Numerical Scheme for Backward Doublay Stochastic Differential Equations with Time Delayed Coefficients

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ABSTRACT

In this paper, we present some assumptions to get the numerical scheme for backward doubly stochastic differential delay equations (shortly-BDSDDEs), and we propose a scheme of BDSDDEs and discuss the numerical convergence and rate of convergence of our scheme.

AMS Mathematics Subject Classification (2010): 60H10, 65C30

Keywords: Brownian motion; Backward doubly stochastic differential equation; Conditional expectation; Time delayed coefficients.

1 Introduction

Backward stochastic differential equations (shortly-BSDEs) have been first presented in Pardoux and Peng [16, 17] in order to proved existence and uniqueness of the adapted solutions and presented a new class of backward doubly stochastic differential equations, further investigations being (see [3, 4, 11, 13]). A lot of mathematicians interested in a numerical methods for approximating solution of BSDEs (see [1, 10, 14, 15, 18, 22]). Xuerong Mao et al. [21] discussed the effects of environmental noise on the delay Lotka-Volterra model. Brahim Boufoussi et al. [2] presented a new class of backward doubly stochastic differential equations, this a new class depend on an integral with respect to an adapted continuous increasing process. Lukasz Delong [5, 6] studied applications of a new class of time-delayed BSDEs and he gives examples of pricing, hedging and portfolio management problems which could be established in the framework of backward stochastic differential delay equation. Wen Lu et al. [19] investigated a class of multivalued backward doubly stochastic differential delay equation, and they proved the existence and uniqueness of the solutions for these equations under Lipschitz condition. Using the Euler-Maruyama method, Xiaotai Wu and Litan Yan [20] defined the numerical solutions of doubly perturbed stochastic delay differential equations driven by Levy process, and they proved the numerical solutions converge to the exact solutions with the local Lipschitz condition. Delong and Imkeller [7] presented a class of BSDEs with time delayed, and they established the existence and uniqueness of a solution for BSDEs with time delayed. Also, they [8] proved the existence and uniqueness as well as the Malliavin's differentiability of the solution for BSDEs with delayed time. Moreover, Diomande and Maticiuc [9] proved the existence and uniqueness of a solution for multivalued BSDEs with time delayed generators. Besides, Lu and Ren [12] established the existence and uniqueness of the solutions for a class of backward doubly stochastic differential equations with time delayed coefficients under Lipschitz condition.

The purpose of this work is to study the numerical convergent of backward doubly stochastic delay differential equations (shortly-BDSDDEs) that has the following

$$Y(t)=\xi+\int_t^T f(s,Y(s),Z(s),Y_s,Z_s)\mathrm{d}s+\int_t^T g(s,Y(s),Z(s),Y_s,Z_s)\mathrm{d}B(s)-\int_t^T Z(s)\mathrm{d}W(s) \qquad (1)$$

where $\{W_t, 0\le t\le T\}$ and $\{B_t, 0\le t\le T\}$ are a Brownian motion defined on the probability space (Ω_1, F_1, P_1) and (Ω_2, F_2, P_2), respectively, and $T<\infty$ is a finite time horizon. The coefficients f and g at time s and the terminal condition ξ depend on the past values of a solution $(Y_s, Z_s)=(Y(s+\theta), Z(s+\theta))_{-T\le\ \theta\le 0}$.

In our work, we extend the approach of BDSDDEs in the general case, and introduce some general assumptions on the numerical convergence of backward doubly stochastic differential equations with time delayed coefficients. Furthermore, we present a numerical scheme based on iterative regression functions which are approximated by projection on vector space of functions. Also, we discuss the numerical convergence and rate of convergence of BDSDDEs Lipschitz condition.

The present paper is organized as follows: In section 2, we present some preliminaries that explain the approximation scheme for BDSDDEs. In section 3, we consider the approximation solution of BDSDDEs and prove some problems that useful for our work. In section 4, we have discussed the numerical convergence and rate of convergence of our scheme.

2 Notations, preliminaries and basic assumptions

In this section, we provide some assumptions and space used in the sequel. Therefore, we consider two independent standard d-dimensional Brownian motions $\{W_t, 0\le t\le T\}$ and $\{B_t, 0\le t\le T\}$, defined on the complete probability spaces (Ω_1, F_1, P_1) and (Ω_2, F_2, P_2), respectively, and a finite time horizon $T<\infty$. We denote

$$F_{t,s}^{B}=\sigma\{B_r-B_s, s\le r\le\ t\},\ F_t^{W}=\sigma\{W_r, 0\le r\le t\}.$$

Moreover, we consider $\Omega=\Omega_1\times\Omega_2$, $F=F_1\otimes F_2$ and $P=P_1\otimes P_2$. In addition, we put

$$F_t = F_t^W \otimes F_{t,s}^B \otimes N,$$

where N is the collection of P-null sets of F. That is to say, the $\sigma-$fields $F_t, 0 \le t \le T$, are $P-$complete, and the family of $\sigma-$algebras $F = \{F_t\}_{t\in[o,T]}$ is neither increasing nor decreasing, it is not constitute a filtration.

We consider the Euclidian norm $|\cdot|$ in $\mathbb{R}^k$ and $\mathbb{R}^{k\times d}$, we use the following spaces

1. Let $L^2_{-T}(\mathbb{R}^{k\times d})$ is the space of measurable function $Z:[-T,0]\to\mathbb{R}^{k\times d}$ such that $\int_{-T}^0 |Z(t)|^2\,\mathrm{d}t<\infty$.
2. Let $L^\infty_{-T}(\mathbb{R}^k)$ is the space of measurable function $Y:[-T,0]\to\mathbb{R}^k$ such that $\sup_{-T\le\ t\le 0}|Y(t)|^2<\infty$.
3. Let $H^2_T(\mathbb{R}^m)$ is the space of F-predictable processes $Y:\Omega\times[0,T]\to\mathbb{R}^m$ such that $E\int_0^T |Y(t)|^2\,\mathrm{d}t<\infty$.
4. Let $S^2_T(\mathbb{R}^k)$ is the space of F -adapted, product measurable processes $Y:\Omega\times[0,T]\to\mathbb{R}^k$ such that $E[\sup_{0\le t\le T}|Y(t)|^2]<\infty$.

