



# OPTIMAL DISTRIBUTED CONTROL OF STOCHASTIC ELLIPTIC SYSTEMS WITH CONSTRAINTS

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## ABSTRACT

The objective of this paper is to study the optimality for stochastic non cooperative elliptic systems. A distributed control problem for a stochastic elliptic systems with constraints on states and controls is studied. First, the existence and uniqueness of the state process for these systems are proved. The necessary and sufficient conditions of optimality are derived for the Dirichlet and Neumann problems.

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## INTRODUCTION

Stochastic systems play an important role in mathematical models of phenomena in many fields of science, engineering, finance, biology, epidemiology and economics. In view probability theory, state of a stochastic system is randomly determined. Recently, Many researchers have been directed to the studies of optimal control of stochastic systems due to their last importance. They deal with stochastic differential equations (SDEs) [6,16].

Zhou was one of the first scientists to have used stochastic partial differential equations (SPDEs). He developed the necessary conditions of optimality (maximum principle) for a very strictly elliptic second order partial differential equations on a  $d$ -dimensional space [15]. The theory of backward and forward partial differential equations (PDEs) has been developed quickly, and it became a powerful tool in the study of PDEs. Moreover, backward and forward SPDEs appeared and was studied in many literatures (see e.g. [1,3]). The optimality for SPDEs is discussed in [15]. There are difficulties in the discussion of the solution methods of systems of forward and backward SPDEs. So, the unique method for the solution is the maximum principle. Researchers used the same steps; they discussed the maximum principle to face the difficulties of investigating the solutions.

Our first interest in this work is to implement new strategies that give a full study of the stochastic control problems. Firstly, we derive the optimality conditions. Secondly, we apply the easiest methods to simplify the technique of the stochastic control problem. Finally, we develop the study of the stochastic optimal control without constraints on the stochastic systems and without forward and backward SPDEs method. In this paper, we discuss the optimal control problem for  $2 \times 2$  stochastic non cooperative elliptic systems with Dirichlet and Neumann conditions. We give the unique solution for these systems (the state process of the system). Then after, we formulate the control problem and proved the existence and uniqueness of the control. Also, we present the set of equations and inequalities that characterizes the distributed control for these systems. Moreover, the optimal control problem of the generalized  $N \times N$  stochastic Neumann elliptic systems with constraints is studied. In all our considered problems the control of distributed type with state constraints is discussed.

This paper is organized as follows: In section 1; we introduce the basic notations and some function spaces defined on  $\mathbb{R}^1$ . Section 2 is devoted to study the optimal control for  $2 \times 2$  stochastic elliptic systems with Dirichlet conditions. In section 3, we also study the problem with Neumann conditions and we introduce the main results of this paper.

## 1. Notations

This section covers the basic definitions and notations, which are necessary to present our work. Let  $G$  be a nonempty open set in  $\mathbb{R}^n$ , the function  $\phi: G \rightarrow \mathbb{R}$  is said to have a compact support if there exists a compact subset  $K$  of  $G$  such that  $\phi(x) = 0$



$\forall x \in G \setminus K$ . Let  $D(G)$  be the space of infinitely differentiable functions  $\phi$  with compact support (space of test functions). The space of distributions is denoted by  $D'(G)$ ,  $D(G)$  is dense in  $L^2(G)$  and  $D(G) \subset L^p(G) \subset D'(G)$ .

For a multi-index  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbf{N}$ , where  $\alpha_i$  are non negative integers and  $\Phi \in D(G)$ , we define

$$D^\alpha(\Phi(x)) = \frac{\partial^{|\alpha|} \Phi}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_n^{\alpha_n}}(x), \quad x = (x_1, x_2, \dots, x_n)$$

where  $|\alpha| = \sum_{i=1}^n \alpha_i$ . The partial derivative  $\partial^{|\alpha|} T$  of a distribution  $T$  is defined by

$$\langle \partial^\alpha T, \Phi \rangle = (-1)^{|\alpha|} \langle \partial^\alpha \Phi, T \rangle, \quad \Phi \in D(G), T \in D'(G).$$

