



Solutions of nonlinear fractional coupled Hirota-Satsuma-KdV Equation

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ABSTRACT:

Our interest in the present work is in implementing the FPSM to stress its power in handling the nonlinear fractional coupled Hirota-Satsuma-KdV Equation. The approximate analytical solution of this type of equations is obtained.

Keywords:

nonlinear fractional coupled Hirota-Satsuma-KdV Equation; fractional power series method; Caputo fractional derivative

INTRODUCTION

During the last decades the nonlinear fractional differential equation has been applied in various scientific and engineering fields, such as electromagnetic theory, fluid mechanics, biology, solid state physics, chemical physics and geochemistry etc[1-16]. In most cases, it is very difficult to obtain exact solutions. So these types of equations must be solved by any numerical methods or approximate methods. Many approaches for the solution of fractional differential equations have been proposed. These methods include the direct algebraic method [4-5], Jacobi elliptic function method [6], tanh-function method [7], variational iteration method and homotopy perturbation method [8,9], Adomian decomposition method [11], sine-cosine method [12], homotopy analysis method [13], the differential transform method [14], and fractional power series method(FPSM)[15].

The FPSM is a powerful tool for solving linear and nonlinear problems. Our interest in the present work is in implementing the FPSM to stress its power in handling the following nonlinear fractional coupled Hirota-Satsuma-KdV Equation:

$$D_t^\alpha u = \frac{1}{4} \frac{\partial^3 u}{\partial x^3} + 3u \frac{\partial u}{\partial x} - 6v \frac{\partial v}{\partial x}, 0 < \alpha \leq 1, \quad (1)$$

$$D_t^\alpha v = -\frac{1}{2} \frac{\partial^3 v}{\partial x^3} - 3u \frac{\partial u}{\partial x}, \quad (2)$$

subject to the conditions

$$u(x,0) = -\tanh^2 x, v(x,0) = \tanh x, \quad (3)$$

where D^α ($1 \geq \alpha > 0$) denote the Caputo fractional derivative of order α .

The Eqs.(1) and (2) arise in many scientific applications such as quantum mechanics and plasma physics [1-3]. It is well known that wave phenomena of plasma are modeled by kind tanh solution.

The present paper has been organized as follows. In section 2, we introduce the basic definitions and properties of fractional calculus and the FPSM is described. In section 3 the FPSM is applied for the Eqs.(1) and (2). Conclusion is



presented in Section 4.

BASIC DEFINITIONS

In this section, we will introduce notations, definitions and some useful lemmas, which play an important role in obtaining the main results of this paper. We begin with some basic definitions [15].

Definition 1. A real function $f(x), x > 0$ is said to be in the space $C_\mu, \mu \in R$ if there exists a real number $p > \mu$, such that $f(x) = x^p f_1(x)$ where $f_1(x) \in C[0, \infty)$ and it is said to be in the space C_n if and only if $f^{(n)} \in C_\mu, n \in N$.

The Riemann-Liouville fractional integral operator is defined as follows:

Definition 2. The Riemann-Liouville fractional integral operator of order $\alpha > 0$ of a function $f(x) \in C_\mu, \mu \geq -1$ is defined as:

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} f(s) ds,$$

$$J^0 f(x) = f(x).$$

Properties of the operator J^α can be found in and we mention only the following: For $\alpha, \beta \geq 0, x > 0$ and $\gamma > -1$:

$$J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x),$$

$$J^\alpha J^\beta f(x) = J^\beta J^\alpha f(x),$$

$$J^\alpha (x^\gamma) = \frac{\Gamma(\gamma+1)}{\Gamma(1+\alpha+\gamma)} x^{\gamma+\alpha}.$$

Definition 3. The fractional derivative of $f(x)$ in Caputo sense is defined as:

$$D^\alpha f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-s)^{m-\alpha-1} f^{(m)}(s) ds,$$

for $m-1 < \alpha \leq m, m \in N^+, x > 0$ and $f \in C_{-1}^m$.

We recall here two of its basic properties :

$$D^\alpha J^\alpha f(x) = f(x),$$



$$J^\alpha D^\alpha f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{x^k}{k!}, \quad x > 0.$$

Definition 4. A power series representation of the form

$$\sum_{n=0}^{\infty} c_n (t-t_0)^{n\alpha} = c_0 + c_1(t-t_0)^\alpha + c_2(t-t_0)^{2\alpha} + \dots,$$

where $0 \leq m-1 < \alpha \leq m, m \in \mathbb{N}^+$ and $t \geq t_0$ is called a fractional power series (FPS) about t_0 , where t is a variable and c_n are the coefficients of the series.

