Volume 13 Number 02
Journal of Advances in Mathematics

STABILITY OF tredecic FUNCTIONAL EQUATION IN MATRIX NORMED SPACES

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ABSTRACT

In this current work, we define and find the general solution of the following tredecic functional equation

$$f(x+7y)-13f(x+6y)+78f(x+5y)-286f(x+4y)+715f(x+3y)$$
$$-1287f(x+2y)+1716f(x+y)-1716f(x)+1287f(x-y)$$
$$-715f(x-2y)+286f(x-3y)-78f(x-4y)+f(x-5y)-f(x-6y)=13!f(y)$$

where $13!=6227020800$. We also investigate and establish the generalized Ulam-Hyers stability of this functional equation in matrix normed spaces by using the fixed point method.

Introduction

In 1940, an interesting topic was presented by S. M. Ulam [18] triggered the study of stability problems for various functional equations. He addressed a question concerning the stability of homomorphism. In the following year, 1941, D. H. Hyers [5] was able to give a partial solution to Ulam's question. The result of Hyers was then generalized by Aoki [1] for additive mappings. In 1978, Th. M. Rassias [14] succeeded in extending the result of Hyers theorem by weakening the condition for the Cauchy difference.

The stability phenomenon that was presented by Th. M. Rassias is called the generalized Hyers-Ulam stability. In 1994, a generalization of the Rassias theorem was obtained by Gavruta [4] by replacing the unbounded Cauchy difference by a general control function. A further generalization of the Hyers-Ulam stability for a large class of mapping was obtained by Isac and Th. M. Rassias [6]. They also presented some applications in non-linear analysis, especially in fixed point theory. This terminology may also be applied to the cases of other functional equations [2, 3, 13, 15, 17, 20]. Also, the generalized Hyers-Ulam stability of functional equations and inequalities in matrix normed spaces has been studied by number of authors [7, 8, 9, 10, 12, 19].

K. Ravi and B. V. Senthil Kumar [16] discussed the general solution of undecic functional equation and proved the stability of this functional equation in quasi β - normed spaces by applying the fixed point method.

In this paper, we introduce the following new functional equation

$$f(x+7y)-13f(x+6y)+78f(x+5y)-286f(x+4y)+715f(x+3y)$$
$$-1287f(x+2y)+1716f(x+y)-1716f(x)+1287f(x-y)$$
$$-715f(x-2y)+286f(x-3y)-78f(x-4y)+f(x-5y)-f(x-6y)=13!f(y) \qquad (1)$$

where $13!=6227020800$ is said to be tredecic functional equation since the function $f(x)=cx^{13}$ is its solution.

In Section 2, we study the tredecic functional equation (1).

In Section 3, using the fixed point technique, we prove the Hyers-Ulam stability of the functional equation (1) in matrix normed spaces.

Tredecic Functional Equation (1)

In this section, we study the tredecic functional equation (1). For this, let us consider A and B be real vector spaces.

Theorem 1 If a mapping $f:A\to B$ satisfies the functional equation (1) for all $x,y\in A$, then $f(2x)=2^{13}f(x)$ for all $x\in A$.

Proof. Letting $x=y=0$ in (1), one gets $f(0)=0$. Replacing $x=0$, $y=x$ and $x=x$, $y=-x$ in (1) and adding the two resulting equations, we get

$$f(-x)=-f(x)$$

Hence, f is an odd mapping. Replacing $x=0$, $y=2x$ and $x=7x$, $y=x$ in (1) and Subtracting Equations the two resulting equations, we get

$$13f(13x)-90f(12x)+286f(11x)-650f(10x)+1287f(9x)$$
$$-1924f(8x)+1716f(7x)-858f(6x)+715f(5x)-858f(4x)$$
$$+78f(3x)-6227020384f(2x)+6227020801f(x)=0 \qquad (2)$$

for all $x \in A$. Replacing (x,y) by $(6x,x)$ in (1)and and multiplying by 13, we get

