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SOLUTIONS OF FRATIONAL EMDEN-FOWLER EQUATIONS BY HOMOTOPY ANALYSIS METHOD

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Abstract:

In this paper, we have solved the singular initial value problems of fractional Emden-Fowler type equations by using the homotopy analysis method. The approximate analytical solution of this type equations are obtained.

Keyword:

fractional Emden-Fowler type equations, homotopy analysis method, Caputo fractional derivative

1.Introduction

Consider the following fractional Emden-Fowler type equations:

$$D^{2\alpha} y + \frac{2}{x} D^{\alpha} y + af(x)g(y) = 0,\ x > 0,\ 0 < \alpha \le 1, \tag{1}$$

subject to the conditions

$$y(0) = A,\ y'(0) = 0, \tag{2}$$

where A and a are constants, $f(x)$ and $g(y)$ are given functions, and the symbol $D^{\lambda}(\lambda > 0)$ denote the fractional derivative of order λ in Caputo sense.

The Eq.(1) was used to model many phenomena in mathematical physics, such as the thermal behavior of a spherical cloud of gas, isothermal gas sphere and theory of thermionic currents etc. Because of the singularity behavior at the origin, the solution of the Eq.(1) is numerically challenging[1-12].

For the case $\alpha = 1$, several researchers have obtained the approximate analytical solution to this equation by using Adomian decomposition method and homotopy perturbation method[1,2,11,12].

The purpose of the present work is to use the homotopy analysis method(HAM) [3,4]to obtain the approximate analytical solutions of the Emden-Fowler type equations. The HAM is an analytical approach to get the series solution of linear or nonlinear fractional differential equations. The method provides great freedom to choose base function to approximate the linear and nonlinear problems. Main advantage of the method is that one can construct a continuous mapping of an initial guess approximation to the exact solution of the given problem through an auxiliary linear operator and to ensure the convergence of the series solution an auxiliary parameter is used. Recently Liao [3,4] has claimed that the difference from the other analytical methods is that one can ensure the convergence of series solution by means of choosing a proper value of convergence-control parameter.

The paper has been organized as follows. In section 2 the Homotopy Analysis Method is described. In section 3 HAM is applied for the fractional Emden-Fowler type equations. Conclusion is presented in Section 4.

2.Homotopy Analysis Method (HAM)

Let us consider the following differential equation

$$N(u(r,t))=0, \qquad (3)$$

where *N* is a linear or nonlinear operator, *r* and *t* are independent variables, *u*(*r*, *t*) is an unknown function, respectively. For simplicity, here we ignore all boundary or initial conditions, which can be treated in the similar way. By means of generalizing the traditional homotopy method, Liao [4] constructs the so called zero order deformation equation

$$(1-p)L\left[\phi(r,t;p)-u_0(x,t)\right]=phH(r,t)N\left[\phi(r,t;p)\right], \qquad (4)$$

where $p\in[0,1]$ is the embedding parameter, h is a nonzero auxiliary parameter, $H(r,t)$ is non zero auxiliary function, L is an auxiliary linear operator, $u_0(r,t)$ is an initial guess of $u(r,t)$, $\phi(r,t;p)$ is a unknown function, respectively. It is important, that one has great freedom to choose auxiliary things in HAM. Obviously, when *p* = 0 and *p* =1, it holds

$$\phi(r,t;0)=u_0(r,t), \phi(r,t;1)=u(r,t),$$

respectively. Thus, as p increases from 0 to 1, the solution $\phi(r,t;p)$ varies from the initial guesses $u_0(r,t)$ to the solution $u(r,t)$. Expanding $\phi(r,t;p)$ in Taylor series with respect to p, we have

$$\phi(r,t;p)=u_0(r,t)+\sum_{k=1}^{+\infty}u_k(r,t)p^k, \qquad (5)$$

where

$$u_k(r,t)=\frac{1}{k!}\frac{\partial^k\phi(r,t;p)}{\partial p^k}\bigg|_{p=0}.$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter h, and the auxiliary function are so properly chosen, the series (5) converges at *p* = 1, then we have

$$u(r,t)=u_0(r,t)+\sum_{k=1}^{+\infty}u_k(r,t). \qquad (6)$$

Define the vector

$$\vec{u}_n=\{u_0(r,t),u_1(r,t),\cdots,u_n(r,t)\}.$$

Differentiating equation (4) k times with respect to the embedding parameter p and then setting $p=0$ and finally dividing them by $k!$, we obtain the kth order deformation equation

$$L[u_k(r,t)-\chi_k u_{k-1}(x,t)] = hH(r,t)R_k(\vec{u}_k, r,t), \quad (7)$$

where

$$R_k(\vec{u}_{k-1}, r,t) = \frac{1}{(k-1)!}\frac{\partial^{k-1}N[\phi(r,t;p)]}{\partial p^{k-1}}\bigg|_{p=0}$$

