = \{b / \ b \text{ is adjacent to } a\}$$

is called the ρ- neighbourhood of a. A digital interval [12] is defined by

$[p,q]_z = \{z \in Z / p \le z \le q\}$, where $p,q \in Z$ and $p<q$.

DOI 10.24297/jam.v13i4.6458
http://cirworld.com

A digital image $X \subset Z^n$ is ρ - connected [1] if and only if for every pair of different points $u,v \in X$, there is a set $\{u_0, u_1, ..., u_r\}$ of points of digital image X such that $u = u_0, v = u_r$ and u_i and u_i and u_{i+1} are ρ - neighbours where $i = 0,1,2,...,r-1.$

Definition 2.2: Let $(X,\rho_0) \subset Z^{n_0,}, (Y,\rho_1) \subset Z^{n_1}$ be digital images and $T: X \to Y$ be a function, then

(i) T is said to be (ρ_0,ρ_1) - continuous [5], if for all ρ_0- connected subset E of X, $f(E)$ is a ρ_1- connected subset of Y.

(ii) For all ρ_0- adjacent points $\{u_0, u_1\}$ of X, either $T(u_0) = T(u_1)$ or $T(u_0)$ and $T(u_1)$ are a ρ_1- adjacent in Y if and only if T is (ρ_0,ρ_1)-continuous .

(iii) If T is (ρ_0,ρ_1) - continuous, bijective and T^{-1} is (ρ_0,ρ_1)- continuous, then T is called (ρ_0,ρ_1) - isomorphism [6]and denoted by $X \cong_{(\rho_0,\rho_1)} Y.$

Definition 2.3 [7]: A sequence $\{x_n\}$ of points of a digital metric space (X,d,ρ) is a Cauchy sequence if for all $\varepsilon > 0$, there exists $\delta \in N$ such that for all $n,m > \delta$, then $d(x_n, x_m) < \varepsilon$.

Definition 2.4 [7]: A sequence $\{x_n\}$ of points of a digital metric space (X,d,ρ) converges to a limit $p \in X$ if for all $\varepsilon > 0$, there exists $\alpha \in \mathbb{N}$ such that for all $n > \delta$, then $d(x_n, p) < \varepsilon$.

Definition 2.5 [7]: A digital metric space (X,d,ρ) is a complete digital metric space if any Cauchy sequence $\{x_n\}$ of points of (X,d,ρ) converges to a point p of (X,d,ρ).

Definition 2.6 [7]: Let (X,d,ρ) be any digital metric space and $T:(X,d,\rho) \to (X,d,\rho)$ be a self digital map. If there exists $\alpha \in (0,1)$ such that for all $x \in X$, $d(Tx,Ty) \le \alpha(x,y),$ then T is called a digital contraction map.

Proposition 2.8 [7]: Every digital contraction map is digitally continuous.

Theorem 2.9 [7]: (Banach Contraction principle) Let (X,d,ρ) be a complete metric space which has a usual Euclidean metric in Z^n. Let, $T: X \to X$ be a digital contraction map. Then T has a unique fixed point, i.e. there exists a unique $p \in X$ such that $T(p) = p.$

3. MAIN RESULTS

Definition 3.1 [10]: Suppose that (X,d,ρ) is a complete digital metric space and $S,T: X \to X$ be maps defined on X. Then S and T are said to be commutative if $S \circ T = T \circ S, \forall x \in X.$

Definition 3.2[10]: The self maps S and T of a digital metric space (X,d,ρ) are said to be weakly commutative iff $d\big(S(T(x)), T(S(x))\big) \le d\big(S(x), T(x)\big)$ for all $x \in X$.

Remark 3.3 [1]: Every pair of commutative maps is weakly commutative but the converse is not true.

Definition 3.4: Definition 3.4: Let S and T be self maps of a digital metric space (X,d,ρ) and $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} S(x_n) = \lim_{n\to\infty} T(x_n) = t$ for some t t in X. Then

(i) S and T are said to be compatible if $\lim_{n\to\infty} d(STx_n, TSx_n) = 0$

(ii) S and T are said to be compatible of type (A) if

DOI 10.24297/jam.v13i4.6458
http://cirworld.com

$\lim_{n\to\infty} d(STx_n, TTx_n) = 0$ and $\lim_{n\to\infty} d(TSx_n, SSx_n) = 0$

(iii) S and T are said to be compatible of type (P) if $\lim_{n\to\infty} d(SSx_n, TTx_n) = 0$

Remark 3.5 : Note that the maps which are commutative are clearly compatible but the converse is not true.