Let  $(\Omega, \mathbf{F}, P)$  be a probability space, where  $\Omega$  is a sample space,  $\mathbf{F}$  is an  $\sigma$ -algebra and  $P$  is a probability measure. We introduce

$$L^2(\Omega, \mathbf{F}, P; G) = \{u : G \times \Omega \rightarrow \mathbf{R} \mid u \text{ is measurable and } \int_\Omega P u P^2 dp < \infty\},$$

with inner product  $(\cdot, \cdot)_{L^2(\Omega, \mathbf{F}, P; G)} : L^2(\Omega, \mathbf{F}, P; G) \times L^2(\Omega, \mathbf{F}, P; G) \rightarrow \mathbf{R}$ , defined as

$$(u, v)_{L^2(\Omega, \mathbf{F}, P; G)} = \left( \int_\Omega \left( \int_G \nabla u(x) \nabla v(x) dx \right) dp \right)$$

The Sobolev space of order  $m$  [4], denoted by  $H^m(\Omega, \mathbf{F}, P; G)$ , is defined by

$$H^m(\Omega, \mathbf{F}, P; G) = \{v \in L^2(\Omega, \mathbf{F}, P; G), \frac{\partial^m v}{\partial x_i^m} \in L^2(\Omega, \mathbf{F}, P; G), 1 \leq i \leq n\},$$

with respect to the inner product

$$\langle f, g \rangle_{H^m(\Omega, \mathbf{F}, P; G)} = \sum_{|\alpha| \leq m} \langle D^\alpha f, D^\alpha g \rangle_{L^2(\Omega, \mathbf{F}, P; G)},$$

where  $\frac{\partial v}{\partial x_i}$  are the derivative in the sense of distribution.

An importance subspace of  $H^m(\Omega, \mathbf{F}, P; G)$  is given by

$$H_0^m(\Omega, \mathbf{F}, P; G) = \{v \in L^2(\Omega, \mathbf{F}, P; G), \frac{\partial^\alpha v}{\partial x_i^\alpha} = 0, |\alpha| \leq m-1\}.$$

The dual space of  $H_0^m(\Omega, \mathbf{F}, P; G)$  is denoted by  $H^{-m}(\Omega, \mathbf{F}, P; G)$ , thus we have the chain;

$$H_0^m(\Omega, \mathbf{F}, P; G) \subseteq L^2(\Omega, \mathbf{F}, P; G) \subseteq H^{-m}(\Omega, \mathbf{F}, P; G).$$

Let  $H$  be a Hilbert space. The bilinear form  $a : H \times H \rightarrow \mathbf{R}$  is said to be coercive if and only if there exists a constant  $c > 0$  such that  $a(u, u) \geq c P u P^2, \forall u \in H$ .

Let  $V$  be a Hilbert space on  $\mathbf{R}$  and  $V'$  be its dual space. The Lax- Milgram lemma [12] states: if the bilinear form  $a(u, v)$  satisfies the continuity and coerciveness conditions for  $u, v \in V$  and the linear form  $f \in V'$  satisfies the continuity condition for the system  $Au = f$  with model  $A$ , then there exists a unique solution  $u$  for the system  $Au = f$  such that  $a(u, v) = f(v)$ .



## 2. $2 \times 2$ Stochastic Elliptic Systems with Constraints and Dirichlet Conditions

In this section, we study the optimal control problem for non cooperative  $2 \times 2$  stochastic elliptic systems with Dirichlet Condition:

$$\begin{cases} -\Delta u_1(x) = W_1(x) - au_1(x) + bu_2(x) & x \in G_1 \\ -\Delta u_2(x) = W_2(x) - bu_1(x) - au_2(x) & x \in G_2 \\ u_1(x) = 0, u_2(x) = 0 & \text{on } \partial G, \end{cases} \quad (2.1)$$

with constraints

$$\begin{cases} \left\{ \frac{\partial u_k(x)}{\partial V_A} \right\}^\pm = t[u_k(x)] & \text{on } \gamma \\ = 0 & \text{on } \gamma \\ \left[ \frac{\partial u_k(x)}{\partial V_A} \right] = 0 & \text{on } \gamma \\ \frac{\partial u_k(x)}{\partial V_A} = \sum_{i,j=1}^n \frac{\partial u_k(x)}{\partial x_j} \cos(v, x_i) \quad k = 1, 2, \end{cases} \quad (2.2)$$