The spaces $H^2_T(\mathbb{R}^{k\times d})$ and $S^2_T(\mathbb{R}^k)$ are done with the norm $\|Z\|^2_{H^2_T}=E\int_0^T|Z(t)|^2\,\mathrm{d}t$ and $\|Y\|^2_{S^2_T}=E[\sup_{0\le t\le T}|Y(t)|^2]$, respectively. In this paper, we consider the following BDSDE with time delayed coefficients

$$\begin{cases} d(Y(t)) = f(t,Y(t),Z(t),Y_t,Z_t)dt + g(t,Y(t),Z(t),Y_t,Z_t)dB(t) - Z(t)dW(t), 0\le t\le T, \\ Y_T = \xi(Y_T,Z_T), -T\le t\le 0, \end{cases}$$

where f and g are Borel-measurable functions at time set depend on the past values of the solution $Y_s=(Y(s+\theta))_{-T\le\theta\le 0}$ and $Z_s=(Z(s+\theta))_{-T\le\theta\le 0}$. We always set $Z(t)=0$ and $Y(t)=Y(0)$ for $t<0$. Now, we make the following assumptions

Assumption (H1): There exist a positive constant K_1 and for all $-\tau\le s<t\le 0$ such that

$$E[|\xi(t)-\xi(s)|^2]\le K_1(t-s).$$

Assumption (H2): Suppose that $f:\Omega\times[0,T]\times\mathbb{R}^k\times\mathbb{R}^{k\times d}\times L^\infty_{-T}(\mathbb{R}^k)\times L^2_{-T}(\mathbb{R}^{k\times d})\to\mathbb{R}^k$ and $g:\Omega\times[0,T]\times L^\infty_{-T}(\mathbb{R}^k)\times L^2_{-T}(\mathbb{R}^{k\times d})\to\mathbb{R}^{k\times d}$ are product measurable, there exist a positive constants K_2, K_3 and K_4, and a finite measure α on $[-\tau,0]$ such that

$$\begin{aligned} &|f(t,Y^1,Z^1,Y^1_t,Z^1_t)-f(t,Y^2,Z^2,Y^2_t,Z^2_t)|^2 \le K_2(|Y^1-Y^2|^2+|Z^1-Z^2|^2) \\ &+K_4\Big(\int_{-T}^0|Y^1(t+\theta)-Y^2(t+\theta)|^2\,\alpha(\mathrm{d}\theta) \\ &+\int_{-T}^0|Z^1(t+\theta)-Z^2(t+\theta)|^2\,\alpha(\mathrm{d}\theta)\Big). \end{aligned}$$

and

$$\begin{aligned} &|g(t,Y^1,Z^1,Y^1_t,Z^1_t)-g(t,Y^2,Z^2,Y^2_t,Z^2_t)|^2\le K_3(|Y^1-Y^2|^2+|Z^1-Z^2|^2) \\ &+K_4\Big(\int_{-T}^0|Y^1(t+\theta)-Y^2(t+\theta)|^2\,\alpha(\mathrm{d}\theta) \\ &+\int_{-T}^0|Z^1(t+\theta)-Z^2(t+\theta)|^2\,\alpha(\mathrm{d}\theta)\Big), \end{aligned}$$

for all $t\in[0,T],(Y^1,Z^1),(Y^2,Z^2)\in\mathbb{R}^k\times\mathbb{R}^{k\times d},(Y^1_t,Z^1_t),(Y^2_t,Z^2_t)\in L^\infty_{-T}(\mathbb{R}^k)\times L^2_{-T}(\mathbb{R}^{k\times d})$.

Assumption (H3)

$$E\int_0^T|f(t,0,0,0,0)|^2\,\mathrm{d}t<\infty, E\int_0^T|g(t,0,0,0,0)|^2\,\mathrm{d}t<\infty.$$

Assumption(H4)

$$f(t,\cdot,\cdot,\cdot)=0, g(t,\cdot,\cdot)=0,$$

for $t<0$.

Assumption (H5): There exists a positive constant K_5 such that

$$|f(Y,Z)|^2 \vee |g(Y,Z)|^2 \le K_5(1+|Y|^2+|Z|^2),$$

where $a\vee b=\max\{a,b\}$.

3 A numerical scheme for BDSDDEs

In this section, we propose a numerical scheme is based upon a descretization of (1). Moreover, for all integers $n,l\ge 1$ and $t\in[0,T]$, let

$$-\tau=t_{-l}<t_{-l+1}<\cdots<0=t_0<t_1<\cdots<t_n=T$$

be a partition of $[-\tau,T]$, and denote

$$\delta=\Delta t_{i+1}=t_{i+1}-t_i=\frac{T}{n}, 1\le i\le n, \Delta B_{t_{i+1}}=B_{t_{i+1}}-B_{t_i}, \Delta W_{t_{i+1}}=W_{t_{i+1}}-W_{t_i},$$

where $i=0,1,\cdot,\cdot,\cdot,n-1$, and $\Delta t=\max_{-\tau\le i\le n-1}\Delta t_i$. Now, on the small interval $[t_i,t_{i+1}]$ the equation

$$Y_{t_i}=Y_{t_{i+1}}+\int_{t_i}^{t_{i+1}}f(s,Y(s),Z(s),Y_s,Z_s)\mathrm{d}s+\int_{t_i}^{t_{i+1}}g(s,Y(s),Z(s),Y_s,Z_s)\mathrm{d}B(s)$$
$$-\int_{t_i}^{t_{i+1}}Z(s)\mathrm{d}W(s). \qquad (2)$$

We can be approximated by the discrete equation

$$Y_{t_i}^n=Y_{t_{i+1}}^n+f(t,Y_i^n(t),Z_i^n(t),Y_i^n(t+\theta),Z_i^n(t+\theta))\delta+g(t,Y_i^n(t),Z_i^n(t),Y_i^n(t+\theta),Z_i^n(t+\theta))\Delta B_{i+1}$$
$$-Z_i^n(t)\Delta W_{i+1},$$

with $Y(T)=\xi(T)$ on $-T\le t\le 0$. Therefore, we consider a class of BDSDDEs as the form

$$Y_i^n(t)=\xi(T)+\int_0^T f(s,Y_i^n(s),Z_i^n(s),Y_i^n(s+\theta),Z_i^n(s+\theta))\mathrm{d}s$$
$$+\int_0^T g(s,Y_i^n(s),Z_i^n(s),Y_i^n(s+\theta),Z_i^n(s+\theta))\mathrm{d}B(s)-\int_0^T Z_i^n(s)\mathrm{d}W(s).$$

Now, let us define the Euler-Maruyama approximate solution by

$$\tilde{Y}(t)=\xi(T)+\int_0^T f(s,\tilde{Y}(s),\tilde{Z}(s),\tilde{Y}(s+\theta),\tilde{Z}(s+\theta))\mathrm{d}s$$
$$+\int_0^T g(s,\tilde{Y}(s),\tilde{Z}(s),\tilde{Y}(s+\theta),\tilde{Z}(s+\theta))\mathrm{d}B(s)-\int_0^T\tilde{Z}(s)\mathrm{d}W(s). \qquad (3)$$

Lemma 3.1 Assume the assumptions (H1)-(H4) hold, for all $n=0,1,\ldots,N-1$, then it holds that

$$\tilde{Y}_n^N(t)=Y_n^N(t) \text{ and } \tilde{Z}_n^N(t)=\frac{1}{h}E_{t_n}\left[\int_{t_n}^{t_{n+1}}\tilde{Z}^N(s)\mathrm{d}s\right]=Z_n^N(t).$$

