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# Two theorems in general metric space with ρ-distance

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## ABSTRACT

In this paper, we prove two theorems about fixed point and coupled coincidence point in generalized $b$-metric space via $\rho$-distance for a mapping satisfying a contraction condition.

## Keywords

Weak Contractions; fixed Points; coupled coincidence points; general metric spaces.

## 1. INTRODUCTION

The Banach contraction principle is the most known fixed point theorems. In 1993, Czerwik.[9] introduced $b$-metric spaces where the triangle inequality generalized as follows: $d(x,z) \leq b[d(x,y)+d(y,z)]$ for all $x,y$ and $z \in X$, $b \geq 1$

In. [8], Branceciri defined a generalized metric space as a metric space in which the triangle inequality is replaced by the rectangular one called quadrilateral inequality $d(x,y) \leq d(x,u)+d(u,v)+d(v,y)$ for all $x,y,u$ and $v \in X$

On the other hand, In. [10], Dhage introduced the notion of $D$-metric spaces on $X^3$:

1. $D(x,y,z)= 0$ if and only if $x=y=z$ (coincidence).
2. $D(x,y,z) = D(p\{x,y,z\})$, for all $x,y,z \in X$ and for any permutation $p\{x,y,z\}$ of $x,y,z$ (symmetry).
3. $D(x,y,z) \leq D(x,y,a) + D(x,a,z) + D(a,y,z)$, for all $x,y,z$, and $a \in X$ (tetrahedral inequality).

and claimed that $D$-metric but, Naidu S.V.R., Rao K.P.R. and Rao N.S. (2004-2005) gave many corrections for Dhage's work in. [14, 15 and 16]. In 2006, Mustafa and Sims. [25] introduce a new concept known as $G$-metric space satisfied the following:

1. $G(x,y,z) = 0$ iff $x = y = z$ for all $x,y,z \in X$.
2. $G(x,x,y) > 0$ for all $x,y \in X$, with $x \neq y$.
3. $G(x,x,y) \leq G(x,y,z)$ for all $x,y,z \in X$, with $z \neq y$.
4. $G(x,y,z) = G(p\{x,y,z\})$, $p$ permutation of $x,y$ and $z$.
5. $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$ for all $x,y,z$ and $a \in X$ (Rectangle inequality).

Mustafa et al. studied many fixed point theorems for mappings satisfying several contractive conditions on complete $G$-metric space. Aghajani et al. [4] introduced new generalizations of $G$-metric spaces called $g_b$-metric space. Mustafa et al. [13] have obtained some coupled coincidence point theorems for $g_b$-metric space. Kada et al. [12] introduced the concept of $w$-

distance on a metric space. Saadati et al. [17] defined an $\rho$-distance on a complete $G$-metric spaces. Gholizadeh et al. [11] state complete partially ordered $G$-metric space with the concept of $\rho$-distance. Shatanawi and Pitea in [19,20] prove some fixed and coupled fixed point theorem for nonlinear contractions used the notion of $\rho$-distance see [1,2,3,5, 6,7]. The aim of this paper is define a new weak contraction mappings defined on a $g_b$-metric space depend on $\rho$-distance and prove some results about the fixed point, coupled coincidence point.

## 2. Preliminaries:

### Definition 2-1: [13]

Let $X$ be a non-empty set and $\gamma: X \times X \times X \to \mathbb{R}^+$ be a function such that for all $x, y, z$ and $a \in X, b \geq 1$

1. $\gamma(x,y,z) = 0$ if $x = y = z$.

2. $\gamma(x,x,y) > 0$ for all $x,y \in X$ with $x \neq y$.

3. $\gamma(x,x,y) \leq \gamma(x,y,z)$ for all $x,y,z \in X$ with $y \neq z$.

4. $\gamma(x,y,z) = \gamma(p\{x,y,z\}), p$ permutation of $x,y$ and $z$.

5. $\gamma(x,y,z) \leq b[\gamma(x,a,a) + \gamma(a,y,z)]$ for all $x,y,z$ and $a \in X, b \geq 1$(Like trihedron).

then the pair $(X, \gamma)$ is called generalized $b$-metric space.

### Definition 2-2: [13]

Let $X$ be a $g_b$-m space. A sequence $\{x_n\}$ in $X$ is said to be:

1. $\gamma$-Cauchy sequence if, for each $\varepsilon > 0$, there is $n_0 \in \mathbb{N}$ such that, for all $m, n, i \geq n_0, \gamma(x_n, x_m, x_i) < \varepsilon$.