Proposition 3.6 : Let S and T be compatible maps of type (A) on a digital metric space (X, d, ρ). If one of S and T is continuous, then S and T are compatible.

Proof : Since S and T are compatible maps of type (A),so

$\lim_{n\to\infty} d(STx_n, TTx_n) = 0$ and $\lim_{n\to\infty} d(TSx_n, SSx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some t in X . Let S is continuous, then

$\lim_{n\to\infty} STx_n \to St$ and $\lim_{n\to\infty} SSx_n \to St$ and hence

$$d(STx_n, TSx_n) \le d(STx_n, SSx_n) + d(SSx_n, TSx_n)$$

Letting $n \to \infty$, we have $\lim_{n\to\infty} d(STx_n, TSx_n) = 0,$ implies S and T are compatible.

Proposition 3.7. Let S and T be compatible maps on a digital metric space (X, d, ρ) into itself. Suppose that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$. Then

(a) $\lim_{n\to\infty} STx_n = Tt$ if T is continuous at t.

(b) $\lim_{n\to\infty} TSx_n = St$ if S is continuous at t.

Proof: (a) Suppose T is continuous at t. Since $\lim_{x\to\infty} Sx_n = \lim_{x\to\infty} Tx_n = t$ for some $t \in X$, we have $\lim_{n\to\infty} TSx_n = Tt$. Since S and T be compatible maps, we have

$d(STx_n, Tt) \le d(STx_n, TTx_n) + d(TTx_n, Tt) \to 0$ as $n \to \infty$ and hence the proof.

(b) The proof of $\lim_{n\to\infty} TSx_n = St$ follows by similar arguments as in (a).

Proposition 3.8 [13]. Let S and T be compatible maps on a digital metric space (X, d, ρ) into itself. If $St = Tt$ for some $t \in X$, then $STt = TSt = SSt = TTt$.

Theorem 3.1 Let A, B, S and T be four self-mappings of a complete digital metric space (X, d, ρ) satisfying the following conditions:

(a) $S(X) \subset B(X)$ and $T(X) \subset A(X)$;

(b) the pairs (A, S) and (B, T) are compatible;

(c) one of S, T, A and B is continuous;

(d) $d(Sx, Ty) \le \left[\phi\left\{\max\left(d(Ax, By), d(Sx, Ax), d(Sx, By)\right)\right\}\right], \forall x, y \in X,$

where $\phi : [0, \infty) \to [0, \infty)$ is a continuous and monotone increasing function such that $\phi(t) < t, \forall t > 0$. Then A, B, S and T have a unique common fixed point in X .

DOI 10.24297/jam.v13i4.6458
http://cirworld.com

Proof. Since $S(X)\subset B(X)$, we can consider a point $x_0\in X$, there exists $x_1\in X$ such that $Sx_0=Bx_1=y_0$. Also, for this point x_1, there exists $x_2\in X$ such that $Tx_1=Ax_2=y_1$. Continuing in this way, we can construct a sequence $\{y_n\}$ in X such that

$y_{2n}=Sx_{2n}=Bx_{2n+1}, y_{2n+1}=Tx_{2n+1}=Ax_{2n+2}$ for each $n\geq 0.$

Now, we have to show that $\{y_n\}$ is Cauchy sequence in (X,d,ρ). Indeed, it follows that, for all

$n\geq 1,$

$$
\begin{aligned}
d(y_{2n},y_{2n+1})&=d(Sx_{2n},Tx_{2n+1})\\
&\leq\left[\phi\left\{\max\left(d(Ax_{2n},Bx_{2n+1}),d\left(Sx_{2n},Ax_{x_{2n}}\right),d(Sx_{2n},Bx_{2n+1})\right)\right\}\right]\\
&=\left[\phi\left\{\max\left(d(y_{2n-1},y_{2n}),d(y_{2n},y_{2n-1}),d(y_{2n},y_{2n})\right)\right\}\right]\\
&=\phi\left(d(y_{2n-1},y_{2n})\right)<d(y_{2n-1},y_{2n})
\end{aligned}
$$

Therefore, the sequence $\{d(y_{2n},y_{2n+1})\}$ is strictly decreasing. Then there exists $r\geq 0$ such that $\lim_{n\to\infty}d(y_{2n},y_{2n+1})=r$. Suppose that $r>0$, then, letting $n\to\infty$ in above equation, $r\leq\phi(r)<r,$ which is impossible . Hence, $r=0$, that is,

$\lim_{n\to\infty}d(y_{2n},y_{2n+1})=0$.