$$13f(13x)-169f(12x)+1014f(11x)-3718f(10x)+9295f(9x)$$
$$-16731f(8x)+22308f(7x)-22308f(6x)+16731f(5x)$$
$$-9295f(4x)+3718f(3x)-1014f(2x)+80951270230f(x)=0 \qquad (3)$$

for all $x \in A$. Subtracting Equations (2) and (3), we arrive at

$$79f(12x)-728f(11x)+3068f(10x)-8008f(9x)+14807f(8x)$$
$$+21450f(6x)-16016f(5x)+8437f(4x)-3640f(3x)$$
$$-20592f(7x)-6227019370f(2x)+87178291030f(x)=0 \qquad (4)$$

for all $x \in A$. Replacing (x,y) by $(5x,x)$ in (1) and multiplying by 79, we get

$$79f(12x)-1027f(11x)+6162f(10x)-22594f(9x)+56485f(8x)$$
$$-101673f(7x)+135564f(6x)-135564f(5x)+101673f(4x)$$
$$-56485f(3x)+22594f(2x)-491934649300f(x)=0 \qquad (5)$$

for all $x \in A$. Subtracting Equations (4) and (5), we have

$$299f(11x)-3094f(10x)+14586f(9x)-41678f(8x)+81081f(7x)$$
$$-114114f(6x)+119548f(5x)-93236f(4x)$$
$$+52845f(3x)-6227041964f(2x)+579112940300f(x)=0 \qquad (6)$$

for all $x \in A$. Replacing (x,y) by $(4x,x)$ in (1) and multiplying by 299, we obtain

$$299f(11x)-3887f(10x)+23322f(9x)-85514f(8x)+213785f(7x)$$
$$-384813f(6x)+513084f(5x)-513084f(4x)$$
$$+384813f(3x)-213486f(2x)-1861879138000f(x)=0 \qquad (7)$$

for all $x \in A$. Subtracting Equations (6) and (7), we obtain

$$793f(10x)-8736f(9x)+43836f(8x)-132704f(7x)+270699f(6x)$$
$$-393536f(5x)+419848f(4x)-6226828478f(2x)$$
$$-331968f(3x)+2440992078000f(x)=0 \qquad (8)$$

for all $x \in A$. Replacing (x,y) by $(3x,x)$ in (1) and multiplying by 793, we arrive at

$$793f(10x)-10309f(9x)+61854f(8x)-226798f(7x)+566995f(6x)$$
$$-1020591f(5x)+1360788f(4x)-1359995f(3x)$$
$$+1010282f(2x)-4938028000000f(x)=0 \qquad (9)$$

for all $x \in A$. Subtracting Equations (8) and (9), we get

$$1573 f(9x) - 18018 f(8x) + 94094 f(7x) - 296296 f(6x) + 627055 f(5x)$$
$$-940940 f(4x) + 1028027 f(3x) - 6227838760 f(2x) + 7379020078000 f(x) = 0 \quad (10)$$

for all $x \in A$. Replacing (x, y) by $(2x, x)$ in (1) and multiplying by 1573, we have

$$1573 f(9x) - 20449 f(8x) + 122694 f(7x) - 449878 f(6x)$$
$$+1124695 f(5x) - 2022878 f(4x) + 2678819 f(3x)$$
$$-2576574 f(2x) - 9795102144000 f(x) = 0 \quad (11)$$

for all $x \in A$. Subtracting Equations (10) and (11), we obtain

$$2431 f(8x) - 28600 f(7x) + 153582 f(6x) - 497640 f(5x) + 1081938 f(4x)$$
$$-1650792 f(3x) - 6225262186 f(2x) + 17174122220000 f(x) = 0 \quad (12)$$

for all $x \in A$. Replacing (x, y) by (x, x) in (1) and multiplying by 2431, we obtain

$$2431 f(8x) - 31603 f(7x) + 189618 f(6x) - 692835 f(5x) + 1706562 f(4x)$$
$$-2939079 f(3x) + 3476330 f(2x) - 15137890000000 f(x) = 0 \quad (13)$$

for all $x \in A$. Subtracting Equations (15) and (17), we arrive at

$$3003 f(7x) - 36036 f(6x) + 195195 f(5x) - 624624 f(4x) + 32312012220000 f(x)$$
$$+1288287 f(3x) - 6228738516 f(2x) = 0 \quad (14)$$

