On adjacent vertex distinguishing total coloring of quadrilateral snake

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ABSTRACT

In this paper, we prove the existence of the adjacent vertex distinguishing total coloring of quadrilateral snake, double quadrilateral snake, alternate quadrilateral snake and double alternate quadrilateral snake in detail. Also, we present an algorithm to obtain the adjacent vertex distinguishing total coloring of these quadrilateral graph family. The minimum number of colors required to give an adjacent vertex distinguishing total coloring (abbreviated as AVDTC) to the graph G is denoted by $\chi_{avt}(G)$.

Keywords: simple graph; adjacent vertex distinguishing total coloring; adjacent vertex distinguishing total chromatic number; quadrilateral snake; double quadrilateral snake; alternate quadrilateral snake and double alternate quadrilateral snake.

AMS Mathematics Subject Classification : 05C85, 05C15, 05C62, 05C76

1 Introduction

In this paper, all graphs are finite, simple and undirected. A graph G consists of a set of vertices $V(G)$ and a set of edges $E(G)$. For every vertex $u,v \in V(G)$, the edge connecting two vertices is denoted by $uv \in E(G)$. The degree of a vertex v of a graph G is denoted by $deg(v)$. Let $\Delta(G)$ denote the *maximum degree* of a graph G. For standard terminology and concepts of graph theory, we refer [1], [2], [3]. For graphs G_1 and G_2, we let $G_1 \cup G_2$ denotes their union, that is, $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$. Let n be any real number. Then $\lfloor n \rfloor$ stands for the largest integer less than or equal to n. [6] It has been shown that, if a simple graph G has two adjacent vertices of maximum degree, then $\psi_{avt}(G) \geq \Delta(G)+2$. Otherwise, if a simple graph G does not have two adjacent vertices of maximum degree, then $\psi_{avt}(G) = \Delta(G)+1$. The adjacent vertex distinguishing total chromatic number of triangular snake family has been obtained in the literature [4].

2 Preliminary Results

In this section, we write some basic definitions and results which are needed for next section.

Definition 2.1 The Quadrilateral snake Q_n is obtained from path P_n with $\{v_1, v_2, \cdots v_n\}$ vertices by replacing each edge of the path by C_4 with a new vertices $\{u_1, u_2, \cdots, u_{n-1}\}$ and $\{w_1, w_2, \cdots, w_{n-1}\}$.

$$V[Q_n] = \left\{ \left(\bigcup_{i=1}^{n} v_i \right) \cup \left(\bigcup_{i=1}^{n-1} (u_i \cup w_i) \right) \right\}$$

$$E[Q_n]=\left\{\bigcup_{i=1}^{n-1}(v_iv_{i+1}\cup v_iu_i\cup v_{i+1}w_i\cup u_iw_i)\right\}$$

Definition 2.2 The Alternate Quadrilateral snake $A(Q_n)$ is obtained from the path P_n with every alternate edge of a path is replaced by C_4. Here, the quadrilateral starts with either the vertex v_1 or with v_2.

Definition 2.3 The Double Quadrilateral snake $D(Q_n)$ consists of two quadrilateral snakes that have a common Path using the new vertices u_i, w_i ,u_i' and w_i' for $i=1,2,\cdots,n-1$. The vertex set and edge set is given by

$$V[D(Q_n)]=\left\{\left(\bigcup_{i=1}^{n}v_i\right)\cup\left(\bigcup_{i=1}^{n-1}(u_i\cup w_i\cup u_i'\cup w_i')\right)\right\}$$

$$E[D(Q_n)]=\left\{\bigcup_{i=1}^{n-1}(v_iv_{i+1}\cup v_iu_i\cup v_{i+1}w_i\cup u_iw_i\cup u_i'w_i'\cup v_iu_i'\cup v_{i+1}w_i')\right\}$$

Definition 2.4 The [5] Alternate Double Quadrilateral snake $DA(Q_n)$ consists of two Alternate Quadrilateral snakes that have a common Path.

