Splitting decomposition homotopy perturbation method to solve one-dimensional Navier-Stokes equation

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Abstract

We have proposed in this research a new scheme to find analytical approximating solutions for Navier-Stokes equation of one dimension. The new methodology depends on combining Adomian decomposition and Homotopy perturbation methods with the splitting time scheme for differential operators . The new methodology is applied on two problems of the test: The first has an exact solution while the other one has no exact solution. The numerical results we obtained from solutions of two problems, have good convergent and high accuracy in comparison with the two traditional Adomian decomposition and Homotopy perturbation methods .

Keywords: Splitting scheme; Adomian decomposition; homotopy perturbation method; Navier-Stokes equation; convergence analysis.

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1- Introduction

Navier–Stokes equations are non-linear partial differential equations that govern the incompressible fluid flow inside specific hollow as it is an equation of parabolic type and the non-linear degree is increased with the increase of Reynolds number . These equations names are taken from the two physicists Claude-Louis Navier and George Gabriel Stokes in the nineteenth century, and resulted from applying Newton second law on the movement of fluids supposing that stress of fluid is totaled of divergence of viscosity (proportional with velocity rate) and pressure. Also, these equations are considered as the most important physical equations which describe a large number of phenomena of different applications in many research fields that may be used in modeling weather, liquid flow in channels and pipes , gas flow round flying bodies, and movement of stars in the galaxy.

Navier-Stokes equations are important mathematically, due to their wide applications, where to this day no one has succeeded in proving the existence of general solution or a permanent for Navier-Stokes equations(privately three-dimensional). Therefore, this kind of problems is named as the existence and smooth flow of Navier-Stokes. These problems are considered as twenty one-century problems and Clay Mathematics Institute was offered a precious prize to solve them[1]. Many scientists and researchers attempted to find solutions for these equations by different methods; For example , Chorin[2] suggested numerical solutions for Navier-Stokes equations by applying differences methods . Jane[3] presented analytic solutions for one dimension parabolic equation by applying Adomian decomposition method . Momani and Odibat[4] used Adomian decomposition method to find analytic solution of a time-fractional Navier-Stokes equation. Musa[5] used finite elements and control volume methods to find numerical solutions for Navier-stokes equation. Al-Saif [6] proposed Adomian decomposition methods to introduce analytical approximate solutions for two-dimensional Navier-Stokes equations. Also, the solutions are found by Li and Wang [7] using finite differences methods, and Donea et. al.[8] used finite element method and Stelian and Anca[9] used finite volume method etc. While, Adomian decomposition method(ADM) and homotopy perturbation method(HPM) are active and strong in finding solutions for physical and mathematical problems ones, we can apply them to solve ordinary/partial differential equations either linear or non-linear for initial-boundary value problems. The first method name is taken from the scientist who discovered it ; namely, G. Adomian, an the second was found for first time by Chinese Mathematician; He[10], who has developed and improved it as stated in references [11-13]. He presented an effective homotopy method according to homotopy topological definition with a traditional effective method

During few last years, many researchers attempted to modify and extend these two methods; for example, Zhang [14] modified ADM to solve a class of non-linear boundary value problems. Jafari and Daftardar-Gejji [15] modified ADM to solve a set of non-linear equations led to find approximation solutions are better than those which were obtained by standrad method in measurements of convergent and accuracy. Luo [16] suggested active methods for ADM which is a two-steps Adomian decomposition method (TSADM) to reach the solution. TSADM reduces the repetitive of mathematical processes that are applied to find the solution and also he made comparison for the results. The results showed that TSADM is an active and efficient method which has high accuracy in finding solutions. Also, in many works[17,18,19,20], the authors used ADM to found analytic and approximate solutions for different problems. In the same direction of modification, the HPM is active to find solutions for non-linear equations[10,21-30].

Depending on the above literature review for attempt of researchers to expand and developed ADM and HPM to solve linear and nonlinear boundary value problems, and depending on our simple information about applications of these methods to solve the problems that under consideration study, we did not find, if any, that is for problems are different from our problem in formulations and conditions. These information encourage us to suggest a new methodology for solving nonlinear problems. The new methodology is constructed by combining the work of ADM and HPM in-turns with the splitting scheme for time differential operators, sampling by (SDHPM) . The power of this manageable method is confirmed by applying it for two selected flow problems as a test to be illustrated by the effectiveness and validity of new

methodology in finding solution of Navier-Stokes equations. The numerical results we obtained are shown that the efficiency and activity of new method, and comparison its high accuracy with standard ADM and HPM.

2-The main idea of the SDHPM method:

Here, we will discuss the basic idea of the two methods ADM & HPM and then describe the basic idea of the new manner.

The main idea of the standard two methods Adomian decomposition method and Homotopy perturbation method to the general nonlinear equation as the form:

$$Lu + Ru + Nu = g \tag{1a}$$

With the initial condition; $u_0 = u(x,0)$ (1b)

The linear terms decomposed into $Lu + Ru$,while the nonlinear terms are represented by Nu , where L is an easily invertible linear operator, R is the remaining linear part, and g is a known analytic function.

We can see the principle steps of this two methods (ADM & HPM) as in [18,10] respectively ; and the convergence of this two methods (ADM & HPM) also we can see in [31,34]respectively. Now, to illustrate the basic idea of the new model , we decomposed the linear differential operator in Equation(1a) into two linear differential operators as the form :

$$L(u) = \alpha L(w) + \beta L(v)\,, \tag{2}$$

where $\alpha + \beta = 1$. by this definition ,we can split Equation(1a) into two kinds of differential operators equations; one is linear and the other is nonlinear as;

$$L(w) + R(w) = 0 \tag{3}$$

$$L(v) + N(v) - g = 0 \tag{4}$$

We apply ADM as explained in [18] on Equation(3) to find the series solution w_n $n = 0,1,2,..$. depending on the initial condition u_0 , then using the result as an initial condition for the series solution v_n $n = 1,2,..$ that are obtained by using HPM [10] for Equation (4) respectively.