We also need the following property:

Lemma 1. Suppose that the FPS $\sum_{n=0}^{\infty} c_n t^{n\alpha}$ has radius of convergence $R > 0$. If $f(t)$ is a function defined by

$f(t) = \sum_{n=0}^{\infty} c_n t^{n\alpha}$ on $0 \leq t < R$, then for $m-1 < \alpha \leq m$ and $0 \leq t < R$, we have:

$$D^\alpha f(t) = \sum_{n=0}^{\infty} c_n \frac{\Gamma(n\alpha + 1)}{\Gamma((n-1)\alpha + 1)} t^{(n-1)\alpha}.$$

THE SOLUTIONS OF Eqs. (1) AND (2)

To solve the Eqs. (1) and (2) by FPSM, suppose that the solution of (1) and (2) takes the form:

$$u(x, t) = \sum_{k=0}^{\infty} a_k(x) t^{\alpha k} = a_0(x) + a_1(x) t^\alpha + a_2(x) t^{2\alpha} + \dots \quad (4)$$

$$v(x, t) = \sum_{k=0}^{\infty} b_k(x) t^{\alpha k} = b_0(x) + b_1(x) t^\alpha + b_2(x) t^{2\alpha} + \dots \quad (5)$$

Using (3), we have

$$a_0(x) = -\tanh^2 x, \quad b_0(x) = \tanh x.$$

Next we determine the $a_k(x), b_k(x)$ ($k = 1, 2, \dots$).

From Lemma 1, we obtain

$$D_x^\alpha u(x, t) = \sum_{k=1}^{\infty} \frac{a_k(x) \Gamma(\alpha k + 1)}{\Gamma(\alpha(k-1) + 1)} t^{\alpha(k-1)}. \quad (6)$$



$$D_x^\alpha v(x,t) = \sum_{k=1}^{\infty} \frac{b_k(x)\Gamma(\alpha k + 1)}{\Gamma(\alpha(k-1) + 1)} t^{\alpha(k-1)}. \quad (7)$$

Substituting (6) and (7) into (4), and comparing the coefficients of t^α in the both side, we get

$$a_1(x) = \frac{8\Gamma(\alpha)e^{2x}(1-e^{2x})}{\Gamma(\alpha+1)(1+e^{2x})^3},$$

$$b_1(x) = \frac{4\Gamma(\alpha)e^{2x}}{\Gamma(\alpha+1)(1+e^{2x})^2},$$

$$a_2(x) = \frac{16e^{2x}\Gamma(\alpha+1)(1-4e^{2x}+e^{4x})}{\Gamma(2\alpha+1)(1+e^{2x})^4},$$

$$b_2(x) = \frac{8e^{2x}\Gamma(\alpha+1)(1-2e^{2x})}{\Gamma(2\alpha+1)(1+e^{2x})^3},$$

$$a_3(x) = \frac{16e^{2x}\Gamma(2\alpha+1)(1-e^{2x})(1-10e^{2x}+e^{4x})}{\Gamma(3\alpha+1)(1+e^{2x})^5},$$

$$b_3(x) = \frac{8e^{2x}\Gamma(2\alpha+1)(1-4e^{2x}+e^{4x})}{\Gamma(3\alpha+1)(1+e^{2x})^4},$$

$$a_4(x) = \frac{32e^{2x}\Gamma(3\alpha+1)(1-26e^{2x}+66e^{4x}-26e^{6x}+e^{8x})}{3\Gamma(4\alpha+1)(1+e^{2x})^6},$$

$$b_4(x) = \frac{16e^{2x}\Gamma(3\alpha+1)(1-11e^{2x}+11e^{4x}-e^{6x})}{3\Gamma(4\alpha+1)(1+e^{2x})^5},$$

and so on.

Thus we obtain the solution

$$u(x,t) = -\tanh^2 x + \frac{8\Gamma(\alpha)e^{2x}(1-e^{2x})}{\Gamma(\alpha+1)(1+e^{2x})^3} t^\alpha + \frac{16e^{2x}\Gamma(\alpha+1)(1-4e^{2x}+e^{4x})}{\Gamma(2\alpha+1)(1+e^{2x})^4} t^{2\alpha}$$



$$\begin{aligned}
 & + \frac{16e^{2x}\Gamma(2\alpha+1)(1-e^{2x})(1-10e^{2x}+e^{4x})}{\Gamma(3\alpha+1)(1+e^{2x})^5} t^{3\alpha} \\
 & + \frac{32e^{2x}\Gamma(3\alpha+1)(1-26e^{2x}+66e^{4x}-26e^{6x}+e^{8x})}{3\Gamma(4\alpha+1)(1+e^{2x})^6} t^{4\alpha} + \dots \\
 v(x,t) = & \tanh x + \frac{4\Gamma(\alpha)e^{2x}}{\Gamma(\alpha+1)(1+e^{2x})^2} t^\alpha \\
 & + \frac{8e^{2x}\Gamma(\alpha+1)(1-2e^{2x})}{\Gamma(2\alpha+1)(1+e^{2x})^3} t^{2\alpha} \\
 & + \frac{8e^{2x}\Gamma(2\alpha+1)(1-4e^{2x}+e^{4x})}{\Gamma(3\alpha+1)(1+e^{2x})^4} t^{3\alpha} \\
 & + \frac{16e^{2x}\Gamma(3\alpha+1)(1-11e^{2x}+11e^{4x}-e^{6x})}{3\Gamma(4\alpha+1)(1+e^{2x})^5} t^{4\alpha} + \dots
 \end{aligned}$$

The exact solution to the problem (1)-(3) when $\alpha = 1$ are given as:

$$u(x,t) = -\tanh^2(x+t),$$

$$v(x,t) = \tanh(x+t).$$

CONCLUSION

The FPSM is a powerful tool for solving the nonlinear fractional coupled Hirota-Satsuma-KdV Equation. Our results show that the FPSM is simple, direct and effective.

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