Definition 2.5 A total k-coloring of G is a mapping $f:V(G)\cup E(G)\rightarrow\{1,2,\cdots,k\}$, $k\in\mathbb{Z}^+$ such that any two adjacent or incident elements in $V(G)\cup E(G)$ have different colors. A proper total k-coloring of G is adjacent vertex distinguishing, if $C_f(u)\neq C_f(v)$ whenever $uv\in E(G)$, the color set of the vertex v (with respect to f), we denote $C_f(v)$ as $C(v)$.

$$\begin{aligned} C(v) &= \{\{f(v)\}\cup\{f(vw)\mid vw\in E(G)\}\} \\ \overline{C}(v) &= \{1,2,\cdots,k\}\setminus C(v) \end{aligned}$$

The well-known $AVDTC$ conjecture, made by Zhang et al [6] says that every simple graph G has $\chi_{avt}(G)\leq\Delta(G)+3$.

3. AVDTC of Q_n and $A(Q_n)$

In this section, we present an algorithm to obtain the adjacent vertex distinguishing total chromatic number of quadrilateral snake and alternate quadrilateral snake and also we discussed their color classes.

Algorithm 3.1 Procedure: Adjacent vertex distinguishing total coloring of Quadrilateral

snake $Q(n)$, for $n\geq 4$.

Input: $G(V(Q_n),E(Q_n))$

for $i=1,2,\cdots,n$ **do**

if $i\equiv 1(mod\ 2)$

$f(v_i)\leftarrow 1$

else

$f(v_i) \leftarrow 2$

end for

for $1 \le i \le n-1$ **do**

$f(u_i w_i) \leftarrow 2$; $f(u_i) \leftarrow 3$; $f(w_i) \leftarrow 4$; $f(u_i v_i) \leftarrow 5$; $f(w_i v_{i+1}) \leftarrow 6$

if $i \equiv 1 \,(mod\ 2)$

$f(v_i v_{i+1}) \leftarrow 3$

else

$f(v_i v_{i+1}) \leftarrow 4$

end for

end procedure

Output: Adjacent vertex distinguishing total colored of Q_n, for $n \ge 4$.

Theorem 3.1 The Quadrilateral snake Q_n admits $AVDTC$ and

$$\chi_{avt}(Q_n) = 6, \ \ n \ge 4.$$

Proof. From the definition (2.1), we have the vertex and edge set of Q_n.

Therefore $|V(Q_n)| = 3n-2$ and $|E(Q_n)| = 4(n-1)$.

Also, $deg(v_1) = 2, deg(v_n) = 2$, $deg(v_i) = 4$ for $2 \le i \le n-1$

and $deg(u_i) = deg(w_i) = 2$ for $i = 1\, to\ n-1$.

The graph is colored using algorithm (3.1). Now the color classes for $n \ge 4$ is given by

$$C(v_1) = \{1,3,5\}\ and\ C(v_n) = \begin{cases} \{2,3,6\}, if\ \ n \equiv 0\,(mod\ 2) \\ \{1,4,6\}, if\ \ n \equiv 1\,(mod\ 2) \end{cases}$$

For $2 \le i \le n-1$

$$\overline{C}(v_i) = \begin{cases} \{1\}, if\ \ i \equiv 0\,(mod\ 2) \\ \{2\}, if\ \ i \equiv 1\,(mod\ 2) \end{cases}$$

For $1 \le i \le n-1$

$$C(u_i) = \{2,3,5\}\ and\ C(w_i) = \{2,4,6\}$$

Clearly, the color classes of any two adjacent vertices are different.

$$\therefore \ \ \chi_{avt}(Q_n) = 6, \quad n \ge 4.$$

Algorithm 3.2 Procedure: Adjacent vertex distinguishing total coloring of $A(Q_n)$

for $n \ge 4$.