for all $x \in A$. Replacing (x, y) by $(0, x)$ in (1) and multiplying by 3003, we obtain

$$3003 f(7x) - 36036 f(6x) + 195195 f(5x) - 624624 f(4x) + 1288287 f(3x)$$
$$-1717716 f(2x) + 18699742170000 f(x) = 0 \quad (15)$$

for all $x \in A$. Subtracting Equations (14) and (15), we get

$$-6227020800 f(2x) + 51011754390000 f(x) = 0 \quad (16)$$

for all $x \in A$. From (16)

$$f(2x) = 2^{13} f(x) \qquad \text{forallx} \in A \quad (17)$$

Hence $f : A \to B$ is a tredecic mapping. This completes the proof.

Stability of Tredecic Functional Equation in Matrix Normed Spaces

In this section, we will investigate the Ulam-Hyers stability for the functional equation (1) in matrix normed spaces by using the fixed point method.

Throughout this section, let us consider $(X, \|.\|_n)$ be a matrix normed space, $(Y, \|.\|_n)$ be a matrix Banach space and let n be a fixed non-negative integer.

For a mapping $f : X \to Y$, define $Df : X^2 \to Y$ and $Df_n : M_n(X^2) \to M_n(Y)$ by, for all $a, b \in X$ and all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$.

Theorem 2 *Let* $l = \pm 1$ *be fixed and* $\psi : X^2 \to [0, \infty)$ *be a function such that there exists a* $\eta < 13$ *with*

$$\psi(a,b) \le 2^{13l}\eta\psi(\frac{a}{2^l},\frac{b}{2^l})\forall \mathrm{a,b} \in \mathrm{X}. \tag{18}$$

Let $f : X \to Y$ be a mapping satisfying

$$\left\| Df_n([x_{ij}],[y_{ij}]) \right\| \le \sum_{i,j=1}^{n} \psi(x_{ij}, y_{ij}) \forall \mathrm{x} = [\mathrm{x_{ij}}], \mathrm{y} = [\mathrm{y_{ij}}] \in \mathrm{M_n(X)}. \tag{19}$$

Then there exists a unique tredecic mapping $\mathrm{T} : X \to Y$ such that

$$\left\| f_n([x_{ij}]) - \mathrm{T}_n([y_{ij}]) \right\|_n \le \sum_{i,j=1}^{n} \frac{\eta^{\frac{1-l}{2}}}{2^{13}(1-\eta)} \overline{\psi}(x_{ij}) \forall \mathrm{x} = [\mathrm{x_{ij}}] \in \mathrm{M_n(X)}, \tag{20}$$

where $\overline{\psi}(x_{ij}) = \frac{1}{13!}\Big[\psi(0,2x_{ij}) + \psi(7x_{ij}, x_{ij}) + 13\psi(6x_{ij}, x_{ij}) + 79\psi(5x_{ij}, x_{ij})$

$+ 299\psi(4x_{ij}, x_{ij}) + 793\psi(3x_{ij}, x_{ij}) + 1573\psi(2x_{ij}, x_{ij})$

$+ 2431\psi(x_{ij}, x_{ij}) + 3003\psi(0, x_{ij})\Big].$

Proof. Substituting n=1 in (19), we obtain

$$\left\| Df(a,b) \right\| \le \psi(a,b) \tag{21}$$

Replacing (a,b) by $(0,2a)$ and in (21), we get

$$\begin{aligned} &\| f(14a) - 12f(12a) + 65f(10a) - 208f(8a) + 429f(6a) \\ &\quad - 572f(4a) - 6227020371f(2a) \| \le \psi(0,2a) \end{aligned} \tag{22}$$

for all $a \in X$. Replacing (a,b) by $(7a,a)$ in (21), we obtain

$$\begin{aligned} &\| f(14a) - 13f(13a) + 78f(12a) - 286f(11a) + 715f(10a) \\ &\quad - 1287f(9a) + 1716f(8a) - 1716f(7a) + 1287f(6a) \\ &\quad - 715f(5a) + 286f(4a) - 78f(3a) + 13f(2a) - 6227020801f(a) \| \le \psi(7a,a) \end{aligned} \tag{23}$$