2. $\gamma$-convergent to a point $x \in X$ if, for each $\varepsilon > 0$, there is $n_0 \in \mathbb{N}$ such that, for all $m, n \geq n_0, \gamma(x_n, x_m, x) < \varepsilon$.

Throughout this paper $(X, \gamma)$ will be a generalized $b$-metric space $b \geq 1$.

### Definition 2-3: [17]

Let $\rho: X \times X \times X \to \mathbb{R}^+$. $\rho$ is called an $\rho$-distance on $X$ if for all $x, y, z$ and $a \in X$:

(a) $\rho(x,y,z) \leq \rho(x,a,a) + \rho(a,y,z)$, for all $x,y,z,a \in X$.

(b) For each $x, y \in X$, $\rho(x,y,.), \rho(x,.,y): X \to \mathbb{R}^+$ are Lower semi-continuous (L.S.C).

(c) $\forall \varepsilon > 0$ there is $\delta > 0$ such that $\rho(x,a,a) \leq \delta$ and $\rho(a,y,z) \leq \delta$ imply

. $\gamma(x,y,z) \leq \varepsilon$

### Lemma 2-4: [17,11]

Let $\rho$ be an $\rho$-distance on $X$ and let $\{x_n\}, \{y_n\}$ are sequences in $X$, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $\mathbb{R}^+$ with $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0$. If $x, y, z$ and $a \in X$ then

(1) If $\rho(y, x_n, x_n) \leq \alpha_n$ and $\rho(x_n, y, z) \leq \beta_n$ for $n \in \mathbb{N}$ then $\gamma(y, y, z) < \varepsilon$ and, $y = z$.

(2) If $\rho(y_n, x_n, x_n) \leq \alpha_n$ and $\rho(x_n, y_m, z) \leq \beta_n$ for $m > n$ then $\gamma(y_n, y_m, z) \to 0$, hence $y_n \to z$.

(3) If $\rho(x_n, x_m, x_i) \leq \alpha_n$ for $i, n, m \in \mathbb{N}$ with $n \leq m \leq i$, then $\{x_n\}$ is a $\gamma$-Cauchy sequence.

(4 ) If $\rho(x_n, a, a) \leq \alpha_n$ , $n \in \mathbb{N}$ then $\{x_n\}$ is a $\gamma$-Cauchy sequence.

### Definition 2-5: [18]

Let $G: X \times X \to X$ and $T: X \to X$ be two mapping. An ordered pair $(x, y) \in X \times X$ is called:

(a) Fixed point if $Tx = x$.

(b) Coupled coincidence point if $T(x) = G(x, y)$ and $T(y) = G(y, x)$.

## 3. Main Results:

The following classes are needed in the next results. Let $\boldsymbol{\mu}$ be a class of functions $\mu: \mathbb{R}^+ \to \mathbb{R}^+$ with

i. $\mu$ is continuous.

ii. $\mu$ non-decreasing.

iii. $\mu(\varepsilon) > 0$ for all $\varepsilon > 0$.

and Let $\Psi$ be a class of functions $\psi: \mathbb{R}^+ \to \mathbb{R}^+$ with

1. $\psi$ non-decreasing.

2. $\psi$ is right continuous.

3. $\psi(t) < 0$ for all $t > 0$.

### Remark 3-1:

If $\psi \in \Psi$ then $\lim_{n\to\infty} \psi^n(t) = 0$ for each $t > 0$ and if $\mu \in \boldsymbol{\mu}$, $\{a_n\} \subseteq \mathbb{R}^+$ and

.$\lim_{n\to\infty} \mu(a_n) = 0$ then $\lim_{n\to\infty} a_n = 0$

**Fixed Point:**

### Theorem 3-2:

Let $\rho$ be an $\rho$-distance, $T: X \to X$ be a mapping and $\mu \in \boldsymbol{\mu}$ , $\psi \in \Psi$ such that

$$\mu\rho(Tx,Ty,Tz) \leq \psi\mu\rho(x,y,z) \text{ for each } x,y,z \in X \qquad (1)$$

Suppose that if $u \neq Tu$ then $\inf\{\rho(x,Tx,u): x \in X\} > 0$

Then $T$ has a unique fixed point.