Proof. From equation

$$\tilde{Y}^N(t)=\tilde{Y}_{n+1}^N(t)+\int_{t_n}^{t_{n+1}}f(s,\tilde{X}_n^N(s),\tilde{Y}_n^N(s),\tilde{Z}_n^N(s),\tilde{X}_n^N(s+\theta),\tilde{Y}_n^N(s+\theta),\tilde{Z}_n^N(s+\theta))\mathrm{d}s$$
$$+\int_t^{t_{n+1}}g(s,\tilde{X}_n^N(s),\tilde{Y}_n^N(s),\tilde{Z}_n^N(s),\tilde{X}_n^N(s+\theta),\tilde{Y}_n^N(s+\theta),\tilde{Z}_n^N(s+\theta))\mathrm{dB(s)}$$
$$-\int_{t_n}^{t_{n+1}}\tilde{Z}_n^N(s)\mathrm{dW(s)}$$

where $t_n\le t\le t_{n+1}$, we have that

$$\tilde{Y}_n^N(t)=\tilde{Y}_{n+1}^N(t)+\int_{t_n}^{t_{n+1}}f(s,\tilde{X}_n^N(s),\tilde{Y}_n^N(s),\tilde{Z}_n^N(s),\tilde{X}_n^N(s+\theta),\tilde{Y}_n^N(s+\theta),\tilde{Z}_n^N(s+\theta))\mathrm{d}s$$

$$+\int_{t_n}^{t_{n+1}} g(s,\widetilde{X}_n^N(s),\widetilde{Y}_n^N(s),\widetilde{Z}_n^N(s),\widetilde{X}_n^N(s+\theta),\widetilde{Y}_n^N(s+\theta),\widetilde{Z}_n^N(s+\theta))\mathrm{dB(s)}$$
$$-\int_{t_n}^{t_{n+1}} \widetilde{Z}_n^N(s)\mathrm{dW(s)},$$

and then, we get that

$$\widetilde{Y}_n^N(t) = E_{t_n}[\widetilde{Y}_{n+1}^N(t) + g(t,\widetilde{X}_n^N(t),\widetilde{Y}_n^N(t),\widetilde{Z}_n^N(t),\widetilde{X}_n^N(t+\theta),\widetilde{Y}_n^N(t+\theta),\widetilde{Z}_n^N(t+\theta))\Delta B_n]$$
$$+hf(t,\widetilde{X}_n^N(t),\widetilde{Y}_n^N(t),\widetilde{Z}_n^N(t),\widetilde{X}_n^N(t+\theta),\widetilde{Y}_n^N(t+\theta),\widetilde{Z}_n^N(t+\theta)).$$

From equation above, we deduce that

$$\int_{t_n}^{t_{n+1}} \widetilde{Z}^N(s)\mathrm{dW(s)}\Delta W_n = \widetilde{Y}_{n+1}^N(t)\Delta W_n$$
$$+\int_{t_n}^{t_{n+1}} f(s,\widetilde{X}_n^N(s),\widetilde{Y}_n^N(s),\widetilde{Z}_n^N(s),\widetilde{X}_n^N(s+\theta),\widetilde{Y}_n^N(s+\theta),\widetilde{Z}_n^N(s+\theta))\mathrm{ds}\Delta W_n$$
$$+\int_{t_n}^{t_{n+1}} g(s,\widetilde{X}_n^N(s),\widetilde{Y}_n^N(s),\widetilde{Z}_n^N(s),\widetilde{X}_n^N(s+\theta),\widetilde{Y}_n^N(s+\theta),\widetilde{Z}_n^N(s+\theta))\mathrm{dB(s)}\Delta W_n$$
$$-\widetilde{Y}_n^N(t)\Delta W_n.$$

By taking the expectation, we get that

$$E_{t_n}[\int_{t_n}^{t_{n+1}} \widetilde{Z}^N(s)\mathrm{dW(s)}\Delta W_n]$$
$$= E_{t_n}[\int_{t_n}^{t_{n+1}} f(s,\widetilde{X}_n^N(s),\widetilde{Y}_n^N(s),\widetilde{Z}_n^N(s),\widetilde{X}_n^N(s+\theta),\widetilde{Y}_n^N(s+\theta),\widetilde{Z}_n^N(s+\theta))\mathrm{ds}\Delta W_n]$$
$$+E_{t_n}[\int_{t_n}^{t_{n+1}} g(s,\widetilde{X}_n^N(s),\widetilde{Y}_n^N(s),\widetilde{Z}_n^N(s),\widetilde{X}_n^N(s+\theta),\widetilde{Y}_n^N(s+\theta),\widetilde{Z}_n^N(s+\theta))\mathrm{dB(s)}\Delta W_n]$$
$$+E_{t_n}[\widetilde{Y}_{n+1}^N(t)\Delta W_n] - E_{t_n}[\widetilde{Y}_n^N(t)\Delta W_n].$$

Therefore, we deduce that

$$E_{t_n}[\int_{t_n}^{t_{n+1}} \widetilde{Z}^N(s)\mathrm{dW(s)}\Delta W_n]$$
$$= hE_{t_n}[f(t,\widetilde{X}_n^N(t),\widetilde{Y}_n^N(t),\widetilde{Z}_n^N(t),\widetilde{X}_n^N(t+\theta),\widetilde{Y}_n^N(t+\theta),\widetilde{Z}_n^N(t+\theta))\Delta W_n]$$
$$+E_{t_n}[g(t,\widetilde{X}_n^N(t),\widetilde{Y}_n^N(t),\widetilde{Z}_n^N(t),\widetilde{X}_n^N(t+\theta),\widetilde{Y}_n^N(t+\theta),\widetilde{Z}_n^N(t+\theta))\Delta B_n\Delta W_n]$$
$$+E_{t_n}[\widetilde{Y}_{n+1}^N(t)\Delta W_n] - E_{t_n}[\widetilde{Y}_n^N(t)\Delta W_n],$$

then

$$E_{t_n}[\int_{t_n}^{t_{n+1}} \widetilde{Z}^N(s)\mathrm{dW(s)}\Delta W_n] = E_{t_n}[g(t,\widetilde{X}_n^N(t),\widetilde{Y}_n^N(t),\widetilde{Z}_n^N(t),\widetilde{X}_n^N(t+\theta),\widetilde{Y}_n^N(t+\theta),\widetilde{Z}_n^N(t+\theta))\Delta B_n\Delta W_n]$$
$$+E_{t_n}[\widetilde{Y}_{n+1}^N(t)\Delta W_n].$$