3- Algorithm Analysis of SDHPM for NSE:

We consider the unsteady state one-dimensional incompressible Navier-Stokes equation as the form :

$$u'_{t'} + u'u'_{x'} + \frac{1}{\rho} p'_{x'} = \nu\, u'_{x'x'} \tag{5}$$

where $u'(x,t)$ is the velocity , ρ is the density, ν is the kinematic viscosity and p'_x pressure term. In order to facilitate the analysis , the following dimensionless variables are considered;

$$u' = \frac{u}{U_0}\,, x' = \frac{x}{L}\,, t' = \frac{U_0}{L} t\,,\ p' = \frac{p}{\rho U_0^2}$$

where U_0 is a reference velocity , and L is a reference length. After we use the new definitions and drop the primes, non-dimensional Navier-Sokes equation (5) becomes:

$$u_t + uu_x + p_x = \frac{1}{\text{Re}} u_{xx} \tag{6}$$

with the initial condition : $u(x,0) = u_0$ (7)

where $\text{Re} = \frac{U_0 L}{\nu}$ is Reynolds number.

Now, we start applying the ADM algorithm for Equation (6) with initial condition(7). Following, we define the linear operators $L_t = \frac{\partial}{\partial t}$, $L_x = \frac{\partial}{\partial x}$, $L_{xx} = \frac{\partial^2}{\partial x^2}$. Therefore, we rewrite Equation (6) with operator form as;

$$L_t u + uL_x u + L_x p = \frac{1}{\text{Re}} L_{xx} u \tag{8}$$

By defining the inverse operator L^{-1} which are given in [18] as $L^{-1}(.) = \int_0^t (.) dt$, we can write the Eq. (8) as;

$$u(x,t) = u(x,0) - L_t^{-1}(u\,L_x u) - L_t^{-1}(L_x p) + \frac{1}{\text{Re}} L_t^{-1}(L_{xx} u) \tag{9}$$

As in [18], the components solutions can be written as;

$$u(x,t)=\sum_{n=0}^{\infty}u_n(x,t)$$

and the nonlinear operator is

$$Nu=\Psi(u)=uL_x u$$

The associated decomposition method is given by

$$u_0=u(x,0) \tag{10a}$$

$$u_{n+1}=-L_t^{-1}(\Psi(u_n))-L_t^{-1}(L_x p)+\frac{1}{\mathrm{Re}}L_t^{-1}(L_{xx}u_n) \tag{10b}$$

We decomposed Ψ according to the series $\sum_{n=0}^{\infty}A_n$, where A_n is calculated by the Adomian's polynomial which is define in [31] then we obtain

$$\left.\begin{aligned}
&A_0=u_0\frac{\partial u_0}{\partial x}\\
&A_1=u_1\frac{\partial u_0}{\partial x}+u_0\frac{\partial u_1}{\partial x}\\
&A_2=u_0\frac{\partial u_2}{\partial x}+u_1\frac{\partial u_1}{\partial x}+u_2\frac{\partial u_0}{\partial x}\\
&A_3=u_3\frac{\partial u_0}{\partial x}+u_2\frac{\partial u_1}{\partial x}+u_1\frac{\partial u_2}{\partial x}+u_0\frac{\partial u_3}{\partial x}\\
&\vdots
\end{aligned}\right\}. \tag{11}$$

and so on. Consequently as in [18] the iterative solutions are;

$$\left.\begin{aligned}
&u_0=u(x,0)=\phi\\
&u_1=L_t^{-1}\left(\left(\frac{1}{\mathrm{Re}}\frac{\partial^2 u_0}{\partial x^2}\right)-p_x\right)-L_t^{-1}(A_0)\\
&u_2=L_t^{-1}\left(\left(\frac{1}{\mathrm{Re}}\frac{\partial^2 u_1}{\partial x^2}\right)-p_x\right)-L_t^{-1}(A_1)\\
&\vdots
\end{aligned}\right\}. \tag{12}$$

and so on. Now , by application HPM to Equation(6), we have

$$H(v,p)=(1-p)\left[\frac{\partial v}{\partial t}-\frac{\partial u_0}{\partial t}\right]+p\left[\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}+p_x-\frac{1}{\mathrm{Re}}\frac{\partial^2 v}{\partial x^2}\right]=0$$

$$or$$

$$\frac{\partial v}{\partial t}-\frac{\partial u_0}{\partial t}=p\left[\frac{1}{\mathrm{Re}}\frac{\partial^2 v}{\partial x^2}-v\frac{\partial v}{\partial x}-p_x-\frac{\partial u_0}{\partial t}\right]=0$$

As in [10]; we assume the solution as a power series in p ; then we have

$$\frac{\partial}{\partial t}(v_0 + pv_1 + p^2 v_2 + \ldots) - \frac{\partial u_0}{\partial t} =$$
$$p\left[\frac{1}{\mathrm{Re}}\frac{\partial^2}{\partial x^2}(v_0 + pv_1 + p^2 v_2 + \ldots) - p_x - \left((v_0 + pv_1 + p^2 v_2 + \ldots)\frac{\partial}{\partial x}(v_0 + pv_1 + p^2 v_2 + \ldots)\right) - \frac{\partial u_0}{\partial t}\right]$$