and

$$\chi_k = \begin{cases} 0, & k \le 1, \\ 1, & k > 1. \end{cases}$$

Applying L^{-1} on both side of equation (7), we get

$$u_k(r,t) = u_{k-1} + hL^{-1}[H(r,t)R_k(\vec{u}_k, r,t)]. \quad (8)$$

In this way, it is easily to obtain u_k for $k \ge 1$, at Mth order, we have

$$u(r,t) = \sum_{k=0}^{M} u_k(r,t).$$

When $M \to \infty$, we get an accurate approximation of the original equation (3). For the convergence of the above method we refer the reader to Liao [4]. If equation (3) admits unique solution, then this method will produce the unique solution. If equation (3) does not possess unique solution, the HAM will give a solution among many other (possible) solutions.

3.The solutions of Eq. (1)

To solve the problem (1) and (2) by HAM, we choose the initial approximation

$$y_0(x) = A,$$

and the linear operator

$$L[\phi(x;p)] = xD^{2\alpha}\phi(x;p). \quad (9)$$

Furthermore, we construct the zeroth-order deformation equation

$$(1-p)L[\phi(x;p)-u_0(x;0)] = ph[xD^{2\alpha}\phi + 2D^{\alpha}\phi + axf(x)g(y)], \quad (10)$$

Obviously, when $p=0$ and $p=1,$ we have

$$\phi(x;0)=y_0(x), \phi(x;1)=y(x).$$

Expanding $\phi(x;p)$ in Taylor series with respect to p ,we have:

$$\phi(r;p)=y_0(x)+\sum_{k=1}^{+\infty} y_k(x)p^k, \tag{11}$$

where $y_k(x)\,(k=1,2,\cdots)$ will be determined later.

Note that the above series contains the convergence control parameter h. If it is chosen so property that the above series is convergent at $p=1,$ then

$$y(x)=\sum_{k=0}^{\infty} y_k(x). \tag{12}$$

Substituting (11) into the zeroth-order deformation Eq.(11), equating the coefficients of the like powers of $p,$ we have kth deformation equation:

$$L[y_k(x)-\chi_k y_{k-1}(x)]=hR_k(\vec{y}_k), k\geq 1, \tag{13}$$

subject to initial condition

$$y_k(0)=0,\ y_k'(0)=0,$$

where

$$R_k(\vec{y}_{k-1})=xD^{2\alpha}y_{k-1}+2D^{\alpha}y_{k-1}+af(x)G_{k-1,}$$

and

$$G_{k-1}=\frac{1}{(k-1)!}\frac{\partial^{k-1}}{\partial p^{k-1}}\left[g(\sum_{i=0}^{\infty}y_i p^i)\right]_{p=0}.$$

The solution of the kth deformation equation (13) for $k\geq 1$ can be obtained

$$y_k(x)=\chi_k y_{k-1}(x)+hL^{-1}[R_k(\vec{y}_k)].$$

Finally, we have

$$y(x)=y_0(x)+y_1(x)+y_2(x)+\cdots.$$

For example, if $A=1,$ and $f(x)=x^m, g(y)=y^n,$ then

$$y_0(x)=1,$$

$$y_1(x)=\frac{ah\Gamma(m+1)}{\Gamma(1+m+2\alpha)}x^{m+2\alpha},$$

$$y_2(x)=\frac{ah(1+h)\Gamma(1+m)}{\Gamma(1+m+2\alpha)}x^{m+2\alpha}$$

$$+\frac{2ah^2\Gamma(m+1)\Gamma(m+\alpha)}{\Gamma(m+1+\alpha)\Gamma(m+3\alpha)}x^{m+3\alpha-1}$$

$$+\frac{na^2h^2\Gamma(m+1)\Gamma(2m+2\alpha+1)}{\Gamma(m+1+2\alpha)\Gamma(2m+4\alpha+1)}x^{2m+4\alpha}.$$

Preceding in this way the rest of the components $y_k, k\geq 3$ can be completely obtained, and the series solutions are thus entirely determined.

By the above solutions, we can obtain the following exact solutions:

$$y(x)=A-\frac{1}{6}ax^3, (f(x)=g(y)=1,\alpha=1)$$

$$y(x)=A-\frac{a}{(m+2)(m+3)}x^{m+2}, (f(x)=x^m, g(y)=1,\alpha=1)$$

which is agreement with the result obtained in [1].

4.Conclusion

We have solved the singular initial value problems of fractional Emden-Fowler type equations based on the HAM. Our purpose has been achieved by formally deriving analytical approximations with a high degree of accuracy. We think that the method has great potential and can be applied to other strong nonlinear fractional differential equation.

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