From the fact $\widetilde{Y}_n^N(t)$ and $f(t,\widetilde{X}_n^N(t),\widetilde{Y}_n^N(t),\widetilde{Z}_n^N(t),\widetilde{X}_n^N(t+\theta),\widetilde{Y}_n^N(t+\theta),\widetilde{Z}_n^N(t+\theta))$ are F_{t_n}-measurable, then

$$E_{t_n}[f(t,\widetilde{X}_n^N(t),\widetilde{Y}_n^N(t),\widetilde{Z}_n^N(t),\widetilde{X}_n^N(t+\theta),\widetilde{Y}_n^N(t+\theta),\widetilde{Z}_n^N(t+\theta))\Delta W_n] = E_{t_n}[\widetilde{Y}_n^N(t)\Delta W_n] = 0.$$

Therefore, we have that

$$E_{t_n}[\int_{t_n}^{t_{n+1}} \widetilde{Z}^N(s)\mathrm{dW(s)}\Delta W_n] = h\widetilde{Z}_n^N(t).$$

By taking the integration, we obtain that

$$E_{t_n}[\int_{t_n}^{t_{n+1}} \widetilde{Z}^N(s)\mathrm{dW(s)}\Delta W_n] = E_{t_n}[\int_{t_n}^{t_{n+1}}\int_{t_n}^{s} \mathrm{dW(u)}\widetilde{Z}^N(s)\mathrm{dW(s)}]$$

$$+E_{t_n}[\int_{t_n}^{t_{n+1}}\int_{t_n}^{s}\widetilde{Z}^N(u)\mathrm{dW(u)dW(s)}]$$
$$+E_{t_n}[\int_{t_n}^{t_{n+1}}\widetilde{Z}^N(s)\mathrm{ds}],$$

then

$$E_{t_n}[\int_{t_n}^{t_{n+1}}\widetilde{Z}^N(s)\mathrm{dW(s)}\Delta W_n]=E_{t_n}[\int_{t_n}^{t_{n+1}}\widetilde{Z}^N(s)\mathrm{ds}].$$

Therefore, we have that

$$\widetilde{Z}_n^N(t)=\frac{1}{h}E_{t_n}[\int_{t_n}^{t_{n+1}}\widetilde{Z}^N(s)\mathrm{ds}].$$

From above equation, we get that

$$\widetilde{Z}_n^N(t)=\frac{1}{h}E_{t_n}[\widetilde{Y}_{n+1}^N(t)\Delta W_n]+\frac{1}{h}E_{t_n}[g(t,\widetilde{X}_n^N(t),\widetilde{Y}_n^N(t),\widetilde{Z}_n^N(t),\widetilde{X}_n^N(t+\theta),\widetilde{Y}_n^N(t+\theta),\widetilde{Z}_n^N(t+\theta))\Delta B_n\Delta W_n].$$

Therefore, for all $n=0,1,...,N-1$, then $(\widetilde{Y}_n^N(t),\widetilde{Z}_n^N(t))=(Y_n^N(t),Z_n^N(t))$.

4 Main results

This section is devoted to discuss numerical convergence and rate of convergence of BDSDDEs.

Theorem 4.1 Suppose that assumption (H2) is fulfilled. Likewise, assume that $\{Y^n(t),Z^n(t)\}$ is a solution of equation (3), then the approximated solution $\{Y^n(t),Z^n(t)\}$ converges to the exact solution $\{Y(t),Z(t)\}$ in the sense that for all $t\in[0,T]$, such that

$$\lim_{n\to\infty}E|Y(t)-Y^n(t)|^2=0$$

and

$$\lim_{n\to\infty}E\int_0^T|Z(t)-Z^n(t)|^2\ \mathrm{dt}=0.$$

Proof. For all $t\in[0,T]$, let $\{Y_i(t),Z_i(t)\}$ and $\{Y_i^n(t),Z_i^n(t)\}$ be the solution of equations (1) and (3), respectively. Therefore, we have that

$$d(Y_i(t)-Y_i^n(t))=[f(t,Y_i(t),Z_i(t),Y_i(t+\theta),Z_i(t+\theta))-f(t,Y_i^n(t),Z_i^n(t),Y_i^n(t+\theta),Z_i^n(t+\theta))]dt$$
$$+[g(t,Y_i(t),Z_i(t),Y_i(t+\theta),Z_i(t+\theta))-g(t,Y_i^n(t),Z_i^n(t),Y_i^n(t+\theta),Z_i^n(t+\theta))]dB(t)$$
$$-[Z_i(t)-Z_i^n(t)]dW(t).$$

Denote $\langle a,b\rangle$ is inner product of two vectors a and b. Now, by applying Ito's formula to $|Y_i-Y_i^n|^2$, we get that

$$|Y_i(t)-Y_i^n(t)|^2=2\int_0^T\langle Y_i(s)-Y_i^n(s),\ f(s,Y_i(s),Z_i(s),Y_i(s+\theta),Z_i(s+\theta))$$
$$-f(s,Y_i^n(s),Z_i^n(s),Y_i^n(s+\theta),Z_i^n(s+\theta))\rangle\mathrm{ds}$$
$$+\int_0^T|g(t,Y_i(t),Z_i(t),Y_i(t+\theta),Z_i(t+\theta))$$
$$-g(t,Y_i^n(t),Z_i^n(t),Y_i^n(t+\theta),Z_i^n(t+\theta))|^2\ \mathrm{ds}$$
$$+2\int_0^T\langle Y_i(s)-Y_i^n(s),g(t,Y_i(t),Z_i(t),Y_i(t+\theta),Z_i(t+\theta))$$
$$-g(t,Y_i^n(t),Z_i^n(t),Y_i^n(t+\theta),Z_i^n(t+\theta))\mathrm{dB(s)}\rangle$$
$$-2\int_0^T\langle Y_i(s)-Y_i^n(s),Z_i(s)-Z_i^n(s)\mathrm{dW(s)}\rangle.$$

By taking the expectation, we get that

$$E|Y_i(t)-Y_i^n(t)|^2=2E\int_0^T\langle Y_i(s)-Y_i^n(s),\ f(s,Y_i(s),Z_i(s),Y_i(s+\theta),Z_i(s+\theta))$$

$$- f(s,Y_i^n(s),Z_i^n(s),Y_i^n(s+\theta),Z_i^n(s+\theta))\rangle \mathrm{ds}$$
$$+E\int_0^T | g(t,Y_i(t),Z_i(t),Y_i(t+\theta),Z_i(t+\theta))$$
$$-g(t,Y_i^n(t),Z_i^n(t),Y_i^n(t+\theta),Z_i^n(t+\theta))|^2 \mathrm{ds}$$
$$+2E\int_0^T \langle Y_i(s)-Y_i^n(s), g(t,Y_i(t),Z_i(t),Y_i(t+\theta),Z_i(t+\theta))$$
$$-g(t,Y_i^n(t),Z_i^n(t),Y_i^n(t+\theta),Z_i^n(t+\theta))\mathrm{dB(s)}\rangle$$
$$-2\int_0^T \langle Y_i(s)-Y_i^n(s), Z_i(s)-Z_i^n(s)\mathrm{dW(s)}\rangle.$$