By equal the term which have the same power p, we get

(13)

$$\left.\begin{aligned}
&p^0 : \frac{\partial v_0}{\partial t} - \frac{\partial u_0}{\partial t} = 0\\
&p^1 : \frac{\partial v_1}{\partial t} + \frac{\partial u_0}{\partial t} + v_0\frac{\partial v_0}{\partial x} + p_x - \frac{1}{RE}\frac{\partial^2 v_0}{\partial x^2} = 0\\
&p^2 : \frac{\partial v_2}{\partial t} + v_0\frac{\partial v_1}{\partial x} + v_1\frac{\partial v_0}{\partial x} - \frac{1}{RE}\frac{\partial^2 v_1}{\partial x^2} = 0\\
&\vdots
\end{aligned}\right\}.$$

and so on. Then the approximate solution can

be find by setting $p = 1$,

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \ldots \qquad (14)$$

The application of SDHPM for Equation(1), by using the splitting method that is represented by Equations(3) and (4) can be done by apply ADM for Equation(3), and use the resulting solution w_1 as initial solution to find the solution v_1 by apply HPM for Equation(4). then after repeated this alternate procedure between two schemes ,we have;

$$\begin{aligned}
&u_0 = u(x,0)\\
&w_1 = L_t^{-1}\left[\frac{2}{\mathrm{Re}}\left(\frac{\partial^2 u_0}{\partial x^2}\right)\right]\\
&v_1 = L_t^{-1}\left[-2\left(w_1\frac{\partial w_1}{\partial x}\right) - 2p_x - \frac{\partial w_1}{\partial t}\right], \text{ where } v_0 = w_1\\
&w_2 = L_t^{-1}\left[\frac{2}{\mathrm{Re}}\left(\frac{\partial^2 w_1}{\partial x^2}\right)\right]\\
&v_2 = L_t^{-1}\left[-2\left(w_1\frac{\partial w_2}{\partial x} + w_2\frac{\partial w_1}{\partial x}\right)\right], \text{ where } v_1 = w_2\\
&\vdots\\
&\text{So on.}
\end{aligned}$$

Substituting above results into Equation(2) with $\alpha = \beta = 0.5$,we obtain

$$u(x,t) = u_0 + u_1 + u_2 + \cdots = \sum_{n=0}^{\infty} u_n(x,t)$$

The convergent of this series will be proved in the next section theoretically. However, for some problems this series can't be determined, so we use an approximation of the solution from truncated series:

$$U_M = \sum_{n=0}^{M} u_n \quad \text{with} \quad \lim_{M\to\infty} U_M = u$$

The acceleration for this convergent means the need to few terms of the above equation, for obtaining the formula which nearby to the exact solution.

4- Analysis of convergence of SDHPM:

In this section, we will study the analysis of convergence in the same manner as [35,36,37] for the decomposition method to the nonlinear Navier-Stokes Equation (6). Let as consider the Hilbert space H which may be defined as $H = L^2(\Omega\times[0,T])$, the set of applications ; $u:\Omega\times[0,T]\to\Re$ with

$$\int_{\Omega\times[0,T]} u^2 d\Omega < +\infty \tag{15}$$

And scalar product and induced norm :

$$(u,v) = \int_{\Omega\times[0,T]} uv d\Omega \quad \text{and} \quad \|u\|^2 = (u,u) \tag{16}$$

where , $\Re$ is real numbers.

We consider the nonlinear Navier-Stokes equation ,then the operator of a nonlinear Navier-Stokes Equation is:

$$L_t u = -uL_x u - p_x + \frac{1}{\text{Re}} L_{xx} u \tag{17}$$

where $L_t = \frac{\partial}{\partial t}, L_x = \frac{\partial}{\partial x}, L_{xx} = \frac{\partial^2}{\partial x^2}$. Following, we define the difference operator $\Delta z = z - \hat{z}$ for any quantity such as z . The Adomian decomposition method is convergent if the following conditions are satisfied ;

$(\text{I}): (L_t(\Delta u), \Delta u) \geq k_1\|\Delta u\|^2, k_1 > 0, \forall u, \hat{u} \in H$

(II) :whatever may be $M > 0$,there exist a constant $C(M) > 0$ such that for $u, \hat{u} \in H$ with $\|u\| \leq M, \|\hat{u}\| \leq M$, we have:

$(L_t(\Delta u), w) \leq C(M, \text{Re})\|\Delta u\|\|w\|$ for every $w \in H$.

Now , we will use the following theorem to satisfy the above conditions as [35,36]

Theorem 1: If (I) and (II) are satisfied , then ADM of Equation (17) convergent.

Proof : It is easy to prove (I) and (II) as the same manner in[6,35,36] obtained on the results: Then condition (I) holds with $k_1 = \delta_2 M - \frac{1}{\text{Re}}\delta_1$, where δ_1, δ_2 are constants and the condition (II) is satisfied with $C(M,\text{Re}) = M - \frac{1}{\text{Re}}$. Hence the prove is complete .

■

Let us consider Equation (1) (after we apply the HPM as in [10]) in the following form:

$$L(v) = L(u_0) + p(g - N(v) - R(v) - L(u_0)) \tag{18}$$

Applying the inverse operator, L^{-1} to both sides of Equation (18) , we obtain

$$v = u_0 + p(L^{-1}g - L^{-1}N(v) - L^{-1}R(v) - u_0) \tag{19}$$

Suppose that

$$v = \sum_{i=0}^{\infty} p^i u_i \quad (20)$$

Substituting (20) into the right-hand side of Equation (19), yields

$$v = u_0 + p(L^{-1}g - (L^{-1}N)\sum_{i=0}^{\infty} p^i u_i - (L^{-1}R)\sum_{i=0}^{\infty} p^i u_i - u_0) \quad (21)$$

If $p \to 1$, the exact solution may be obtained by using Equation (14) as;

$$u = L^{-1}(g) - \left[\sum_{i=0}^{\infty}(L^{-1}N)(u_i)\right] - \left[\sum_{i=0}^{\infty}(L^{-1}R)(u_i)\right]$$

To study the convergence of this method , let us state the following theorem.