Input:

$$V[A(Q_n)] \leftarrow \begin{cases} \left(\bigcup_{i=1}^{n} v_i\right) \cup \left(\bigcup_{i=1}^{\left\lfloor \frac{n}{2} \right\rfloor} (u_i \cup w_i)\right), & for\ n \equiv 1 (mod\ 2) \\ \begin{cases} \left(\bigcup_{i=1}^{n} v_i\right) \cup \left(\bigcup_{i=1}^{\left(\frac{n}{2}\right)} (u_i \cup w_i)\right), & if\ \ deg(v_1) = deg(v_n) = 2 \\ \left(\bigcup_{i=1}^{n} v_i\right) \cup \left(\bigcup_{i=1}^{\left\lfloor \frac{n-1}{2} \right\rfloor} (u_i \cup w_i)\right), & if\ \ deg(v_1) = deg(v_n) = 1 \end{cases}, & for\ n \equiv 0 (mod\ 2) \end{cases}$$

for $i = 1\, to\ n$ **do**

if $i \equiv 1\,(mod\ 2)$

$f(v_i) \leftarrow 1$

else

$f(v_i) \leftarrow 2$

end for

for $i = 1\, to\ n-1$ **do**

if $i \equiv 1\,(mod\ 2)$

$f(v_i v_{i+1}) \leftarrow 3$

else

$f(v_i v_{i+1}) \leftarrow 4$

end for

if $n \equiv 1\,(mod\ 2)$

for $i = 1\, to \left\lfloor \frac{n}{2} \right\rfloor$ **do**

$f(u_i) \leftarrow 3$; $f(w_i) \leftarrow 4$; $f(u_i w_i) \leftarrow 2$

if $deg(v_n) = 1$

$f(v_{2i-1} u_i) = f(v_{2i} w_i) \leftarrow 5$

else

$f(v_{2i+1} w_i) = f(v_{2i} u_i) \leftarrow 5$

end for

else

if $deg(v_n) = deg(v_1) = 2$

for $i = 1\, to \left(\frac{n}{2}\right)$ **do**

$f(u_i) \leftarrow 3;\ f(w_i) \leftarrow 4;\ f(u_i w_i) \leftarrow 2;\ f(v_{2i-1} u_i) = f(v_{2i} w_i) \leftarrow 5$

end for

else

for $i = 1\, to \left\lfloor \frac{n-1}{2} \right\rfloor$ **do**

$f(u_i) \leftarrow 3;\ f(w_i) \leftarrow 4;\ f(u_i w_i) \leftarrow 2;\ f(v_{2i+1} w_i) = f(v_{2i} u_i) \leftarrow 5$

end for

end procedure

Output: Adjacent vertex distinguishing total colored of $A(Q_n)$, for $n \geq 4$.

Theorem 3.2 The Alternate Quadrilateral snake $A(Q_n)$ admits $AVDTC$ and

$$\chi_{avt}(A(Q_n)) = 5, \quad for\ n \geq 4.$$

Proof. The vertex set of Alternate Quadrilateral snake $A(Q_n)$ is given in the algorithm (3.2). Here we have two cases for n is even or odd. In each case, the quadrilateral starts with either the vertex of the path v_1 or v_2.

Case-1. $When\ n \equiv 1 (mod\ 2)$.

Suppose the quadrilateral starts with v_1, then $deg(v_n) = 1$ and if the quadrilateral starts with v_2, then $deg(v_1) = 1$. The edge set of $A(Q_n)$ is given by

$$E(A(Q_n)) = \begin{cases} \left(\bigcup_{i=1}^{n-1} v_i v_{i+1}\right) \cup \left(\bigcup_{i=1}^{\left\lfloor \frac{n}{2} \right\rfloor} ((v_{2i-1} u_i) \cup (v_{2i} w_i) \cup (u_i w_i))\right), & if\ \ deg(v_n) = 1 \\ \left(\bigcup_{i=1}^{n-1} v_i v_{i+1}\right) \cup \left(\bigcup_{i=1}^{\left\lfloor \frac{n}{2} \right\rfloor} ((v_{2i+1} w_i) \cup (v_{2i} u_i) \cup (u_i w_i))\right), & if\ \ deg(v_1) = 1 \end{cases}$$

$$|V(A(Q_n))| = 2n - 1\ and\ |E(A(Q_n))| = \frac{5(n-1)}{2}$$

The color classes of $A(Q_n)$ is given by

For $2 \leq i \leq n-1$

$$\overline{C}(v_i) = \begin{cases} \{2\}, & if\ i \equiv 1\,(mod\ 2) \\ \{1\}, & if\ i \equiv 0\,(mod\ 2) \end{cases}$$