Making use of the Young's inequality, we derive that

$$E|Y_i(t)-Y_i^n(t)|^2 \le 4KE\int_0^T |Y_i(s)-Y_i^n(s)|^2 \mathrm{ds} + \frac{1}{4K}E\int_0^T | f(s,Y_i(s),Z_i(s),Y_i(s+\theta),Z_i(s+\theta))$$
$$- f(s,Y_i^n(s),Z_i^n(s),Y_i^n(s+\theta),Z_i^n(s+\theta))|^2 \mathrm{ds}$$
$$+E\int_0^T | g(s,Y_i(s),Z_i(s),Y_i(s+\theta),Z_i(s+\theta))$$
$$-g(s,Y_i^n(s),Z_i^n(s),Y_i^n(s+\theta),Z_i^n(s+\theta))|^2 \mathrm{ds}$$
$$+4KE\int_0^T |Y_i(s)-Y_i^n(s)|^2 \mathrm{ds} + \frac{1}{4K}E\int_0^T | g(s,Y_i(s),Z_i(s),Y_i(s+\theta),Z_i(s+\theta))$$
$$-g(s,Y_i^n(s),Z_i^n(s),Y_i^n(s+\theta),Z_i^n(s+\theta))|^2 \mathrm{ds} - \frac{1}{4K}E\int_0^T |Y_i(s)-Y_i^n(s)|^2 \mathrm{ds}$$
$$-4KE\int_0^T |Z_i(s)-Z_i^n(s)|^2 \mathrm{ds}.$$

By using assumption (H2), we have that

$$E|Y_i(t)-Y_i^n(t)|^2 \le 4KE\int_0^T |Y_i(s)-Y_i^n(s)|^2 \mathrm{ds} + \frac{1}{4K}E\int_0^T K_2 |Y_i(s)-Y_i^n(s)|^2 \mathrm{ds}$$
$$+\frac{1}{4K}E[\int_0^T K_4 \int_{-T}^0 |Y_i(s+\theta)-Y_i^n(s+\theta)|^2 \alpha(\mathrm{d}\theta)\mathrm{ds}]$$
$$+\frac{1}{4K}E[\int_0^T K_4 \int_{-T}^0 |Z_i(s+\theta)-Z_i^n(s+\theta)|^2 \alpha(\mathrm{d}\theta)\mathrm{ds}]$$
$$+E\int_0^T K_2 |Y_i(s)-Y_i^n(s)|^2 \mathrm{ds}$$
$$+E[\int_0^T K_4 \int_{-T}^0 |Y_i(s+\theta)-Y_i^n(s+\theta)|^2 \alpha(\mathrm{d}\theta)\mathrm{ds}]$$
$$+E\int_0^T K_2 |Z_i(s)-Z_i^n(s)|^2 \mathrm{ds}$$
$$+E[\int_0^T K_4 \int_{-T}^0 |Z_i(s+\theta)-Z_i^n(s+\theta)|^2 \alpha(\mathrm{d}\theta)\mathrm{ds}]$$
$$+\frac{1}{4K}E\int_0^T K_2 |Z_i(s)-Z_i^n(s)|^2 \mathrm{ds}$$
$$+4KE\int_0^T |Y_i(s)-Y_i^n(s)|^2 \mathrm{ds} + \frac{1}{4K}E\int_0^T K_2 |Y_i(s)-Y_i^n(s)|^2 \mathrm{ds}$$
$$+\frac{1}{4K}E\int_0^T K_2 |Z_i(s)-Z_i^n(s)|^2 \mathrm{ds}$$

$$+\frac{1}{4K}E[\int_0^T K_4\int_{-T}^0 |Y_i(s+\theta)-Y_i^n(s+\theta)|^2\ \alpha(\mathrm{d}\theta)\mathrm{ds}]$$
$$+\frac{1}{4K}E[\int_0^T K_4\int_{-T}^0 |Z_i(s+\theta)-Z_i^n(s+\theta)|^2\ \alpha(\mathrm{d}\theta)\mathrm{ds}]$$
$$-\frac{1}{4K}E\int_0^T |Y_i(s)-Y_i^n(s)|^2\ \mathrm{ds}-4KE\int_0^T |Z_i(s)-Z_i^n(s)|^2\ \mathrm{ds}.$$

By changing the integration order, we obtain that

$$E\,|Y_i(t)-Y_i^n(t)|^2\le 4KE\int_0^T |Y_i(s)-Y_i^n(s)|^2\ \mathrm{ds}+\frac{K_2}{4K}E\int_0^T |Y_i(s)-Y_i^n(s)|^2\ \mathrm{ds}$$
$$+\frac{K_2}{4K}E\int_0^T |Z_i(s)-Z_i^n(s)|^2\ \mathrm{ds}+\frac{K_4}{2K}E[\int_{-T}^0\int_0^T |Y_i(s+\theta)-Y_i^n(s+\theta)|^2\ \mathrm{ds}\alpha(\mathrm{d}\theta)]$$
$$+\frac{K_4}{4K}E[\int_{-T}^0\int_0^T |Z_i(s+\theta)-Z_i^n(s+\theta)|^2\ \mathrm{ds}\alpha(\mathrm{d}\theta)]+K_2E\int_0^T |Y_i(s)-Y_i^n(s)|^2\ \mathrm{ds}$$
$$+K_2E\int_0^T |Z_i(s)-Z_i^n(s)|^2\ \mathrm{ds}+K_4E[\int_{-T}^0\int_0^T |Y_i(s+\theta)-Y_i^n(s+\theta)|^2\ \mathrm{ds}\alpha(\mathrm{d}\theta)]$$
$$+K_4E[\int_{-T}^0\int_0^T |Z_i(s+\theta)-Z_i^n(s+\theta)|^2\ \mathrm{ds}\alpha(\mathrm{d}\theta)]+4KE\int_0^T |Y_i(s)-Y_i^n(s)|^2\ \mathrm{ds}$$
$$+\frac{K_2}{4K}E\int_0^T |Y_i(s)-Y_i^n(s)|^2\ \mathrm{ds}+\frac{K_2}{4K}E\int_0^T |Z_i(s)-Z_i^n(s)|^2\ \mathrm{ds}$$
$$+\frac{K_4}{2K}E[\int_{-T}^0\int_0^T |Y_i(s+\theta)-Y_i^n(s+\theta)|^2\ \mathrm{ds}\alpha(\mathrm{d}\theta)]$$
$$+\frac{K_4}{4K}E[\int_{-T}^0\int_0^T |Z_i(s+\theta)-Z_i^n(s+\theta)|^2\ \mathrm{ds}\alpha(\mathrm{d}\theta)]$$
$$-\frac{1}{4K}E\int_0^T |Y_i(s)-Y_i^n(s)|^2\ \mathrm{ds}-4KE\int_0^T |Z_i(s)-Z_i^n(s)|^2\ \mathrm{ds}.$$