**Proof:**

Let $x_0 \in X$ and $x_{n+1} = Tx_n, \forall n \in \mathbb{N}$

if there is $n \in \mathbb{N}$ for which $x_{n+1} = x_n$ then $x_n$ is fixed point of $T$.

in the following, we assume $x_{n+1} \neq x_n, \forall n \in \mathbb{N}$

by condition (1)

$$\mu\rho(x_n,x_{n+1},x_{n+1}) = \mu\rho(Tx_{n-1},Tx_n,Tx_n)$$

$$\leq \psi\mu\rho(x_{n-1},x_n,x_n)$$

⋮

$$\leq \psi^n\mu\rho(x_0,x_1,x_1)$$

thus $\lim_{n\to\infty} \mu\rho(x_n,x_{n+1},x_{n+1}) = 0$. Then by remark (2-1) implies

$$\lim_{n\to\infty} \rho(x_n,x_{n+1},x_{n+1}) = 0 \qquad (2)$$

also

$$\lim_{n\to\infty} \rho(x_{n+1},x_n,x_n) = 0 \qquad (3)$$

Assume that $\{x_n\}$ is not a $\gamma$-Cauchy sequence, so, there is an $\varepsilon > 0$ and $\{x_{n_k}\}$, $\{x_{m_k}\}$ subsequences of $\{x_n\}$ with $m_k \geq n_k \geq k$ such that

$$\rho(x_{n_k},x_{m_k},x_{m_k}) \geq \varepsilon \qquad (4)$$

$$\rho(x_{n_k},x_{m_k-1},x_{m_k-1}) < \varepsilon \qquad (5)$$

the next step getting from conditions (4) and (5)

$$\varepsilon \leq \rho(x_{n_k},x_{m_k},x_{m_k})$$

$$\leq \rho(x_{n_k},x_{m_k-1},x_{m_k-1}) + \rho(x_{m_k-1},x_{m_k},x_{m_k})$$

$$< \varepsilon + \rho(x_{m_k-1},x_{m_k},x_{m_k})$$

then letting $k \to \infty$ in the above inequality and using (2)

$\lim_{k\to\infty} \rho(x_{n_k}, x_{m_k}, x_{m_k}) = \varepsilon^+$

if $\eta = \limsup \rho(x_{n_k+1}, x_{m_k+1}, x_{m_k+1}) \geq \varepsilon$

then there exists $\{k_r\}$ such that

$\rho\left(x_{n_{k_r}+1}, x_{m_{k_r}+1}, x_{m_{k_r}+1}\right) \to \eta \geq \varepsilon \text{ as } r \to \infty.$

since $\mu$ is continuous and non-decreasing

$\mu(\varepsilon) \leq \mu(\eta) = \lim_{r\to\infty} \mu\rho\left(x_{n_{k_r}+1}, x_{m_{k_r}+1}, x_{m_{k_r}+1}\right)$

$\leq \lim_{r\to\infty} \psi\mu\rho\left(x_{n_{k_r}}, x_{m_{k_r}}, x_{m_{k_r}}\right) = \psi\mu(\varepsilon)$

note that $\mu\rho\left(x_{n_{k_r}}, x_{m_{k_r}}, x_{m_{k_r}}\right) \to \mu(\varepsilon)$, and $\psi$ is right continuous.

thus $\mu(\varepsilon) = 0.$ This is a contradiction and

$$\lim_{k\to\infty} \sup\rho(x_{n_k+1}, x_{m_k+1}, x_{m_k+1}) < \varepsilon \qquad (6)$$

this implies that

$\varepsilon \leq \rho(x_{n_k}, x_{m_k}, x_{m_k})$

$\leq \rho(x_{n_k}, x_{n_k+1}, x_{n_k+1}) + \rho(x_{n_k+1}, x_{m_k+1}, x_{m_k+1}) + \rho(x_{m_k+1}, x_{m_k}, x_{m_k})$

by (2),(3) and (6)

$\varepsilon \leq \lim_{k\to\infty} p(x_{n_k}, x_{n_k+1}, x_{n_k+1}) + \lim_{k\to\infty} \sup\rho(x_{n_k+1}, x_{m_k+1}, x_{m_k+1})$

$+ \lim_{k\to\infty} \rho(x_{m_k+1}, x_{m_k}, x_{m_k})$

$= \lim_{k\to\infty} \sup\rho(x_{n_k+1}, x_{m_k+1}, x_{m_k+1}) < \varepsilon$

a contradiction, then

$\lim_{m,n\to\infty} \rho(x_n, x_m, x_m) = 0$

then $\{x_n\}$ is $\gamma$-Cauchy sequence. Since $X$ complete, there exists $u \in X$ such that