Theorem (2): (Sufficient Condition of Convergence)[34]

Supposes that X and Y are Banach space and $N : X \to Y$ is a contractive nonlinear mapping , that is

$$\forall\, w, w^* \in X; \left\| N(w) - N(w^*) \right\| \le \gamma \left\| w - w^* \right\| \quad ,0 < \gamma < 1.$$

Then according to Banach's fixed point theorem N has a unique fixed point u , that is $N(u) = u.$

Assume that the sequence generated by homotopy perturbation method can be written as;

$$W_n = N(W_{n-1}), \qquad W_{n-1} = \sum_{i=0}^{n-1} w_i, \qquad n = 1,2,3,...$$

and suppose that:

$$W_0 = w_0 \in \mathrm{B}_r(w) \text{ where } \mathrm{B}_r(w) = \left\{ w^* \in X \mid \left\| w^* - w \right\| < r \right\} \quad (22)$$

Then we have :

$$(i) \quad W_n \in \mathrm{B}_r(w),. \qquad (ii) \quad \lim_{n\to\infty} W_n = w.$$

Proof : We can see the proof in [34]

Depend on above theorems and its proofs the converges of SDHPM(sufficient condition of convergence) is to be hold. on other hand the combining between two theorems give us guarantee for convergent of the solutions that are obtained by SDHPM. In the next section, we will illustrate the convergence of Splitting Adomian decomposition homotopy perturbation method numerically and theoretically by apply the above theorem.

5- Numerical Test and Discussion:

The theoretical analysis of SDHPM done in the previous sections (2) will be applied in this section to find the approximate solution of two test problem: the first have exact solution and the second without exact solution.

Test of problem 1[38]:The one-dimensional NSE (6) with exact solution and initial

$$u = \frac{-2e^{\frac{c\,\mathrm{Re}(\zeta_0 + x + ct)}{2}}}{c^2\,\mathrm{Re}^2}$$

condition $u_0 = \dfrac{-2e^{\frac{c\,\mathrm{Re}(\zeta_0 + x)}{2}}}{c^2\,\mathrm{Re}^2}$,with pressure term $p_x = \dfrac{-2e^{c\,\mathrm{Re}(\zeta_0 + x + ct)}}{c^3\,\mathrm{Re}^3} + \dfrac{e^{\frac{c\,\mathrm{Re}(\zeta_0 + x + ct)}{2}}}{2\,\mathrm{Re}}$

The iterative solutions for this problem by using SDHPM can be found of Equation(6) after we splitting the linear operator of time of this equation as in Equation(3,4), then by using ADM to the linear part of Equation(6) with respect to the initial condition u_0, then we have the first approximate solution w_1 , then let that w_1 as the initial condition of the nonlinear part of Equation(6) and applied HPM we find the first approximate solution v_1 , and then take the average of this two solutions we have the first approximate solution of Equation(6) u_1, and so on.

$$u_0=\frac{-2e^{\frac{c\,\mathrm{Re}(\zeta_0+x)}{2}}}{c^2\,\mathrm{Re}^2}$$

$$u_1=\frac{2e^{c\,\mathrm{Re}\zeta_0}e^{c\,\mathrm{Re}x}(e^{c^2\,\mathrm{Re}t}-1)}{c^5\,\mathrm{Re}^4}-\frac{te^{\frac{c\,\mathrm{Re}(\zeta_0+x)}{2}}}{2\,\mathrm{Re}}-\frac{2te^{c\,\mathrm{Re}(\zeta_0+x)}}{c^3\,\mathrm{Re}^3}-\frac{2e^{\frac{c\,\mathrm{Re}(\zeta_0+x)}{2}}}{c^2\,\mathrm{Re}^2}-\frac{e^{\frac{c\,\mathrm{Re}\zeta_0}{2}}e^{\frac{c\,\mathrm{Re}x}{2}}(e^{\frac{c^2\,\mathrm{Re}t}{2}}-1)}{c^2\,\mathrm{Re}^2}$$

$$u_2=\frac{2e^{c\,\mathrm{Re}\zeta_0}e^{c\,\mathrm{Re}x}(e^{c^2\,\mathrm{Re}t}-1)}{c^5\,\mathrm{Re}^4}-\frac{2e^{\frac{c\,\mathrm{Re}(\zeta_0+x)}{2}}}{c^2\,\mathrm{Re}^2}-\frac{c^2t^2e^{\frac{c\,\mathrm{Re}(\zeta_0+x)}{2}}}{8}-\frac{te^{\frac{c\,\mathrm{Re}(\zeta_0+x)}{2}}}{2\,\mathrm{Re}}-\frac{2te^{c\,\mathrm{Re}(\zeta_0+x)}}{c^3\,\mathrm{Re}^3}-$$

$$\frac{(1-e^{c^2\,\mathrm{Re}t})e^{\frac{3c\,\mathrm{Re}\zeta_0}{2}}e^{\frac{3c\,\mathrm{Re}x}{2}}-16c^3\,\mathrm{Re}^2\,e^{c\,\mathrm{Re}\zeta_0}e^{c\,\mathrm{Re}x}+12c^4\,\mathrm{Re}^2\,e^{\frac{3c\,\mathrm{Re}\zeta_0}{2}}e^{\frac{3c\,\mathrm{Re}x}{2}}+24c^2\,\mathrm{Re}\,te^{\frac{3c\,\mathrm{Re}\zeta_0}{2}}e^{\frac{3c\,\mathrm{Re}x}{2}}}{2c^8\,\mathrm{Re}^6}+$$