And then, we obtain that

$$E\,|Y_i(t)-Y_i^n(t)|^2\le 4KE\int_0^T |Y_i(s)-Y_i^n(s)|^2\ \mathrm{ds}+\frac{K_2}{4K}E\int_0^T |Y_i(s)-Y_i^n(s)|^2\ \mathrm{ds}$$
$$+\frac{K_2}{4K}E\int_0^T |Z_i(s)-Z_i^n(s)|^2\ \mathrm{ds}+\frac{K_4}{4K}E[\int_{-T}^0\int_\theta^{T+\theta} |Y_i(s)-Y_i^n(s)|^2\ \mathrm{ds}\alpha(\mathrm{d}\theta)]$$
$$+\frac{K_4}{4K}E[\int_{-T}^0\int_\theta^{T+\theta} |Z_i(s)-Z_i^n(s)|^2\ \mathrm{ds}\alpha(\mathrm{d}\theta)]+K_2E\int_0^T |Y_i(s)-Y_i^n(s)|^2\ \mathrm{ds}$$
$$+K_2E\int_0^T |Z_i(s)-Z_i^n(s)|^2\ \mathrm{ds}+K_4E[\int_{-T}^0\int_\theta^{T+\theta} |Y_i(s)-Y_i^n(s)|^2\ \mathrm{ds}\alpha(\mathrm{d}\theta)]$$
$$+K_4E[\int_{-T}^0\int_\theta^{T+\theta} |Z_i(s)-Z_i^n(s)|^2\ \mathrm{ds}\alpha(\mathrm{d}\theta)]+4KE\int_0^T |Y_i(s)-Y_i^n(s)|^2\ \mathrm{ds}$$
$$+\frac{K_2}{4K}E\int_0^T |Y_i(s)-Y_i^n(s)|^2\ \mathrm{ds}+\frac{K_2}{4K}E\int_0^T |Z_i(s)-Z_i^n(s)|^2\ \mathrm{ds}$$
$$+\frac{K_4}{4K}E[\int_{-T}^0\int_\theta^{T+\theta} |Y_i(s)-Y_i^n(s)|^2\ \mathrm{ds}\alpha(\mathrm{d}\theta)]$$
$$+\frac{K_4}{4K}E[\int_{-T}^0\int_\theta^{T+\theta} |Z_i(s)-Z_i^n(s)|^2\ \mathrm{ds}\alpha(\mathrm{d}\theta)]$$

$$-\frac{1}{4K}E\int_0^T|Y_i(s)-Y_i^n(s)|^2\,\mathrm{ds}-4KE\int_0^T|Z_i(s)-Z_i^n(s)|^2\,\mathrm{ds}.$$

And then, we have that

$$\begin{aligned}E|Y_i(t)-Y_i^n(t)|^2\le{}& 4KE\int_0^T|Y_i(s)-Y_i^n(s)|^2\,\mathrm{ds}+\frac{K_2}{4K}E\int_0^T|Y_i(s)-Y_i^n(s)|^2\,\mathrm{ds}\\&+\frac{K_2}{4K}E\int_0^T|Z_i(s)-Z_i^n(s)|^2\,\mathrm{ds}+\frac{K_4\beta}{4K}E\int_0^T|Y_i(s)-Y_i^n(s)|^2\,\mathrm{ds}\\&+\frac{K_4\beta}{4K}E\int_0^T|Z_i(s)-Z_i^n(s)|^2\,\mathrm{ds}+K_2E\int_0^T|Y_i(s)-Y_i^n(s)|^2\,\mathrm{ds}\\&+K_2E\int_0^T|Z_i(s)-Z_i^n(s)|^2\,\mathrm{ds}+K_4\beta E\int_0^T|Y_i(s)-Y_i^n(s)|^2\,\mathrm{ds}\\&+K_4\beta E\int_0^T|Z_i(s)-Z_i^n(s)|^2\,\mathrm{ds}+4KE\int_0^T|Y_i(s)-Y_i^n(s)|^2\,\mathrm{ds}\\&+\frac{K_2}{4K}E\int_0^T|Y_i(s)-Y_i^n(s)|^2\,\mathrm{ds}+\frac{K_2}{4K}E\int_0^T|Z_i(s)-Z_i^n(s)|^2\,\mathrm{ds}\\&+\frac{K_4\beta}{4K}E\int_0^T|Y_i(s)-Y_i^n(s)|^2\,\mathrm{ds}+\frac{K_4\beta}{4K}E\int_0^T|Z_i(s)-Z_i^n(s)|^2\,\mathrm{ds}\\&-\frac{1}{4K}E\int_0^T|Y_i(s)-Y_i^n(s)|^2\,\mathrm{ds}-4KE\int_0^T|Z_i(s)-Z_i^n(s)|^2\,\mathrm{ds},\end{aligned}$$

where $\beta=\int_{-T}^0\alpha(\mathrm{d}\theta)$,

$$\int_\theta^{T+\theta}|Y_i(s)-Y_i^n(s)|^2\,\mathrm{ds}\le\int_0^T|Y_i(s)-Y_i^n(s)|^2\,\mathrm{ds}$$

and

$$\int_\theta^{T+\theta}|Z_i(s)-Z_i^n(s)|^2\,\mathrm{ds}\le\int_0^T|Z_i(s)-Z_i^n(s)|^2\,\mathrm{ds}.$$

Therefore, we have that

$$\begin{aligned}E|Y_i(t)-Y_i^n(t)|^2\le{}&(8K+\frac{K_2}{2K}+\frac{K_4\beta}{4K}+K_2+K_4\beta+\frac{K_4}{4K}-\frac{1}{4K})E\int_0^T|Y_i(s)-Y_i^n(s)|^2\,\mathrm{ds}\\&+(\frac{K_2}{2K}+\frac{K_4\beta}{4K}+K_2+K_4\beta+\frac{K_4}{4K}-4K)E\int_0^T|Z_i(s)-Z_i^n(s)|^2\,\mathrm{ds}.\end{aligned}$$

By choosing

$$C_1=8K+\frac{K_2}{2K}+\frac{K_4\beta}{4K}+K_2+K_4\beta+\frac{K_4}{4K}-\frac{1}{4K}$$

and

$$C_2=\frac{K_2}{2K}+\frac{K_4\beta}{4K}+K_2+K_4\beta+\frac{K_4}{4K},$$

$C_1,C_2>0$ and $C_2\le K$, we deduce that

$$E|Y_i(t)-Y_i^n(t)|^2\le C_1E\int_0^T|Y_i(s)-Y_i^n(s)|^2\,\mathrm{ds}+(C_2-4K)E\int_0^T|Z_i(s)-Z_i^n(s)|^2\,\mathrm{ds}.$$