By the completeness of X, there exists $z\in X$ such that $y_n\to\infty$ as $n\to\infty$. Consequently, the subsequences $Sx_{2n},Ax_{2n},Tx_{2n+1}$ and Bx_{2n+1} of $\{y_n\}$ also converge to a point $z\in X$. Now, suppose that A is continuous. Then $\{AAx_{2n}\}$ and $\{ASx_{2n}\}$ converge to Az A as $n\to\infty$. Since the mappings A and S are compatible on X, it follows from Proposition 3.7 that $\{SAx_{2n}\}$ converge to Az as $n\to\infty$. Now, we claim that $Az=z$.

Consider,

$$d(SAx_{2n},Tx_{2n+1})\leq\left[\phi\left\{\max\left(d(AAx_{2n},Bx_{2n+1}),d(SAx_{2n},AAx_{2n}),d(SAx_{2n},Bx_{2n+1})\right)\right\}\right]$$

Letting $n\to\infty$, we have

$$d(Az,z)\leq\left[\phi\left\{\max\left(d(Az,z),d(Az,Az),d(Az,z)\right)\right\}\right]=d(Az,z),$$

implies $Az=z.$ Again, from (d), we obtain

$$d(Sz,Tx_{2n+1})\leq\left[\phi\left\{\max\left(d(Az,Bx_{2n+1}),d(Sz,Az),d(Sz,Bx_{2n+1})\right)\right\}\right]$$

Letting $n\to\infty$, we get $Sz=z.$ Thus, since $S(X)\subset B(X)$, there exists $u\in X$ such that

$z=Az=Sz=Bu.$. By using (d), we can obtain

$$d(z,Tu)=d(Sz,Tu)\leq\left[\phi\left\{\max\left(d(Az,Bu),d(Sz,Az),d(Sz,Bu)\right)\right\}\right]=0\Rightarrow z=Tu.$$

Since B and T are compatible on X and $z=Tu=Bu$, by Proposition 3.8, we have $TBu=BTu$ and hence $Bz=TBu=BTu=Tz$. Also, we have

DOI 10.24297/jam.v13i4.6458
http://cirworld.com

$$d(z,Bz)=d(z,Tz)=d(Sz,Tz)\le\left[\phi\left\{\max\left(d(Az,Bz),d(Sz,Az),d(Sz,Bz)\right)\right\}\right]$$

yields $z=Tz=Bz$ and so z is a common fixed point of A,B,S and T. Similarly, we can use above assertion in case of continuity of B or S or T and uniqueness of the common fixed point follows directly from the condition (d) and hence the proof follows.

Theorem 3.2. Let A,B,S and T be four self-mappings of a complete digital metric space (X,d,ρ) satisfying the following conditions:

(a) $S(X)\subset B(X)$ and $T(X)\subset A(X)$;

(b) the pairs (A,S) and (B,T) are compatible;

(c) one of S,T,A and B is continuous;

(d) $d(Sx,Ty)\le\lambda\left\{\max\left(d(Ax,By),d(Sx,Ax),d(Sx,By)\right)\right\},\forall x,y\in X,$

For some $\lambda\in[0,1)$. Then A,B,S and T have a unique common fixed point in X.

Proof. If we set $\phi(t)=\lambda t$, we get the result.

Theorem 3.3. Let A,B,S and T be four self-mappings of a complete digital metric space (X,d,ρ) satisfying the conditions (a), (c) and (d). If the pairs (A,S) and (B,T) are compatible of type (A), then A,B,S and T have a unique common fixed point in X.

Proof. Theorem is direct consequence of proposition 3.6 and theorem 3.1.