$\lim_{n\to\infty} x_n = u$

suppose $u \neq Tu$

now, for $\varepsilon > 0$ and by (L.S.C) of $\rho$, we get

$$\rho(x_n, x_m, u) \leq \lim_{p\to\infty} \inf \rho(x_n, x_m, x_p) \leq \varepsilon \qquad (7)$$

considering $m = n + 1$ in (7), we get

$$\rho(x_n, Tx_n, u) \leq \varepsilon$$

on the other hand, we get

$$0 < \inf\{\rho(x, Tx, u): x \in X\}$$

$$\leq \inf\{\rho(x_n, Tx_n, u): n \geq n_0\} \leq \varepsilon$$

this implies that $\inf\{\rho(x, Tx, u): x, y \in X\} = 0$

which is contradiction with hypothesis, therefore $u = Tu$

Suppose $u_1$ and $u_2$ are two fixed points of $T$, we have

$$\mu\rho(u_1, u_2, u_2) = \mu\rho(Tu_1, Tu_2, Tu_2)$$

$$\leq \psi\mu\rho(u_1, u_2, u_2)$$

thus, $\mu\rho(u_1, u_2, u_2) = 0$ and $\rho(u_1, u_2, u_2) = 0$

similarly $\rho(u_1, u_2, u_1) = 0$

then, by lemma (2-4) pent (1), we get $u_1 = u_2$. ■

## Coupled Coincidence Point:

### Theorem 3-3:

Let ρ be an ρ-distance, $G: X \times X \to X$ and $T: X \to X$ be a mappings with properties $G(X \times X) \subseteq Tx$ and $TX$ complete subspace of $X$. Consider $\mu \in \boldsymbol{\mu}$, $\psi \in \Psi$ such that

$$\mu\rho(G(x,y), G(u,v), G(z,w)) \leq \psi\mu\rho(Tx, Tu, Tz) \text{ for each } x, y, u, v, z, w \in X \qquad (8)$$

If $G(u,v) \neq Tu$ or $G(v,u) \neq Tv$ then

$$\inf\{\rho(Tx, G(x,y), Tu) + \rho(Ty, G(y,x), Tv): x, y \in X\} > 0$$

Then $G$ and $T$ have a unique coupled coincidence point.

**Proof:**

Let $x_0, y_0 \in X$, since $G(X \times X) \subseteq TX$, we can choose $x_1, y_1 \in X$ such that $Tx_1 = G(x_0, y_0)$ and $Ty_1 = G(y_0, x_0)$. Again from $G(X \times X) \subseteq TX$, we can choose $x_2, y_2 \in X$ such that $Tx_2 = G(x_1, y_1)$ and $Ty_2 = G(y_1, x_1)$

continuing in the process, we can construct two sequences $\{x_n\}$ and $\{y_n\}$ in $X$ such that

$Tx_{n+1} = G(x_n, y_n)$ and $Ty_{n+1} = G(y_n, x_n)$

by (8)

$$\mu\rho(Tx_n, Tx_{n+1}, Tx_{n+1}) = \mu\rho\big(G(x_{n-1}, y_{n-1}), G(x_n, y_n), G(x_n, y_n)\big)$$

$$\le \psi\mu\rho(Tx_{n-1}, Tx_n, Tx_n)$$

$$\vdots$$

$$\le \psi^n(\mu\rho(Tx_0, Tx_1, Tx_1)$$

then $\lim_{n\to\infty}[\mu\rho(Tx_n, Tx_{n+1}, Tx_{n+1})] = 0$

by remark (2-1) implies

$$\lim_{n\to\infty}[\rho(Tx_n, Tx_{n+1}, Tx_{n+1})] = 0 \qquad (9)$$

and

$$\lim_{n\to\infty}[\rho(Tx_{n+1}, Tx_n, Tx_n)] = 0 \qquad (10)$$

also

$$\lim_{n\to\infty}[\rho(Ty_n, Ty_{n+1}, Ty_{n+1})] = 0 \qquad (11)$$

and

$$\lim_{n\to\infty}[\rho(Ty_{n+1}, Ty_n, Ty_n)] = 0 \qquad (12)$$

Assume that at least one of $\{Tx_n\}$ or $\{Ty_n\}$ is not a $\gamma$-Cauchy sequence, so, there is an $\varepsilon > 0$ and $\{Tx_{n_k}\}, \{Tx_{m_k}\}$ subsequences of $\{Tx_n\}$ and $\{Ty_{n_k}\}, \{Ty_{m_k}\}$ subsequences of $\{Ty_n\}$ with $m_k \ge n_k \ge k$ such that