For $i = 1,2,\cdots,\left\lfloor \frac{n}{2} \right\rfloor$

$$C(u_i) = \{2,3,5\}\ and\ C(w_i) = \{2,4,5\}$$

If $deg(v_n) = 1$ then

$$C(v_1) = \{1,3,5\}\ and\ C(v_n) = \{1,4\}$$

If $deg(v_1) = 1$, then

$$C(v_1) = \{1,3\}\ and C(v_n) = \{1,4,5\}$$

Case-2. $When\ n \equiv 0\,(mod\ 2)$.

Here, we have the two cases that the quadrilatral starts with v_1, then $deg(v_1, v_n) = 2$ and starts with v_2, then $deg(v_1, v_n) = 1$.

$$E(A(Q_n)) = \begin{cases} \left(\bigcup_{i=1}^{n-1} v_i v_{i+1}\right) \cup \left(\bigcup_{i=1}^{\left(\frac{n}{2}\right)} ((v_{2i-1}u_i) \cup (v_{2i}w_i) \cup (u_i w_i))\right), & if\ \ deg(v_1) = deg(v_n) = 2 \\ \left(\bigcup_{i=1}^{n-1} v_i v_{i+1}\right) \cup \left(\bigcup_{i=1}^{\left\lfloor\frac{n-1}{2}\right\rfloor} ((v_{2i+1}w_i) \cup (v_{2i}u_i) \cup (u_i w_i))\right), & if\ \ deg(v_1) = deg(v_n) = 1 \end{cases}$$

If $deg(v_1) = deg(v_n) = 2$, then

$$|V(A(Q_n))| = 2n\ and\ |E(A(Q_n))| = \frac{5n-2}{2}$$

The color classes of $A(Q_n)$ is given by

For $1 \le i \le \left(\frac{n}{2}\right)$

$$C(u_i) = \{2,3,5\}\ and C(w_i) = \{2,4,5\}$$

For $2 \le i \le n-1$

$$\overline{C}(v_i) = \begin{cases} \{2\}, & if\ i \equiv 1\,(mod\ 2) \\ \{1\}, & if\ i \equiv 0\,(mod\ 2) \end{cases}$$

$$C(v_1) = \{1,3,5\}\ and C(v_n) = \{2,3,5\}$$

If $deg(v_1) = deg(v_n) = 1$, then

$$|V(A(Q_n))| = 2n-2 \; and \; |E(A(Q_n))| = \frac{5n-8}{2}$$

The color classes are for $i = 1,2,\cdots \left\lfloor \frac{n-1}{2} \right\rfloor$

$$C(v_1) = \{1,3\}, C(v_n) = \{2,3\}, \; C(u_i) = \{2,3,5\} \; and \; C(w_i) = \{2,4,5\}$$

For $2 \le i \le n-1$

$$\overline{C}(v_i) = \begin{cases} \{1\}, \; if \; i \equiv 0 \, (mod\ 2) \\ \{2\}, \; if \; i \equiv 1 \, (mod\ 2) \end{cases}$$

Clearly, the color classes of any two adjacent vertices are different.

$$\therefore \chi_{avt}(A(Q_n)) = 5, \quad for\ n \ge 4.$$

4 AVDTC of $D(Q_n)$ and $DA(Q_n)$

In this section, we present an algorithm to obtain the adjacent vertex distinguishing total chromatic number of Double Quadrilateral snake and Double Alternate Quadrilateral snake and also we discussed their color classes in detail.

Algorithm 4.1 Procedure: Adjacent vertex distinguishing total coloring of Double Quadrilateral snake $D(Q_n)$, for $n \ge 4$.