And then, we get that

$$E|Y_i(t)-Y_i^n(t)|^2\le C_1E\int_0^T|Y_i(s)-Y_i^n(s)|^2\,\mathrm{ds}.$$

Now, using the Gronwall's inequality, and for all $i=0,\cdots,n$ and $t\in[0,T]$, we drive that

$$\lim_{n\to\infty}E[\max_{i=0,\cdots,n}|Y_i(t)-Y_i^n(t)|^2]=0.$$

Hence $\lim_{n\to\infty} E|Y(t)-Y^n(t)|^2 = 0$, and consequently

$$\lim_{n\to\infty} E\int_0^T |Z(t)-Z^n(t)|^2 \,\mathrm{dt} = 0.$$

Theorem 4.2 Assume the assumptions (H1)-(H5) are fulfilled, then it holds that

$$\sup_{0\le t\le T} E|Y(t)-\widetilde{Y}(t)|^2 + E[\int_0^T |Z(s)-\widetilde{Z}(s)|^2 \,\mathrm{ds}] \le C|\delta|.$$

Proof. By applying $It\hat{o}'s$ formula $|Y(t)-\widetilde{Y}(t)|^2$, we have that

$$\begin{aligned}
&|Y(t)-\widetilde{Y}(t)|^2 + \int_0^T |Y(s)-\widetilde{Y}(s)|^2 \,\mathrm{ds} + \int_0^T |Z(s)-\widetilde{Z}(s)|^2 \,\mathrm{ds} \le |\xi(T)-\widetilde{\xi}(T)|^2 \\
&+2\int_0^T \langle Y(s)-\widetilde{Y}(s), f(s,Y(s),Z(s),Y_s,Z_s) - f(s,\widetilde{Y}(s),\widetilde{Z}(s),\widetilde{Y}_s,\widetilde{Z}_s)\rangle \mathrm{ds} \\
&+\int_0^T |g(s,Y(s),Z(s),Y_s,Z_s) - g(s,\widetilde{Y}(s),\widetilde{Z}(s),\widetilde{Y}_s,\widetilde{Z}_s)|^2 \,\mathrm{ds} \\
&+2\int_0^T \langle Y(s)-\widetilde{Y}(s),(g(s,Y(s),Z(s),Y_s,Z_s) - g(s,\widetilde{Y}(s),\widetilde{Z}(s),\widetilde{Y}_s,\widetilde{Z}_s))\mathrm{dB(s)}\rangle \\
&-2\int_0^T \langle Y(s)-\widetilde{Y}(s),(Z(s)-\widetilde{Z}(s))\mathrm{dW(s)}\rangle,
\end{aligned}$$

where $t\in[0,T]$. By Youn's inequality and assumption (H2), we get that

$$\begin{aligned}
&2\int_0^T \langle Y(s)-\widetilde{Y}(s), f(s,Y(s),Z(s),Y_s,Z_s) - f(s,\widetilde{Y}(s),\widetilde{Z}(s),\widetilde{Y}_s,\widetilde{Z}_s)\rangle \mathrm{ds} \le \gamma\int_0^T |Y(s)-\widetilde{Y}(s)|^2 \,\mathrm{ds} \\
&+\frac{1}{\gamma}\int_0^T |f(s,Y(s),Z(s),Y_s,Z_s) - f(s,\widetilde{Y}(s),\widetilde{Z}(s),\widetilde{Y}_s,\widetilde{Z}_s)|^2 \,\mathrm{ds} \\
&\le \gamma\int_0^T |Y(s)-\widetilde{Y}(s)|^2 \,\mathrm{ds} + \frac{1}{\gamma}\int_0^T [K_2|Y(s)-\widetilde{Y}(s)|^2 + K_2|Z(s)-\widetilde{Z}(s)|^2 \\
&+K_4\int_{-T}^0 |Y(s+\theta)-\widetilde{Y}(s+\theta)|^2 \alpha(\mathrm{d}\theta) + K_4\int_{-T}^0 |Z(s+\theta)-\widetilde{Z}(s+\theta)|^2 \alpha(\mathrm{d}\theta)]\mathrm{ds} \\
&= \gamma\int_0^T |Y(s)-\widetilde{Y}(s)|^2 \,\mathrm{ds} + \frac{K_2}{\gamma}\int_0^T |Y(s)-\widetilde{Y}(s)|^2 \,\mathrm{ds} + \frac{K_2}{\gamma}\int_0^T |Z(s)-\widetilde{Z}(s)|^2 \,\mathrm{ds} \\
&+\frac{K_4}{\gamma}\int_0^T\int_{-T}^0 |Y(s+\theta)-\widetilde{Y}(s+\theta)|^2 \alpha(\mathrm{d}\theta)\mathrm{ds} + \frac{K_4}{\gamma}\int_0^T\int_{-T}^0 |Z(s+\theta)-\widetilde{Z}(s+\theta)|^2 \alpha(\mathrm{d}\theta)\mathrm{ds},
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^T |g(s,Y(s),Z(s),Y_s,Z_s) - g(s,\widetilde{Y}(s),\widetilde{Z}(s),\widetilde{Y}_s,\widetilde{Z}_s)|^2 \,\mathrm{ds} \le K_3\int_0^T |Y(s)-\widetilde{Y}(s)|^2 \,\mathrm{ds} \\
&+K_3\int_0^T |Z(s)-\widetilde{Z}(s)|^2 \,\mathrm{ds} + K_4\int_0^T\int_{-T}^0 |Y(s+\theta)-\widetilde{Y}(s+\theta)|^2 \alpha(\mathrm{d}\theta)\mathrm{ds} \\
&+K_4\int_0^T\int_{-T}^0 |Z(s+\theta)-\widetilde{Z}(s+\theta)|^2 \alpha(\mathrm{d}\theta)\mathrm{ds}.
\end{aligned}$$

By changing of integration order argument, we obtain that

$$\begin{aligned}
&\int_0^T\int_{-T}^0 |Y(s+\theta)-\widetilde{Y}(s+\theta)|^2 \alpha(\mathrm{d}\theta)\mathrm{ds} = \int_{-T}^0\int_0^T |Y(s+\theta)-\widetilde{Y}(s+\theta)|^2 \,\mathrm{ds}\alpha(\mathrm{d}\theta) \\
&= \int_{-T}^0\int_\theta^{T+\theta} |Y(t)-\widetilde{Y}(t)|^2 \,\mathrm{dt}\alpha(\mathrm{d}\theta) \le \beta\int_0^T |Y(t)-\widetilde{Y}(t)|^2 \,\mathrm{dt},
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^T\int_{-T}^0 |Z(s+\theta)-\widetilde{Z}(s+\theta)|^2 \alpha(\mathrm{d}\theta)\mathrm{ds} = \int_{-T}^0\int_0^T |Z(s+\theta)-\widetilde{Z}(s+\theta)|^2 \,\mathrm{ds}\alpha(\mathrm{d}\theta) \\
&= \int_{-T}^0\int_\theta^{T+\theta} |Z(t)-\widetilde{Z}(t)|^2 \,\mathrm{dt}\alpha(\mathrm{d}\theta) \le \beta\int_0^T |Z(t)-\widetilde{Z}(t)|^2 \,\mathrm{dt},
\end{aligned}$$