$$\frac{16c^3\,\mathrm{Re}^2\,e^{c\,\mathrm{Re}\zeta_0}e^{c\,\mathrm{Re}x}e^{\frac{c^2\,\mathrm{Re}t}{2}}-8c^5\,\mathrm{Re}^3\,te^{c\,\mathrm{Re}\zeta_0}e^{c\,\mathrm{Re}x}}{2c^8\,\mathrm{Re}^6}-\frac{e^{\frac{c\,\mathrm{Re}\zeta_0}{2}}e^{\frac{c\,\mathrm{Re}x}{2}}(e^{\frac{c^2\,\mathrm{Re}t}{2}}-1)}{c^2\,\mathrm{Re}^2}$$

$$\vdots$$

Figure (1) explained the comparison for the three methods of one dimensional NSE at $t=0.1$ with different values of Re ;Figure (2) shows the solutions of one dimensional NSE for $\mathrm{Re}=40$ with different values of time $t=0.1,1,1.5,2$ and Table(1) explained the absolute error comparison of the present study ,ADM and HPM at center grid point.

Table 1: Absolute errors comparison of the present study, ADM and HPM at center grid for problem1

Re	t	Method → ADM	HPM	SDHPM
100	0.1	8.851×10^{-6}	1.395×10^{-6}	3.371×10^{-7}
	1.0	4.298×10^{-4}	2.149×10^{-4}	3.373×10^{-5}
40	0.1	3.663×10^{-4}	2.962×10^{-5}	7.352×10^{-6}
	1.0	8.051×10^{-3}	3.221×10^{-3}	7.497×10^{-4}
20	0.1	1.135×10^{-3}	1.397×10^{-3}	3.522×10^{-5}
	1.0	2.182×10^{-2}	1.216×10^{-2}	3.396×10^{-3}

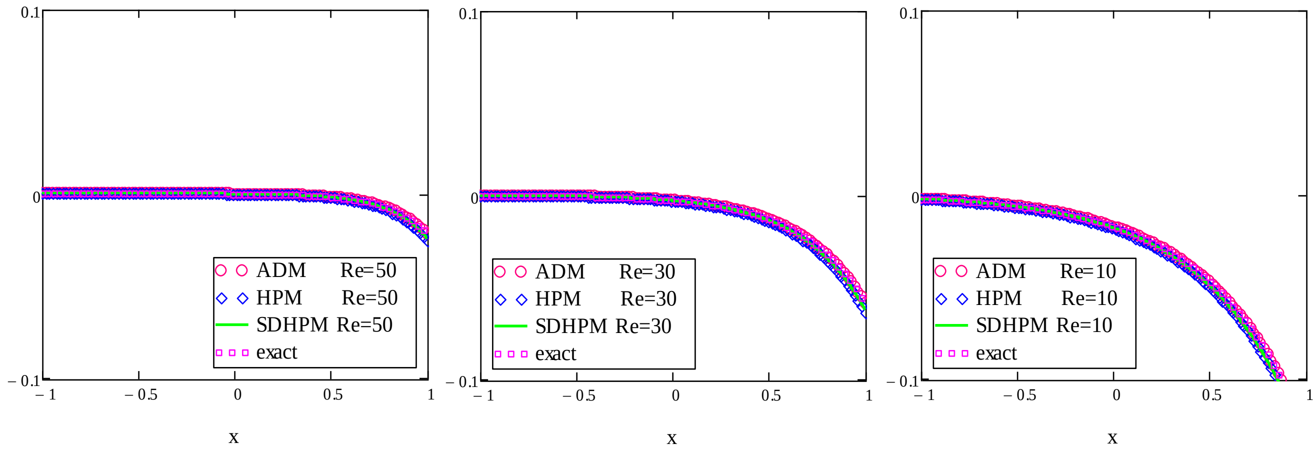

Fig.1. Solutions comparison of one dimensional NSE(p1) for t=.1 with different values of Re.

Through the table of error notes that the effect and the accuracy of the present study comparison with the other method (ADM, HPM), and the accuracy of this method increasing with increase the Reynolds number . And from Figure (1), we can also note that when $t = 0.1$ and Re increase then the accuracy and convergence of this methods also increase, and from Figure (2) we show that the approximate solution convergence to the exact solution at $t = 1$ in present study, but in ADM, HPM the approximate solution move away from the exact solution at the same time. Then from the above results, we can say that the present study is faster convergence and more accurate than the other two methods ADM and HPM.