for all $a \in X$. It follows from (22) and (23), we arrive at

$$\begin{aligned} &\| 13f(13a) - 90f(12a) + 286f(11a) - 650f(10a) + 1287f(9a) \\ &\quad - 1924f(8a) + 1716f(7a) - 858f(6a) + 715f(5a) - 858f(4a) \\ &\quad + 78f(3a) - 6227020384f(2a) + 6227020801f(a) \| \le \psi(0,2a) + \psi(7a,a) \end{aligned} \tag{24}$$

for all $a \in X$. Replacing (a,b) by $(6a,a)$ in (21) and multiplying by 13, we get

$$\begin{aligned} &\| 13f(13a) - 169f(12a) + 1014f(11a) - 3718f(10a) + 9295f(9a) \\ &\quad - 16731f(8a) + 22308f(7a) - 22308f(6a) + 16731f(5a) \\ &\quad - 9295f(4a) + 3718f(3a) - 1014f(2a) + 80951270230f(a) \| \le 13\psi(6a,a) \end{aligned} \tag{25}$$

for all $a \in X$. It follows from (24) and (25), we arrive at

$$\| 79f(12a) - 728f(11a) + 3068f(10a) - 8008f(9a) + 14807f(8a) - 20592f(7a)$$

$$+21450 f(6a)-16016 f(5a)+8437 f(4a)-3640 f(3a)$$
$$-6227019370 f(2a)+87178291030 f(a)\| \le \psi(0,2a)+\psi(7a,a)+13\psi(6a,a) \quad (26)$$

for all $a \in X$. Replacing (a,b) by $(5a,a)$ in (21) and multiplying by 79, we get

$$\|79 f(12a)-1027 f(11a)+6162 f(10a)-22594 f(9a)+56485 f(8a)$$
$$-101673 f(7a)+135564 f(6a)-135564 f(5a)+101673 f(4a)$$
$$-56485 f(3a)+22594 f(2a)-491934649300 f(a)\| \le 79\psi(5a,a) \quad (27)$$

for all $a \in X$. It follows from (26) and (27), we arrive at

$$\|299 f(11a)-3094 f(10a)+14586 f(9a)-41678 f(8a)+81081 f(7a)$$
$$-114114 f(6a)+119548 f(5a)-93236 f(4a)-6227041964 f(2a)+52845 f(3a)$$
$$+579112940300 f(a)\| \le \psi(0,2a)+\psi(7a,a)+13\psi(6a,a)+79\psi(5a,a) \quad (28)$$

for all $a \in X$. Replacing (a,b) by $(4a,a)$ in (21) and multiplying by 299, we get

$$\|299 f(11a)-3887 f(10a)+23322 f(9a)-85514 f(8a)+213785 f(7a)$$
$$-384813 f(6a)+513084 f(5a)-513084 f(4a)$$
$$+384813 f(3a)-213486 f(2a)-1861879138000 f(a)\| \le 299\psi(4a,a) \quad (29)$$

for all $a \in X$. It follows from (28) and (29), we obtain

$$\|793 f(10a)-8736 f(9a)+43836 f(8a)-132704 f(7a)+270699 f(6a)$$
$$-393536 f(5a)+419848 f(4a)-6226828478 f(2a)+2440992078000 f(a)$$
$$-331968 f(3a)\| \le \psi(0,2a)+\psi(7a,a)+13\psi(6a,a)+79\psi(5a,a)+299\psi(4a,a) \quad (30)$$

for all $a \in X$. Replacing (a,b) by $(3a,a)$ in (21) and multiplying by 793, we arrive at

$$\|793 f(10a)-10309 f(9a)+61854 f(8a)-226798 f(7a)+566995 f(6a)$$
$$-1020591 f(5a)+1360788 f(4a)-1359995 f(3a)$$
$$+1010282 f(2a)-4938028000000 f(a)\| \le 793\psi(3a,a) \quad (31)$$