Input: $G(V(D(Q_n)), E(D(Q_n)))$

for $i = 1,2,\cdots,n$ **do**

if $i \equiv 1(mod\ 2)$

$f(v_i) \leftarrow 1$

else

$f(v_i) \leftarrow 2$

end for

for $1 \le i \le n-1$ **do**

$f(u_i w_i) = f(u_i' w_i') \leftarrow 2$; $f(u_i) = f(u_i') \leftarrow 3$; $f(w_i) = f(w_i') \leftarrow 4$

$f(u_i v_i) \leftarrow 5$; $f(u_i' v_i) \leftarrow 7$; $f(w_i v_{i+1}) \leftarrow 6$; $f(w_i' v_{i+1}) \leftarrow 8$

if $i \equiv 1\,(mod\ 2)$

$f(v_i v_{i+1}) \leftarrow 3$

else

$f(v_i v_{i+1}) \leftarrow 4$

end for

end procedure

Output: Adjacent vertex distinguishing total colored of $D(Q_n)$, for $n \geq 4$.

Theorem 4.1 The Double Quadrilateral snake $D(Q_n)$ admits $AVDTC$ and

$$\chi_{avt}(D(Q_n)) = 8, \ n \geq 4.$$

Proof. From the definition (2.3), We have the vertex set and edge set of $D(Q_n)$. Therefore $|V(D(Q_n))| = 5n-4$ and $|E(D(Q_n))| = 7(n-1)$. Also, $deg(v_1) = deg(v_n) = 3$, $deg(v_i) = 6$ for $2 \leq i \leq n-1$ and $deg(u_i) = deg(u_i') = deg(w_i) = deg(w_i') = 2$ for $i = 1 \, to \, n-1$. We have colored the graph using algorithm (4.1), for $n \geq 4$. Now the color classes are given by

$$C(v_1) = \{1,3,5,7\} \, and C(v_n) = \begin{cases} \{2,3,6,8\}, if \ n \equiv 0 \ (mod \ 2) \\ \{1,4,6,8\}, if \ n \equiv 1 \ (mod \ 2) \end{cases}$$

For $2 \leq i \leq n-1$

$$\overline{C}(v_i) = \begin{cases} \{1\}, if \ i \equiv 0 \ (mod \ 2) \\ \{2\}, if \ i \equiv 1 \ (mod \ 2) \end{cases}$$

For $1 \leq i \leq n-1$

$$C(u_i) = \{2,3,5\}, \ C(u_i') = \{2,3,7\}, \ C(w_i) = \{2,4,6\}, \ C(w_i') = \{2,4,8\}$$

Clearly, the color set of any two adjacent vertices are different.

$$\therefore \quad \chi_{avt}(D(Q_n)) = 8, \quad n \geq 4.$$

Algorithm 4.2 Procedure: Adjacent vertex distinguishing total coloring of

$DA(Q_n)$ for $n \geq 4$.

Input:

$$V[A(Q_n)] \leftarrow \begin{cases} \left(\bigcup_{i=1}^{n} v_i\right) \cup \left(\bigcup_{i=1}^{\lfloor \frac{n}{2} \rfloor} (u_i \cup w_i \cup u_i' \cup w_i')\right), & for \ n \equiv 1 (mod \ 2) \\ \begin{cases} \left(\bigcup_{i=1}^{n} v_i\right) \cup \left(\bigcup_{i=1}^{(\frac{n}{2})} (u_i \cup w_i \cup u_i' \cup w_i')\right), & if \ deg(v_1) = deg(v_n) = 3 \\ \left(\bigcup_{i=1}^{n} v_i\right) \cup \left(\bigcup_{i=1}^{\lfloor \frac{n-1}{2} \rfloor} (u_i \cup w_i \cup u_i' \cup w_i')\right), & if \ deg(v_1) = deg(v_n) = 1 \end{cases}, & for \ n \equiv 0 (mod \ 2) \end{cases}$$

for $i = 1 \, to \, n$ **do**

if $i \equiv 1 \, (mod\ 2)$

$f(v_i) \leftarrow 1$

else

$f(v_i) \leftarrow 2$

end for

for $i = 1\, to\ n-1$ **do**

if $i \equiv 1 \, (mod\ 2)$

$f(v_i v_{i+1}) \leftarrow 3$

else

$f(v_i v_{i+1}) \leftarrow 4$

end for

if $n \equiv 1 \, (mod\ 2)$

for $i = 1\, to \left\lfloor \frac{n}{2} \right\rfloor$ **do**

$f(u_i) = f(u_i') \leftarrow 3$; $f(w_i) = f(w_i') \leftarrow 4$; $f(u_i w_i) = f(u_i' w_i') \leftarrow 2$