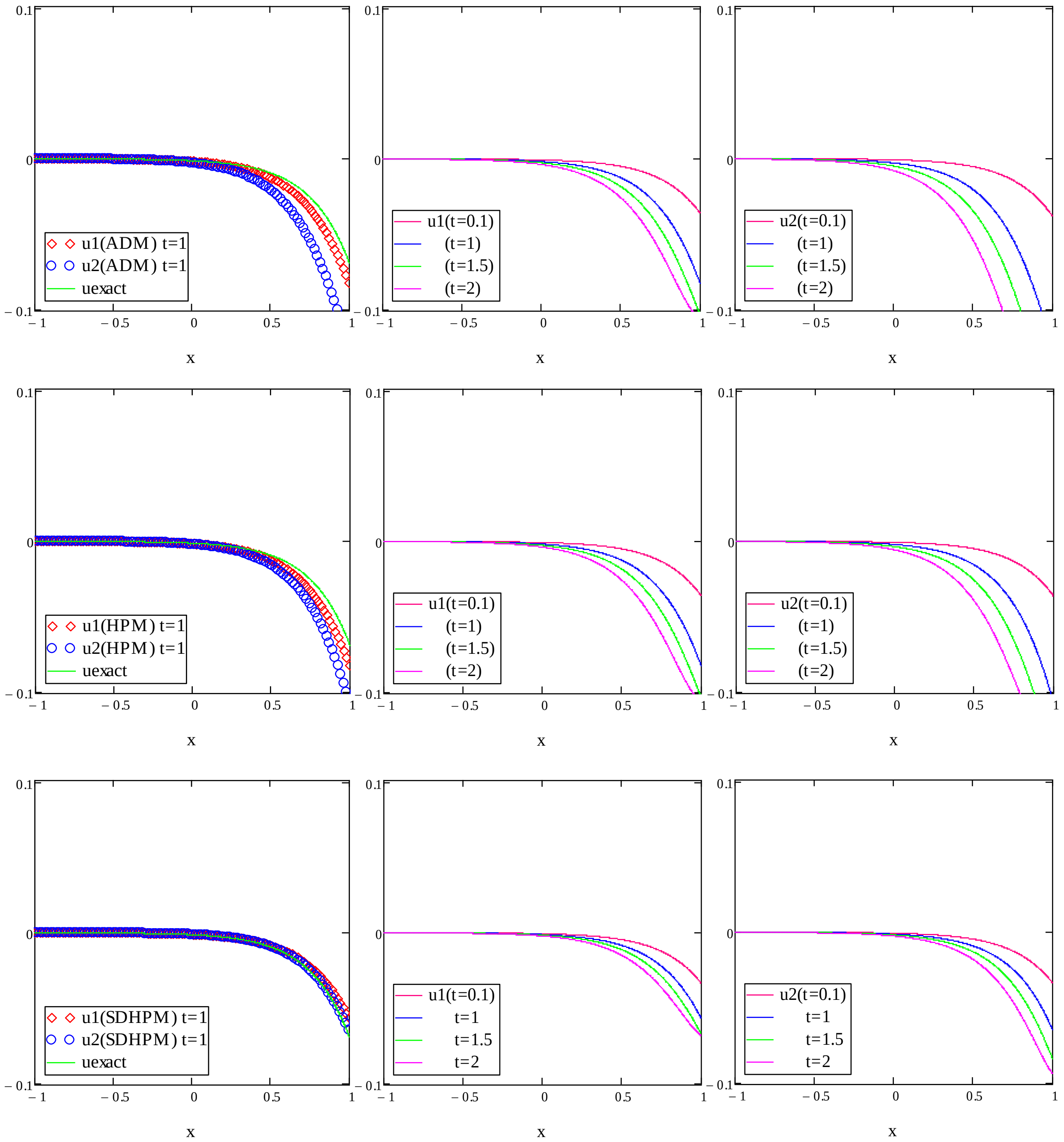

Fig. 2. Solutions of one dimensional NSE(p1) for Re=40 with different values of time.

Test of problem 2 [6]:

The equation of flow which is applied in the present study test represented by the one dimensional NSE (5).

We consider the laminar flow of viscous fluid Equation(5) behind a one-dimensional grid, with x -axis normal to the grid and the velocity field is assumed to be such that $u := U_0 + u$ where U_0 is the mean velocity(reference of velocity) in the x -direction. Thus; the one-dimensional Navier-Stokes equation with a periodicity in one direction, which may represent the wake of a one dimensional grid as the same as Equations(5) with replacing the coefficients of convective terms in x-direction by $U_0 + u$;that is: $(U_0 + u)\dfrac{\partial u}{\partial x}$ and in the non-dimensional NSE(6) these terms become $(1+u)\dfrac{\partial u}{\partial x}$.The initial condition and the pressure term of this problem are gives respectively ;

$$u_0 = 1 - e^{\lambda x} \ \& \ p_x = \frac{-\lambda e^{\lambda x}(\lambda - \mathrm{Re} + \mathrm{Re}\, e^{\lambda x})}{\mathrm{Re}}, \quad \text{where; } \lambda = \frac{(\mathrm{Re} - \sqrt{\mathrm{Re}^2 + 16\pi^2})}{2}$$

The iterative solutions of this problem by using SDHPM are respectively;

$$u_0 = 1 - e^{\lambda x}$$

$$u_1 = (1 - \lambda t\, e^{\lambda x})(1 - e^{\lambda x}) - 4\lambda^5 t^3 e^{2\lambda x}(3\,\mathrm{Re}^2)^{-1}$$

$$u_2 = (1 - \lambda t\, e^{\lambda x})(1 - e^{\lambda x}) - \lambda^4 t^3 e^{2\lambda x}[(10\,\mathrm{Re} - 20\lambda)\,\mathrm{Re} + (12\lambda^4 t^2 - 15\,\mathrm{Re}^2)e^{\lambda x}](15\,\mathrm{Re}^3)^{-1} - \lambda^4 t^3 e^{\lambda x}(3t^{-1} + 4\lambda e^{\lambda x})(3\,\mathrm{Re}^2)^{-1}$$

$$\vdots$$

Figure (3) explains the comparison of the three methods for one dimensional NSE at $t = 0.1$ with different values of Re while Figure (4) show that the solutions of one dimensional NSE for $\mathrm{Re} = 100$ with different values of time $t = 0.1, 1, 2, 3, 4, 5$. Table (2) explain the absolute errors comparison of the present study, ADM and HPM at center grid point.