for all $a \in X$. It follows from (30) and (31), we get

$$\|1573 f(9a)-18018 f(8a)+94094 f(7a)-296296 f(6a)+627055 f(5a)$$
$$-940940 f(4a)+1028027 f(3a)-6227838760 f(2a)+7379020078000 f(a)\|$$
$$\le \psi(0,2a)+\psi(7a,a)+13\psi(6a,a)+79\psi(5a,a)+299\psi(4a,a)+793\psi(3a,a) \quad (32)$$

for all $a \in X$. Replacing (a,b) by $(2a,a)$ in (21) and multiplying by 1573, we arrive at

$$\|1573 f(9a)-20449 f(8a)+122694 f(7a)-449878 f(6a)$$
$$+1124695 f(5a)-2022878 f(4a)+2678819 f(3a)$$

$$-2576574 f(2a)-9795102144000 f(a)\| \leq 1573\psi(2a,a) \tag{33}$$

for all $a \in X$. It follows from (32) and (33), we obtain

$$\|2431 f(8a)-28600 f(7a)+153582 f(6a)-497640 f(5a)+1081938 f(4a)$$
$$-1650792 f(3a)-6225262186 f(2a)+17174122220000 f(a)\|$$
$$\leq \psi(0,2a)+\psi(7a,a)+13\psi(6a,a)+79\psi(5a,a)+299\psi(4a,a)$$
$$+793\psi(3a,a)+1573\psi(2a,a) \tag{34}$$

for all $a \in X$. Replacing (a,b) by (a,a) in (21) and multiplying by 2431, we obtain

$$\|2431 f(8a)-31603 f(7a)+189618 f(6a)-692835 f(5a)+1706562 f(4a)$$
$$-2939079 f(3a)+3476330 f(2a)-15137890000000 f(a)\| \leq 2431\psi(a,a) \tag{35}$$

for all $a \in X$. It follows from (37) and (35), we arrive at

$$\|3003 f(7a)-36036 f(6a)+195195 f(5a)-624624 f(4a)+32312012220000 f(a)$$
$$+1288287 f(3a)-6228738516 f(2a)\|$$
$$\leq \psi(0,2a)+\psi(7a,a)+13\psi(6a,a)+79\psi(5a,a)+299\psi(4a,a)$$
$$+793\psi(3a,a)+1573\psi(2a,a)+2431\psi(a,a) \tag{36}$$

for all $a \in X$. Replacing (a,b) by $(0,a)$ in (21) and multiplying by 3003, we obtain

$$\|3003 f(7a)-36036 f(6a)+195195 f(5a)-624624 f(4a)+1288287 f(3a)$$
$$-1717716 f(2a)+18699742170000 f(a)\| \leq 3003\psi(0,a) \tag{37}$$

for all $a \in X$. It follows from (36) and (37), we get

$$\|-6227020800 f(2a)+51011754390000 f(a)\|$$
$$\leq \psi(0,2a)+\psi(7a,a)+13\psi(6a,a)+79\psi(5a,a)+299\psi(4a,a) \tag{38}$$
$$+793\psi(3a,a)+1573\psi(2a,a)+2431\psi(a,a)+3003\psi(0,a)$$

for all $a \in X$. From (38)

$$\|-f(2a)+2^{13} f(a)\| \leq \frac{1}{13!}\big[\psi(0,2a))+\psi(7a,a)+13\psi(6a,a)+79\psi(5a,a)$$
$$+299\psi(4a,a)+793\psi(3a,a)+1573\psi(2a,a)$$
$$+2431\psi(a,a)+3003\psi(0,a)\big] \tag{39}$$

Therefore,

$$\| f(2a)-2^{13} f(a)\| \leq \overline{\psi}(a) \forall \mathrm{a} \in \mathrm{X}. \tag{40}$$

Thus

$$\left\| f(a) - \frac{1}{2^{13l}} f(2^l a) \right\| \leq \frac{\eta^{\left(\frac{1-l}{2}\right)}}{2^{13}} \overline{\psi}(a) \qquad \forall a \in X. \tag{41}$$