Applications of Common Fixed Point Theorems in Digital Metric Space

In this section, we give an application of digital contractions to solve the problem related to image compression. The aim of image compression is to reduce redundant image information in the digital image. When we store an image we may come across certain type of problems like either memory data is usually too large or stored image has not more information than original image. Also, the quality of compressed image can be poor. For this reason, we must pay attention to compress a digital image. Fixed point theorem can be used for image compression of a digital image.

CONCLUSION :

The aim of this paper is to introduce common fixed point theorems for the digital metric spaces, using compatible maps and its variants. This concept may come handy in the image processing and redefinement of the image storage. The redefinition of the same slot of memory is a proposed application of the proposed concept of the paper.

REFRENCES:

[1] Rani A, Jyoti K and Rani A.2016. Common fixed point theorems in digital metric spaces, International Journal of Scientific & Engineering Research, Volume 7, Issue 12, December-2016 ISSN 2229-5518

[2] Boxer L. 1994. Digitally Continuous Functions, Pattern Recognition Letters, 15 (1994), 833-839.

[3] Boxer L. 2005. Properties of Digital Homotopy , J. Math. Imaging V is., 22(2005), 19-26.

[4] Boxer L. 1991. A Classical Constructions for The Digital Fundamental Group, J. Math. Imaging Vis., 10(1999), 51-62

[5] Boxer L.2010. Continuous Maps on Digital Simple Closed Curves, Appl. Math., 1(2010), 377-386.

[6] Das KM, Naik KV. 1979. Common fixed-point theorems for commuting maps on a metric space, Proc. Amer. Math. Soc. 1979, 77, 369-373.

[7] Ege O, K. 2015. Banach Fixed Point Theorem for Digital Images, J. Nonlinear Sci. Appl., 8(2015), 237-245.

[8] Ege O, K. 2013. Lefschetz Fixed Point Theorem for Digital Images, Fixed Point Theory Appl., 2013(2013), 13 pages.

[9] Ege O, K. 2014. Applications of The Lefschetz Number to Digital Images, Bull. Belg. Math. Soc. Simon Stevin, 21(2014), 823-839.

DOI 10.24297/jam.v13i4.6458
http://cirworld.com

[10] Jain D, and Upadhaya A.C. 2017. Weakly commuting Mappings in Digital Metric Spces, International Advanced Research Journal in Science, Engineering and Technology, Vol 4, Issue 8, August 2017

[11] Jungck G. 1976. Commuting mappings and fixed point, Amer. Math. Monthly 1976, 83, 261-263.

[12] Jungck G.1986. Compatible mappings and common fixed points, International J. Math. Math. Sci. 1986, 9(4), 771-779.

[13] Jungck G, Murthy PP, Cho, YJ. 1993. Compatible mappings of type (A) and common fixed points, Math. Japon. 1993, 38, 381 – 390.

[14] Rani A, Kumar S, Kumar N, Garg SK. 2012. Common Fixed Point Theorems for Compatible and Weakly Compatible Maps in G-Metic Spaces, Applied Math. 2012, 3, 1128-1134.

[15] Rosenfeld A. 1979. Digital Topology, Amer. Math. Monthly, 86(1979), 76-87.

[16] Sessa S and Fisher B. 1986. ON COMMON FIXED POINTS OF WEAKLY COMMUTING MAPPINGS AND SET-VALUED MAPPINGS, internat. J. Math. & Math. Sci. Vol. 9 No. 2 (1986) 323-329

[17] Sumitra. 2013. COUPLED FIXED POINT THEOREMS FOR COMPATIBLE MAPS AND ITS VARIANTS IN FUZZY METRIC SPACES, Jordan Journal of Mathematics and Statistics (JJMS) 6(2), 2013, pp. 141-155

[18] Sumitra. 2012. Coupled Fixed Point Theorem for Weakly Compatible Mappings in Intuitionistic Fuzzy Metric Spaces, International Journal of Computer Applications (0975 – 8887) Volume 50– No.23, July 2012

[19] Z-Dos`enovic` T. 2009. A common fixed point theorem for compatible mappings in fuzzy metric spaces using implicit relation, Acta Math. Hang. 2009, 125(4), 357-368.

DOI 10.24297/jam.v13i4.6458
http://cirworld.com