Table 2:Absolute errors comparison of the present study,ADM and HPM at center grid for problem2

Re	t	Method →	ADM	HPM	SDHPM
100	0.1		1.892×10^{-3}	2.877×10^{-3}	5.880×10^{-14}
	1.0		2.044×10^{-2}	3.122×10^{-2}	**5.653×10^{-11}**
40	0.1		1.893×10^{-3}	2.793×10^{-3}	9.188×10^{-13}
	1.0		2.045×10^{-2}	3.130×10^{-2}	**8.834×10^{-10}**
20	0.1		1.893×10^{-3}	2.780×10^{-3}	7.350×10^{-12}
	1.0		2.046×10^{-2}	3.145×10^{-2}	**7.067×10^{-9}**

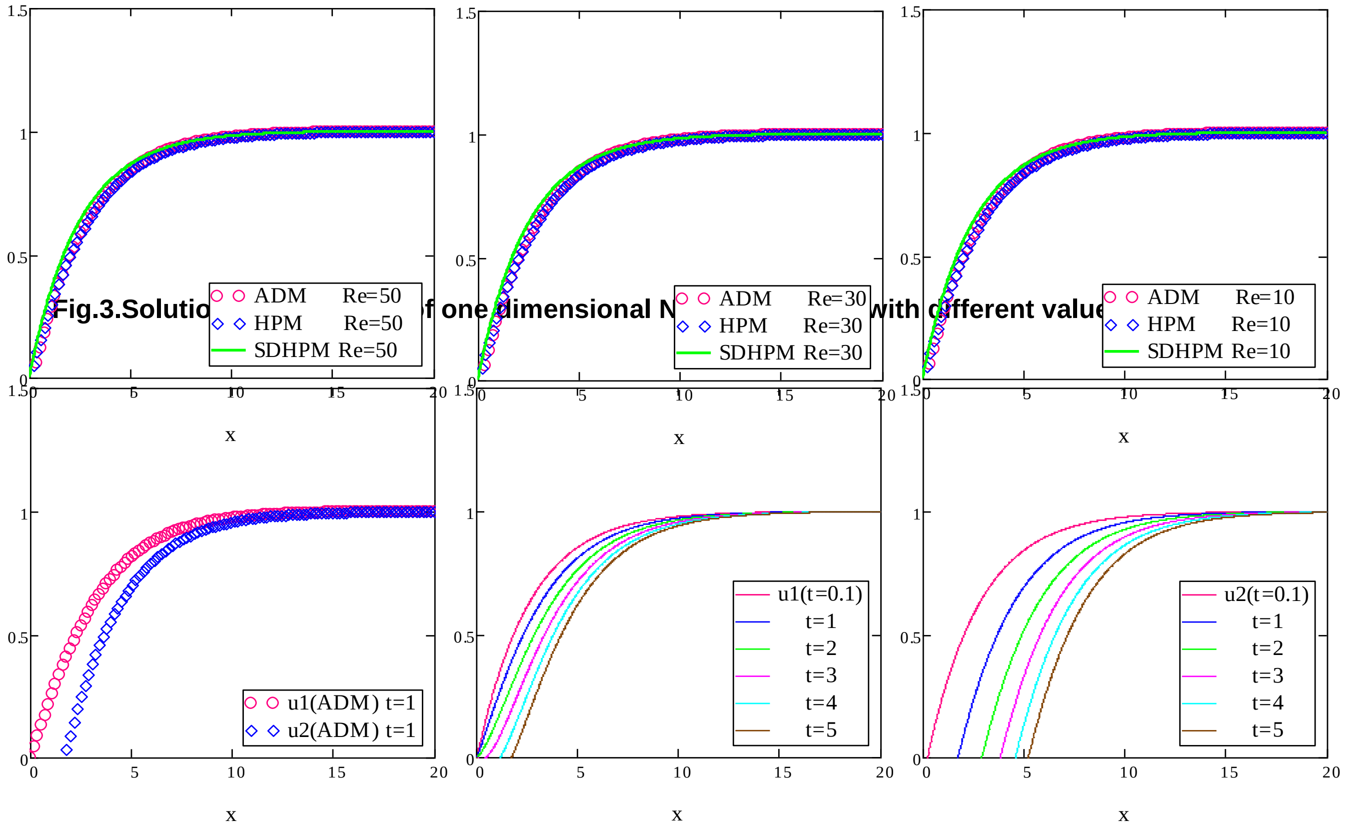

Fig.3.Solutio f one dimensional N with different value

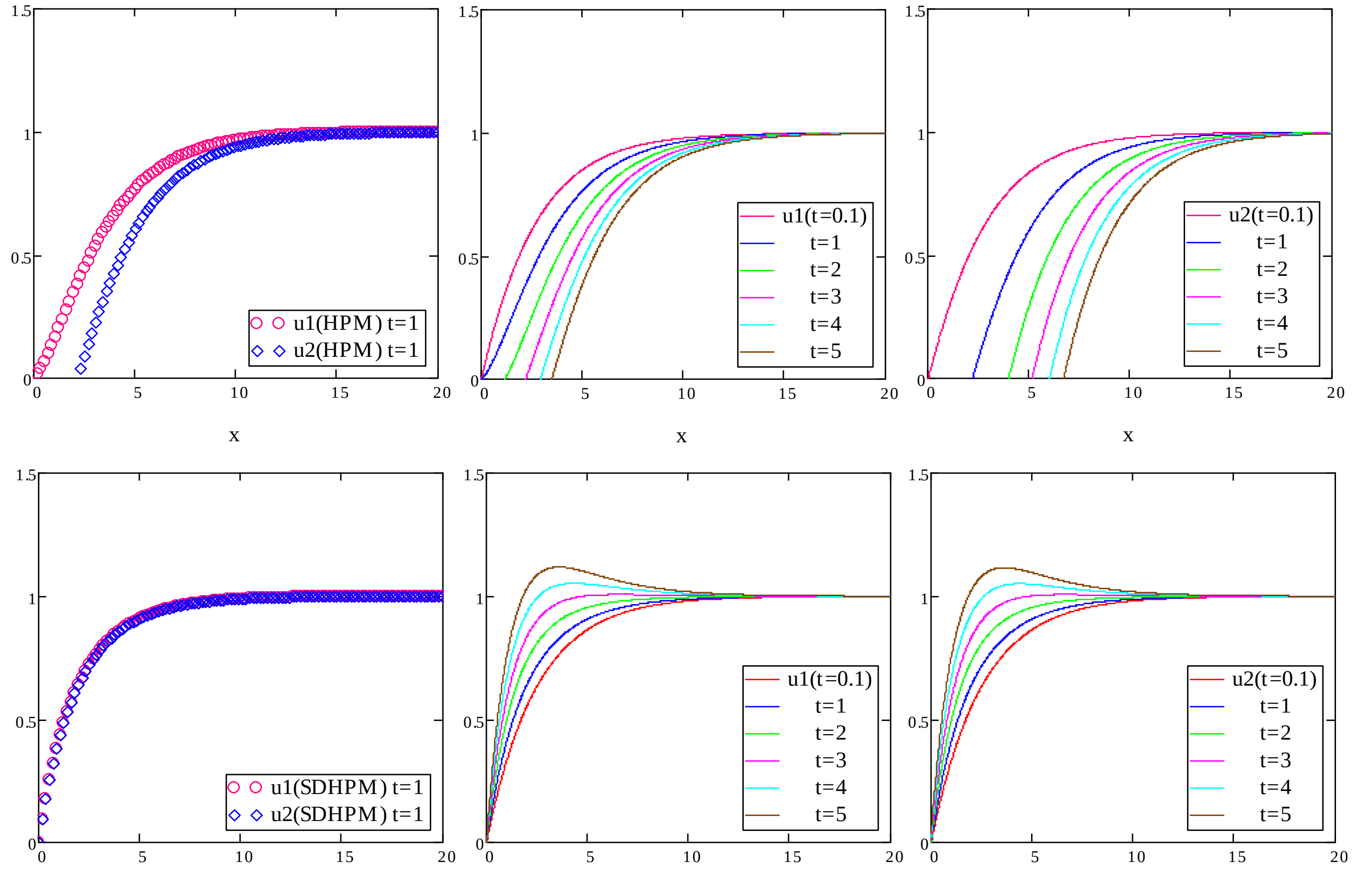

Fig. 4. Solutions of one dimensional NSE(p2) for Re=100 with different values of time.

From the table of error which explain the comparison of the absolute error of three methods (ADM, HPM, SDHPM) through it we note that when the Reynolds number increase then the accuracy and convergence of this method also increases; and the new method it's the best methods with a good convergence and high accuracy. From Figure (4) when $\mathrm{Re}=100$ and the time ($t=1$) we note that the approximate solutions of present study are stay converge together while in the other methods (ADM, HPM) the approximate solutions diverge from each other. And when $x=10$ the solutions up to one value. Therefore, we can say that the new method is the best way and with high accuracy and good convergence.

According to the theorems of convergence that are mentioned in section(4), the convergence of splitting Adomian decomposition homotopy perturbation method for the non-linear NSE (6) will be illustrate here. By using definitions (15) and(16) and suppose that

$$W_n = N(W_{n-1}), \qquad W_{n-1} = u_{n-1}$$

$$u_n = \sum_{j=0}^{n} \int_0^t \left(\frac{1}{\mathrm{Re}} \frac{\partial^2 u_j}{\partial x^2} - \frac{\partial p}{\partial x} - \sum_{k=0}^{j} u_k \frac{\partial u_{j-k}}{\partial x} \right) dt \qquad n = 1,2,3,\ldots$$