We consider the set $M = \{f : X \to Y\}$ and introduce the generalized metric ρ on M as follows:

$$\rho(f, g) = \inf\{\mu \in R_+ : \| f(a) - g(a) \| \leq \mu \overline{\psi}(a), \forall a \in X\},$$

It is easy to check that (M, ρ) is a complete generalized metric (see also [11]). Define the mapping $P : M \to M$ by

$$Pf(a) = \frac{1}{2^{13l}} f(2^l a) \qquad \forall f \in M and a \in X.$$

Let $f, g \in M$ and ν be an arbitrary constant with $\rho(f, g) = \nu$. Then

$$\| f(a) - g(a) \| \leq \nu \overline{\psi}(a) \text{ for all } a \in X.$$

Utilizing (18), we find that

$$\| Pf(a) - Pg(a) \| = \left\| \frac{1}{2^{13l}} f(2^l a) - \frac{1}{2^{13l}} g(2^l a) \right\| \leq \eta \nu \overline{\psi}(a) \text{ for all } a \in X.$$

Hence it holds that $\rho(Pf, Pg) \leq \eta \nu$, that is, $\rho(Pf, Pg) \leq \eta \rho(f, g)$ for all $f, g \in M$.

It follows from (41) that $\rho(f, Pf) \leq \frac{\eta^{\left(\frac{1-l}{2}\right)}}{2^{13}}$.

Therefore according to Theorem 2.2 in [3], there exists a mapping $T : X \to Y$ which satisfying:

1. T is a unique fixed point of P in the set $S = \{g \in M : \rho(f, g) < \infty\}$, which is satisfied

$$T(2^l a) = 2^{13l} T(a) \qquad \forall a \in X. \tag{42}$$

In other words, there exists a μ satisfying

$$\| f(a) - g(a) \| \leq \mu \overline{\psi}(a) \forall a \in X.$$

2. $\rho(P^k f, T) \to 0$ as $k \to \infty$. This implies that

$$\lim_{k \to \infty} \frac{1}{2^{13kl}} f(2^{kl} a) = T(a) \qquad \forall a \in X.$$

3. $\rho(f, T) \leq \frac{1}{1-\eta} \rho(f, Pf)$, which implies the inequality $\rho(f, T) \leq \frac{\eta^{\frac{1-l}{2}}}{2^{13}(1-\eta)}$.

$$\text{So} \quad \| f(a) - T(a) \| \leq \frac{\eta^{\frac{1-l}{2}}}{2^{13}(1-\eta)} \overline{\psi}(a) \qquad \forall a \in X. \tag{43}$$

It follows from (18) and (19) that

$$\| DT(a, b) \| = \lim_{k \to \infty} \frac{1}{2^{13kl}} \| Df(2^{kl} a, 2^{kl} b) \|$$

$$\leq \lim_{k\to\infty} \frac{1}{2^{13kl}} \psi(2^{kl} a, 2^{kl} b) \leq \lim_{k\to\infty} \frac{2^{kl}\eta^{k}}{2^{13kl}} \psi(a,b) = 0$$

for all $a,b \in X$. Hence

for all $a,b \in X$. Therefore, the mapping $\mathrm{T}: X \to Y$ is tredecic mapping. By Lemma 2.1 in [9] and (43),

$$\left\| f_n([x_{ij}]) - \mathrm{T}_n([x_{ij}]) \right\| \leq \sum_{i,j=1}^{n} \left\| f(x_{ij}) - \mathrm{T}(x_{ij}) \right\|$$

$$\leq \sum_{i,j=1}^{n} \frac{\eta^{\frac{1-l}{2}}}{2^{13}(1-\eta)} \overline{\psi}(x_{ij}) \forall \mathrm{x} = [\mathrm{x_{ij}}] \in \mathrm{M_n(X)},$$

where $\overline{\psi}(x_{ij}) = \frac{1}{13!}\big[\psi(0,2x_{ij}) + \psi(7x_{ij}, x_{ij}) + 13\psi(6x_{ij}, x_{ij}) + 79\psi(5x_{ij}, x_{ij})$

$$+299\psi(4x_{ij}, x_{ij}) + 793\psi(3x_{ij}, x_{ij}) + 1573\psi(2x_{ij}, x_{ij})$$

$$+2431\psi(x_{ij}, x_{ij}) + 3003\psi(0, x_{ij})\big].$$

Thus $\mathrm{T}: X \to Y$ is a unique tredecic mapping satisfying (20).