if $deg(v_n) = 1$

$f(v_{2i-1} u_i) = f(v_{2i} w_i) \leftarrow 5$; $f(v_{2i-1} u_i') = f(v_{2i} w_i') \leftarrow 6$

else

$f(v_{2i+1} u_i) = f(v_{2i} w_i) \leftarrow 5$; $f(v_{2i+1} u_i') = f(v_{2i} w_i') \leftarrow 6$

end for

else

if $deg(v_n) = deg(v_1) = 3$

for $i = 1\, to\ \frac{n}{2}$ **do**

$f(u_i) = f(u_i') \leftarrow 3$; $f(w_i) = f(w_i') \leftarrow 4$; $f(u_i w_i) = f(u_i' w_i') \leftarrow 2$

$f(v_{2i-1} u_i) = f(v_{2i} w_i) \leftarrow 5$; $f(v_{2i-1} u_i') = f(v_{2i} w_i') \leftarrow 6$

end for

else

for $i = 1\, to \left\lfloor \frac{n-1}{2} \right\rfloor$ **do**

$$f(u_i) = f(u_i') \leftarrow 3;\ f(w_i) = f(w_i') \leftarrow 4;\ f(u_i w_i) = f(u_i' w_i') \leftarrow 2$$

$$f(v_{2i+1} w_i) = f(v_{2i} u_i) \leftarrow 5;\ f(v_{2i+1} w_i') = f(v_{2i} u_i') \leftarrow 6$$

end for

end procedure

Output: Adjacent vertex distinguishing total colored of $DA(Q_n)$, for $n \geq 4$.

Theorem 4.2 The Double Alternate Quadrilateral snake $DA(Q_n)$ admits $AVDTC$ and $\chi_{avt}(DA(Q_n)) = 6, \quad for\ n \geq 4.$

Proof. The vertex set of Double Alternate Quadrilateral snake is given in the algorithm (4.2). Here we discussed two cases that when n is even or odd. In each case, the quadrilateral starts with either the vertex v_1 or with v_2.

Case-1. $When\ n \equiv 1\,(mod\ 2)$.

$$E[DA(Q_n)] = \begin{cases} \left(\bigcup_{i=1}^{n-1} v_i v_{i+1}\right) \cup \left(\bigcup_{i=1}^{\lfloor \frac{n}{2} \rfloor} (v_{2i-1}u_i \cup v_{2i}w_i \cup u_i w_i \cup v_{2i-1}u_i' \cup v_{2i}w_i' \cup u_i' w_i')\right), & if\ \ deg(v_n) = 1 \\ \left(\bigcup_{i=1}^{n-1} v_i v_{i+1}\right) \cup \left(\bigcup_{i=1}^{\lfloor \frac{n}{2} \rfloor} (v_{2i+1}w_i \cup v_{2i}u_i \cup u_i w_i \cup v_{2i+1}w_i' \cup v_{2i}u_i' \cup u_i' w_i')\right), & if\ \ deg(v_1) = 1 \end{cases}$$

$$|V(DA(Q_n))| = 3n-2\ and\ |E(DA(Q_n))| = 4n-4$$

The color classes of $DA(Q_n)$ for $n \geq 4$ is given by

For $2 \leq i \leq n-1$

$$\overline{C}(v_i) = \begin{cases} \{2\}, & if\ \ i \equiv 1\,(mod\ 2) \\ \{1\}, & if\ \ i \equiv 0\,(mod\ 2) \end{cases}$$

For $i = 1,2,\cdots,\left\lfloor \frac{n}{2} \right\rfloor$

$$C(u_i) = \{2,3,5\}\ and\ C(w_i) = \{2,4,5\}$$
$$C(u_i') = \{2,3,6\}\ and\ C(w_i') = \{2,4,6\}$$