$$\rho\big(Tx_{n_k}, Tx_{m_k}, Tx_{m_k}\big) \ge \varepsilon \qquad (13)$$

$$\rho\big(Tx_{n_k}, Tx_{m_k-1}, Tx_{m_k-1}\big) < \varepsilon \qquad (14)$$

the next step getting from conditions (13) and (14)

$$\varepsilon \le \rho(Tx_{n_k}, Tx_{m_k}, Tx_{m_k})$$

$$\le \rho\big(Tx_{n_k}, Tx_{m_k-1}, Tx_{m_k-1}\big) + \rho\big(Tx_{m_k-1}, Tx_{m_k}, Tx_{m_k}\big)$$

$$< \varepsilon + \rho\big(Tx_{m_k-1}, Tx_{m_k}, Tx_{m_k}\big)$$

and by (9)as $k \to \infty$,

$$\lim_{k\to\infty} \rho(Tx_{n_k}, Tx_{m_k}, Tx_{m_k}) = \varepsilon^+$$

if $\eta = \lim_{k\to\infty} \sup \rho(Tx_{n_k+1}, Tx_{m_k+1}, Tx_{m_k+1}) \geq \varepsilon$

then there exists $\{k_r\}$ such that

$$\rho\left(Tx_{n_{k_r}+1}, Tx_{m_{k_r}+1}, Tx_{m_{k_r}+1}\right) \to \eta \geq \varepsilon \text{ as } r \to \infty$$

since $\mu$ is continuous and non-decreasing

$$\mu(\varepsilon) \leq \mu(\eta) = \lim_{r\to\infty} \mu\rho\left(Tx_{n_{k_r}+1}, Tx_{m_{k_r}+1}, Tx_{m_{k_r}+1}\right)$$

$$< \lim_{r\to\infty} \psi\mu\rho\left(Tx_{n_{k_r}}, Tx_{m_{k_r}}, Tx_{m_{k_r}}\right)$$

$$= \psi\mu(\varepsilon)$$

note that $\mu\rho\left(Tx_{n_{k_r}}, Tx_{m_{k_r}}, Tx_{m_{k_r}}\right) \to \mu(\varepsilon)$

and $\psi$ is right continuous. Thus $\mu(\varepsilon) = 0$. This is a contradiction and

$$\lim_{k\to\infty} \sup\rho(Tx_{n_k+1}, Tx_{m_k+1}, Tx_{m_k+1}) < \varepsilon \qquad (15)$$

this implies that

$$\varepsilon \leq \rho(Tx_{n_k}, Tx_{m_k}, Tx_{m_k})$$

$$\leq \rho(Tx_{n_k}, Tx_{n_k+1}, Tx_{n_k+1}) + \rho(Tx_{n_k+1}, Tx_{m_k+1}, Tx_{m_k+1}) + \rho(Tx_{m_k+1}, Tx_{m_k}, Tx_{m_k})$$

by (9),(10) and (15)

$$\varepsilon \leq \lim_{k\to\infty} p(Tx_{n_k}, Tx_{n_k+1}, Tx_{n_k+1}) + \lim_{k\to\infty} \sup\rho(Tx_{n_k+1}, Tx_{m_k+1}, Tx_{m_k+1})$$

$$+ \lim_{k\to\infty} \rho(Tx_{m_k+1}, Tx_{m_k}, Tx_{m_k})$$

$$= \lim_{k\to\infty} \sup\rho(Tx_{n_k+1}, Tx_{m_k+1}, Tx_{m_k+1}) < \varepsilon$$

a contradiction, then

$$\lim_{m,n\to\infty} \rho(Tx_n, Tx_m, Tx_m) = 0$$

also

$$\lim_{m,n\to\infty} \rho(Ty_n, Ty_m, Ty_m) = 0$$

therefore by lemma (1-4) part (3) $\{Tx_n\}$ and $\{Ty_n\}$ are $\gamma$-Cauchy sequence, since $TX$ is $\gamma$-complete, there exists $u, v \in X$ such that

$$\lim_{n\to\infty} Tx_n = Tu \text{ and } \lim_{n\to\infty} Ty_n = Tv$$

suppose $G(u,v) \neq Tu$ or $G(v,u) \neq Tv$

Now, for $\varepsilon > 0$ and by (L.S.C) of $\rho$, we get

$$\rho(Tx_n, Tx_m, Tu) \leq \lim_{p\to\infty} \inf \rho(Tx_n, Tx_m, Tx_p) \leq \varepsilon \qquad (16)$$

$$\rho(Ty_n, Ty_m, Tv) \leq \lim_{p\to\infty} \inf \rho(Ty_n, Ty_m, Ty_p) \leq \varepsilon \qquad (17)$$