Corollary 1 Let $l = \pm 1$ be fixed and let t, ϑ be positive real numbers with $t \neq 13$. Let $f: X \to Y$ be a mapping such that

$$\left\| \mathrm{D}f_n([x_{ij}],[y_{ij}]) \right\|_n \leq \sum_{i,j=1}^{n} \vartheta\left(\left\| x_{ij} \right\|^t + \left\| y_{ij} \right\|^t \right) \forall \mathrm{x} = [\mathrm{x_{ij}}], \mathrm{y} = [\mathrm{y_{ij}}] \in \mathrm{M_n(X)}. \quad (44)$$

Then there exists a unique tredecic mapping $\mathrm{T}: X \to Y$ such that

$$\left\| f_n([x_{ij}]) - \mathrm{T}_n([x_{ij}]) \right\|_n \leq \sum_{i,j=1}^{n} \frac{\vartheta_s}{l(2^{13} - 2^t)} \left\| x_{ij} \right\|^t \quad \forall \quad x = [x_{ij}] \in M_n(X),$$

where $\vartheta_s = \frac{\vartheta}{13!}[3003 + 2432(2^t) + 1573(3^t) + 793(4^t) + 299(5^t) + 79(6^t) + 13(7^t) + 8^t]$

Proof. The proof follows from Theorem 2 by taking $\psi(a,b) = \vartheta(\|a\|^t + \|b\|^t)$ for all $a,b \in X$. Then we can choose $\eta = 2^{l(t-13)}$, and we can obtain the required result.

Example 3 *Let* $\psi: \mathrm{R} \to \mathrm{R}$ *be a function defined by*

$$\psi(x) = \begin{cases} \vartheta x^{13}, & if\, |x| < 1 \\ \vartheta, & otherwise \end{cases}$$

where $\vartheta > 0$ is a constant, and define a function $f: \mathrm{R} \to \mathrm{R}$ by

$$f(x) = \sum_{n=0}^{\infty} \frac{\psi(2^n x)}{2^{13n}}$$

for all $x \in \mathrm{R}$. Then f satisfies the inequality

$$\| f(x+7y) - 13f(x+6y) + 78f(x+5y) - 286f(x+4y) + 715f(x+3y)$$

$$-1287f(x+2y)+1716f(x+y)-1716f(x)+1287f(x-y)$$
$$-715f(x-2y)+286f(x-3y)-78f(x-4y)+f(x-5y)$$
$$-f(x-6y)-13!f(y)\| \le \frac{(6227028992)\vartheta}{8191}(8192)^2(|x|^{13}+|y|^{13}) \tag{45}$$

for all $x, y \in \mathrm{R}$. Then there do not exists a tredecic mapping $\mathrm{T}:\mathrm{R}\to\mathrm{R}$ and a constant $\lambda > 0$ such that

$$|f(x)-\mathrm{T}(x)| \le \lambda |x|^{13} \qquad \forall \mathrm{x} \in \mathrm{R}. \tag{46}$$

Proof. It is easy to see that f is bounded by $\frac{8192\vartheta}{8191}$ on R.

Next we have to show that f satisfies (45).

If $x = y = 0$, then (45) is trivial.

If $|x|^{13}+|y|^{13} \ge \frac{1}{2^{13}}$, then L.H.S of (45) is less than $\frac{(6227028992)(8192)\vartheta}{8191}$.