If $deg(v_n) = 1$ then

$$C(v_1) = \{1,3,5,6\}\ and C(v_n) = \{1,4\}$$

If $deg(v_1) = 1$, then

$$C(v_1) = \{1,3\}\ and\ C(v_n) = \{1,4,5,6\}$$

Case-2. $When\ n \equiv 0\,(mod\ 2)$.

$$E(DA(Q_n)) = \begin{cases} \left(\bigcup_{i=1}^{n-1} v_i v_{i+1}\right) \cup \left(\bigcup_{i=1}^{(\frac{n}{2})} (v_{2i-1}u_i \cup v_{2i}w_i \cup v_{2i-1}u'_i \cup v_{2i}w'_i \cup u_i w_i \cup u'_i w'_i)\right), if\ deg(v_1) = deg(v_n) = 3 \\ \left(\bigcup_{i=1}^{n-1} v_i v_{i+1}\right) \cup \left(\bigcup_{i=1}^{\lfloor\frac{n-1}{2}\rfloor} (v_{2i+1}w_i \cup v_{2i}u_i \cup v_{2i+1}w'_i \cup v_{2i}u'_i \cup u_i w_i \cup u'_i w'_i)\right), if\ deg(v_1) = deg(v_n) = 1 \end{cases}$$

If $deg(v_1) = deg(v_n) = 3$, then

$$|V(DA(Q_n))| = 3n\ and\ |E(DA(Q_n))| = 4n-1$$

The color classes of $DA(Q_n)$ for $n \geq 4$ is given by

For $1 \leq i \leq \left(\frac{n}{2}\right)$

$$C(u_i) = \{2,3,5\}, C(w_i) = \{2,4,5\}, C(u'_i) = \{2,3,6\}, C(w'_i) = \{2,4,6\}$$

For $2 \leq i \leq n-1$

$$\overline{C}(v_i) = \begin{cases} \{2\},\ if\ i \equiv 1\,(mod\ 2) \\ \{1\},\ if\ i \equiv 0\,(mod\ 2) \end{cases}$$

$$C(v_1) = \{1,3,5,6\}\ and C(v_n) = \{2,3,5,6\}$$

If $deg(v_1) = 1$ and $deg(v_n) = 1$, then

$$|V(DA(Q_n))| = 3n-4\ and\ |E(DA(Q_n))| = 4n-7$$

The color classes are

$$C(v_1) = \{1,3\}\ and C(v_n) = \{2,3\}$$

for $i = 1,2,\cdots\left\lfloor\frac{n-1}{2}\right\rfloor$

$$C(u_i) = \{2,3,5\}\ and\ C(w_i) = \{2,4,5\}$$
$$C(u'_i) = \{2,3,6\}\ and\ C(w'_i) = \{2,4,6\}$$

For $2 \leq i \leq n-1$

$$\overline{C}(v_i) = \begin{cases} \{1\},\ if\ i \equiv 0\,(mod\ 2) \\ \{2\},\ if\ i \equiv 1\,(mod\ 2) \end{cases}$$

Clearly, the color classes of any two adjacent vertices are different.

$$\therefore \chi_{avt}(DA(Q_n)) = 6,\quad for\ n \geq 4.$$

Conclusion.

We found the adjacent vertex distinguishing total chromatic number of Quadrilateral snake family.

References

[1] N.L. Biggs, Algebraic Graph Theory, Cambridge University Press, 2^{nd} edition, Cambridge, 1993.

[2] Harary, F., Graph Theory (Addison-Wesley, Reading, Mass. 1969).

[3] K.K.Parthasarathy, Basic Graph Theory, Tata McGraw Hill, 1994.

[4] K.Thirusangu and R.Ezhilarasi "On Adjacent Vertex Distinguishing total coloring of Triangular snake", (communicated).

[5] S.K. Vaidya and N. B. Vyas, "Product cordial labeling for alternate snake graphs", Malaya J. Mat. 2(3)(2014) 188-196.

[6] ZHANG Zhongfu, CHEN Xiang 'en, L. Jingwen, YAO Bing, LU Xinzhong and WANG Jianfang. "On Adjacent-Vertex- Distinguishing total coloring of graphs", Science in China Ser.A Mathematics 2005, Vol. 48 (No.3), 289-299.