Considering $m = n + 1$ in (16) and (17), we get

$$\rho(Tx_n, G(x_n, y_n), Tu) + \rho(Ty_n, G(y_n, x_n), Tv) \leq 2\varepsilon$$

on the other hand, we get

$$0 < \inf\{\rho(Tx, G(x,y), Tu) + \rho(Ty, G(y,x), Tv) : x, y \in X\}$$

$$\leq \inf\{\rho(Tx_n, G(x_n, y_n), Tu) + \rho(Ty_n, G(y_n, x_n), Tv) : n \geq n_0\} \leq 2\varepsilon$$

this implies that $\inf\{\rho(Tx, G(x,y), Tu) + \rho(Ty, G(y,x), Tv) : x, y \in X\} = 0$

which is contradiction with hypothesis, therefore $G(u,v) = Tu$ and $G(v,u) = Tv$

Now we prove the uniqueness

assume that $(u,v)$ and $(u^*, v^*)$ be a another coupled coincidence point of $G$ and $T$

by (8)

$$\mu\rho(Tu^*, Tu, Tu) = \mu\rho(G(u^*, v^*), G(u,v), G(u,v))$$

$$\leq \psi\mu\rho(Tu^*, Tu, Tu)$$

then $\mu\rho(Tu^*, Tu, Tu) = 0$ then $\rho(Tu^*, Tu, Tu) = 0$

similarly $\rho(Tu, Tu^*, Tu) = 0$

then by lemma (2-4) pent (1), then $Tu = Tu^*$

similarly we can show that $Tv = Tv^*$.

now, by (3.8)

$$\mu\rho(Tu, Tu, Tv) = \mu\rho(G(u,v), G(u,v), G(v,u))$$

$\leq \psi\mu\rho(Tu, Tu, Tv)$

then $\mu\rho(Tu, Tu, Tv) = 0$ then $\rho(Tu, Tu, Tv) = 0$

also $\rho(Tu, Tv, Tu) = 0$

then, by lemma (2-4) pent (1), then $Tu = Tv$. ■

The following example illustrate theorem (2-2)

**Example 3-4:**

Consider $(X, \gamma)$ $g_b$-m space with $b = 1$ define as follows

$X = \{0,1,2, \dots\}$ define $\gamma: X \times X \times X \to \mathbb{R}^+$ by

$$\gamma(x, y, z) = \begin{cases} 0 & \text{if } x = y = z \\ x + y + z & \text{if } x \neq y \text{ or } y \neq z \text{ or } x \neq z \end{cases}$$

-distance, $\rho: X \times X \times X \to X$, $\rho(x, y, z) = x + 2\max\{y, z\}\rho$ is $\rho$

Define $T: X \to X$

$$Tx = \begin{cases} 0 & \text{if } x = 0,1 \\ x - 1 & \text{if } x \geq 2 \end{cases}$$

and $\mu: \mathbb{R}^+ \to \mathbb{R}^+, \mu(t) = 4t,\ \psi: \mathbb{R}^+ \to \mathbb{R}^+, \psi(t) = t,\ t > 0$

If $u \neq Tu$ then

$$\inf\{\rho(x, Tx, u): x \in X\} \geq \inf\{x + 2u: x \in X\} \geq 2u > 0$$

for $x, y, z \in X$, with $y \geq z$, then

$$\rho(x, y, z) = x + 2y \text{ and } \rho(Tx, Ty, Tz) = x - 1 + 2(y - 1)$$

Since

$$4[x - 1 + 2(y - 1)] \leq 4[x + 2y]$$

We have

$$\mu\rho(Tx, Ty, Tz) \leq \psi\mu\rho(x, y, z)$$

thus all hypotheses of theorem (3-2) are satisfied and $x = 0$ is the unique fixed point of $T$.

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