Suppose that $0<|x|^{13}+|y|^{13}<\frac{1}{2^{13}}$, then there exists a non-negative integer k such that

$$\frac{1}{2^{13(k+1)}} \le |x|^{13}+|y|^{13} < \frac{1}{2^{13k}}, \tag{47}$$

so that $2^{13(k-1)}x^{13}<\frac{1}{2^{13}}, 2^{13(k-1)}y^{13}<\frac{1}{2^{13}}$, and

$2^n(x), 2^n(y), 2^n(x+7y), 2^n(x+6y), 2^n(x+5y), 2^n(x+4y),\quad 2^n(x+3y), 2^n(x+2y), 2^n(x+y), 2^n(x-y), 2^n($

for all $n = 0,1,2,\ldots,k-1$. Hence

$$\psi(2^n(x+7y))-13\psi(2^n(x+6y))+78\psi(2^n(x+5y))-286\psi(2^n(x+4y))$$
$$+715\psi(2^n(x+3y))-1287\psi(2^n(x+2y))+1716\psi(2^n(x+y))$$
$$-1716\psi(2^n x)+1287\psi(2^n(x-y))-715\psi(2^n(x-2y))+286\psi(2^n(x-3y))$$
$$-78\psi(2^n(x-4y))+\psi(2^n(x-5y))-\psi(2^n(x-6y))-13!\psi(2^n y)=0$$

for $n = 0,1,2,\ldots,k-1$. From the definition of f and (47), we obtain that

$$|f(x+7y)-13f(x+6y)+78f(x+5y)-286f(x+4y)+715f(x+3y)$$
$$-1287f(x+2y)+1716f(x+y)-1716f(x)+1287f(x-y)-715f(x-2y)$$
$$+286f(x-3y)-78f(x-4y)+f(x-5y)-f(x-6y)-13!f(y)|$$
$$\le \sum_{n=0}^{\infty}\frac{1}{2^{13n}}|\psi(2^n(x+7y))-13\psi(2^n(x+6y))+78\psi(2^n(x+5y))$$
$$-286\psi(2^n(x+4y))+715\psi(2^n(x+3y))-1287\psi(2^n(x+2y))$$
$$+1716\psi(2^n(x+y))-1716\psi(2^n x)+1287\psi(2^n(x-y))$$

$$-715\psi(2^n(x-2y))+286\psi(2^n(x-3y))-78\psi(2^n(x-4y))$$
$$+\psi(2^n(x-5y))-\psi(2^n(x-6y))-13!\psi(2^n y)\Big|$$
$$\leq \sum_{n=k}^{\infty}\frac{(6227028992)\vartheta}{2^{13n}}=\frac{(8192)(6227028992)\vartheta}{2^{13k}8191}$$
$$\leq \frac{(6227028992)}{8191}(8192)^2\vartheta(|x|^{13}+|y|^{13}).$$

Thus f satisfies (45) for all $x, y, z \in \mathbb{R}$ with $0<|x|^{13}+|y|^{13}<\frac{1}{2^{13}}$.

Now, we claim that the tredecic functional equation (1) is not stable for $t=13$ in corollary 1. Suppose that there exists a tredecic mapping $\mathrm{T}:\mathbb{R}\to\mathbb{R}$ and a constant $\lambda>0$ satisfying (46). Then there exists a constant $c\in\mathbb{R}$ such that $\mathrm{T}(x)=cx^{19}$ for any $x\in\mathbb{R}$. Thus we obtain the following inequality

$$|f(x)|\leq(\lambda+|c|)|x|^{13} \tag{48}$$

Let $m\in\mathbb{N}$ with $m\varepsilon>\lambda+|c|$. If $x\in(0,\frac{1}{2^{m-1}})$, then $2^n x\in(0,1)$ for all $n=0,1,2,\ldots,m-1$, and for this case we get

$$f(x)=\sum_{n=0}^{\infty}\frac{\psi(2^n x)}{2^{13n}}\geq\sum_{n=0}^{m-1}\frac{\vartheta(2^n x)^{13}}{2^{13n}}=m\vartheta x^{13}>(\lambda+|c|)|x|^{13}$$

which is a contradiction to (48). Thus the tredecic functional equation (1) is not stable for $t=13$.

Conclusion

In this investigation, we identified a general solution of tredecic functional equation and establised the generalized Ulam - Hyers stability of this functional equation in matrix normed spaces by using the fixed point method and also provided an example for non-stability.

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