where $\beta=\int_{-T}^{0}\alpha(d\theta)$. Therefore, we drive that

$$
\begin{aligned}
&|Y(t)-\widetilde{Y}(t)|^2+\int_0^T|Y(s)-\widetilde{Y}(s)|^2\,\mathrm{ds}+\int_0^T|Z(s)-\widetilde{Z}(s)|^2\,\mathrm{ds}\,|\leq\xi(T)-\widetilde{\xi}(T)|^2\\
&+\gamma\int_0^T|Y(s)-\widetilde{Y}(s)|^2\,\mathrm{ds}+\frac{K_2}{\gamma}\int_0^T|Y(s)-\widetilde{Y}(s)|^2\,\mathrm{ds}+\frac{K_2}{\gamma}\int_0^T|Z(s)-\widetilde{Z}(s)|^2\,\mathrm{ds}\\
&+\frac{K_4\beta}{\gamma}\int_0^T|Y(s)-\widetilde{Y}(s)|^2\,\mathrm{ds}+\frac{K_4\beta}{\gamma}\int_0^T|Z(s)-\widetilde{Z}(s)|^2\,\mathrm{ds}\\
&+K_3\int_0^T|Y(s)-\widetilde{Y}(s)|^2\,\mathrm{ds}+K_3\int_0^T|Z(s)-\widetilde{Z}(s)|^2\,\mathrm{ds}\\
&+K_4\beta\int_0^T|Y(s)-\widetilde{Y}(s)|^2\,\mathrm{ds}+K_4\beta\int_0^T|Z(s)-\widetilde{Z}(s)|^2\,\mathrm{ds}\\
&+2\int_0^T\langle Y(s)-\widetilde{Y}(s),(g(s,Y(s),Z(s),Y_s,Z_s)-g(s,\widetilde{Y}(s),\widetilde{Z}(s),\widetilde{Y}_s,\widetilde{Z}_s))\mathrm{dB(s)}\rangle\\
&-2\int_0^T\langle Y(s)-\widetilde{Y}(s),(Z(s)-\widetilde{Z}(s))\mathrm{dW(s)}\rangle.
\end{aligned}
$$

Therefore, we have that

$$
\begin{aligned}
&|Y(t)-\widetilde{Y}(t)|^2+\int_0^T|Y(s)-\widetilde{Y}(s)|^2\,\mathrm{ds}+\int_0^T|Z(s)-\widetilde{Z}(s)|^2\,\mathrm{ds}\leq|\xi(T)-\widetilde{\xi}(T)|^2\\
&+\gamma\int_0^T|Y(s)-\widetilde{Y}(s)|^2\,\mathrm{ds}+\frac{K_2}{\gamma}\int_0^T|Y(s)-\widetilde{Y}(s)|^2\,\mathrm{ds}+\frac{K_2}{\gamma}\int_0^T|Z(s)-\widetilde{Z}(s)|^2\,\mathrm{ds}\\
&+\frac{K_4\beta}{\gamma}\int_0^T|Y(s)-\widetilde{Y}(s)|^2\,\mathrm{ds}+\frac{K_4\beta}{\gamma}\int_0^T|Z(s)-\widetilde{Z}(s)|^2\,\mathrm{ds}\\
&+K_3\int_0^T|Y(s)-\widetilde{Y}(s)|^2\,\mathrm{ds}+K_3\int_0^T|Z(s)-\widetilde{Z}(s)|^2\,\mathrm{ds}\\
&+K_4\beta\int_0^T|Y(s)-\widetilde{Y}(s)|^2\,\mathrm{ds}+K_4\beta\int_0^T|Z(s)-\widetilde{Z}(s)|^2\,\mathrm{ds}\\
&+2\lambda_1\int_0^T|Y(s)-\widetilde{Y}(s)|^2\,\mathrm{ds}+\frac{2K_3}{\lambda_1}\int_0^T|Y(s)-\widetilde{Y}(s)|^2\,\mathrm{ds}\\
&+\frac{2K_3}{\lambda_1}\int_0^T|Z(s)-\widetilde{Z}(s)|^2\,\mathrm{ds}+\frac{2K_4\beta}{\lambda_1}\int_0^T|Y(s)-\widetilde{Y}(s)|^2\,\mathrm{ds}\\
&+\frac{2K_4\beta}{\lambda_1}\int_0^T|Z(s)-\widetilde{Z}(s)|^2\,\mathrm{ds}+2\lambda_2\int_0^T|Y(s)-\widetilde{Y}(s)|^2\,\mathrm{ds}\\
&+\frac{2}{\lambda_2}\int_0^T|Z(s)-\widetilde{Z}(s)|^2\,\mathrm{ds},
\end{aligned}
$$

where $\lambda_1,\lambda_2>0$, by taking the expectation and choosing

$$C_1=1-\gamma-\frac{K_2}{\gamma}-\frac{K_4\beta}{\gamma}-K_3-K_4\beta-2\lambda_1-\frac{2K_3}{\lambda_1}-\frac{2K_4\beta}{\lambda_1}-2\lambda_2$$

and

$$C_2=1-\frac{K_2}{\gamma}-\frac{K_4\beta}{\gamma}-K_3-K_4\beta-\frac{2K_3}{\lambda_1}-\frac{2K_4\beta}{\lambda_1}-\frac{2}{\lambda_2},$$

we have that

$$E|Y(t)-\widetilde{Y}(t)|^2+C_1E\int_0^T|Y(s)-\widetilde{Y}(s)|^2\,\mathrm{ds}+C_2E\int_0^T|Z(s)-\widetilde{Z}(s)|^2\,\mathrm{ds}\leq E|\xi(T)-\widetilde{\xi}(T)|^2.$$

For sufficiently small K_3 and K_4 choosing $\gamma, \lambda_1, \lambda_2 > 0$ such that $C_1 > 0$ and $C_2 > 0$, then there exists a constant $C > 0$ depending on $\gamma, K_1, K_2, K_3, K_4, \beta, \lambda_1$ and λ_2 such that

$$\sup_{0 \le t \le T} E|Y(t) - \tilde{Y}(t)|^2 + E\int_0^T |Y(s) - \tilde{Y}(s)|^2 \, ds + E\int_0^T |Z(s) - \tilde{Z}(s)|^2 \, ds \le CE|\xi(T) - \tilde{\xi}(T)|^2$$

. By assumption (H1), we have that

$$\sup_{0 \le t \le T} E|Y(t) - \tilde{Y}(t)|^2 + E\int_0^T |Z(s) - \tilde{Z}(s)|^2 \, ds \le C|\delta|.$$

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