with the theorem (2) (Sufficient Condition of Convergence) for the nonlinear mapping N, a sufficient condition for convergence of the SDHPM is the strict contraction of N, we have :

$$\|u_0 - u\| = \left\| \frac{-2e^{\frac{c\,\mathrm{Re}(\zeta_0 + x)}{2}}}{c^2\,\mathrm{Re}^2} - \frac{-2e^{\frac{c\,\mathrm{Re}(\zeta_0 + x + ct)}{2}}}{c^2\,\mathrm{Re}^2} \right\|$$

$$\|u_1 - u\| \le \|u_0 - u\|\gamma \quad \gamma = 3.15e-7 < 1,$$

$$\|u_2 - u\| \le \|u_0 - u\|\gamma^2 \quad \gamma = 7.16e-10 < 1,$$

$$\vdots$$

$$\|u_n - u\| \le \|u_0 - u\|\gamma^n .$$

Therefore , $\lim_{n\to\infty}\|u_n - u\| \le \lim_{n\to\infty}\|u_0 - u\|\gamma^n = 0.$ be hold for the problem 1. Also, for the problem2,we have

$$\|u_1 - u_0\| = \|(((1-\lambda t\, e^{\lambda x})(1-e^{\lambda x}) - 4\lambda^5 t^3 e^{2\lambda x}(3\operatorname{Re}^2)^{-1})) - (1-e^{\lambda x})\|$$
$$\|u_2 - u_1\| \le \|u_1 - u_0\|\gamma \quad \gamma = 3.17e-8 < 1,$$
$$\|u_3 - u_2\| \le \|u_1 - u_0\|\gamma^2 \quad \gamma = 4.91e-9 < 1,$$
$$\vdots$$
$$\|u_n - u_{n-1}\| \le \|u_1 - u_0\|\gamma^n.$$

Therefore , $\lim_{n\to\infty}\|u_n - u_{n-1}\| \le \lim_{n\to\infty}\|u_1 - u_0\|\gamma^n = 0.$

Conclusions

The Splitting Adomian decomposition Homotopy perturbation method is used here to find the analytic approximate solution of one dimensional Navier Stokes equation that describes two test problem: one of them with the exact solution and the other without an exact solution. The results show that the SDHPM has high accuracy and efficiency to finding the analytic approximate solutions of one-dimensional NSE comparison with the standard two methods ADM and HPM. And the accuracy of solutions by using three methods increase with increasing Reynolds numbers with high convergence. Therefore ; based on the foregoing, we can say that the new method is the best and with good convergence and high accuracy to find the approximate analytic solutions at least for the one-dimensional NSE.

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