where  $a, b \in \mathbb{R}^+$  and  $G_1$  and  $G_2$  are two bounded, continuous and strictly domains in  $\mathbb{R}^n$  such that  $G_1 \cap G_2 = \emptyset$ . Here,  $G = G_1 \cup G_2$  with boundary  $\partial G = (\partial G_1 \cup \partial G_2) \setminus \gamma$ ,  $\gamma = \partial G_1 \cap \partial G_2 \neq \emptyset$ ,  $\cos(v, x_i) = i$  th direction cosine of  $v$ ,  $v$  being the normal at  $\gamma$ ,  $[u] = u^+ - u^-$ , where  $u^+ = \{u\}^+ = u(x)$  under  $x \in \partial G_2 \cap \gamma$ ,  $u^- = \{u\}^- = u(x)$  under  $x \in \partial G_1 \cap \gamma$ , and

$$0 \leq t \leq t(x) \leq v, t \in C(\gamma), v \text{ is a constant.} \quad (2.3)$$

While  $u(x) = (u_1(x), u_2(x)) \in H_0^1(\Omega, \mathcal{F}, P; G)$  is a state process and  $W(x) = (W_1(x), W_2(x))$  is a brownian motion (wiener process). In the following subsection, we present the existence and uniqueness of (2.1).

### 2.1 Existence and Uniqueness for Solution

In this subsection, we study the existence and uniqueness of solution for (2.1), (2.2). Since

$$H_0^1(\Omega, \mathcal{F}, P; G) \subseteq L^2(\Omega, \mathcal{F}, P; G) \subseteq H^{-1}(\Omega, \mathcal{F}, P; G),$$

then we have chain

$$[H_0^1(\Omega, \mathcal{F}, P; G)]^2 \subseteq [L^2(\Omega, \mathcal{F}, P; G)]^2 \subseteq [H^{-1}(\Omega, \mathcal{F}, P; G)]^2.$$

We define the following bilinear form on the sobolev space  $[H_0^1(\Omega, \mathcal{F}, P; G)]^2$ ,



$$\begin{aligned}
 b(u, \Phi) &= \mathbf{E} \left( \int_G (-\Delta u_1(x) \Phi_1(x) + a u_1(x) \Phi_1(x) - b u_2(x) \Phi_1(x)) dx \right) \\
 &+ \mathbf{E} \left( \int_G (-\Delta u_2(x) \Phi_2(x) + b u_1(x) \Phi_2(x) + a u_2(x) \Phi_2(x)) dx \right) \\
 &+ \mathbf{E} \left( \int_\gamma t[u_1(x)][\Phi_1(x)] + t[u_2(x)][\Phi_2(x)] d\gamma \right), \quad (2.4)
 \end{aligned}$$

the linear form is given by

$$L(\Phi) = \mathbf{E} \left( \int_G W_1(x) \Phi_1(x) + W_2(x) \Phi_2(x) dx \right) \quad (2.5)$$

where  $W_1(x)$  and  $W_2(x)$  are the components of the Wiener process. By Lax-Milgram lemma, we prove the following theorem.

**Theorem 2.1.** The bilinear form (2.4) satisfies the stochastic coerciveness condition and then there exists a unique solution  $u \in [H_0^1(\Omega, \mathbf{F}, P; G)]^2$  of system (2.1), conversely, if there exists a unique solution  $u \in [H_0^1(\Omega, \mathbf{F}, P; G)]^2$  such that  $b(u, \Phi) = L(\Phi)$ , then we get the system (2.1).

**Proof.** The bilinear form (2.4) can be written as

$$\begin{aligned}
 b(u, \Phi) &= \mathbf{E} \left( \int_G (-\Delta u_1(x) \Phi_1(x) + a u_1(x) \Phi_1(x) - b u_2(x) \Phi_1(x)) dx \right) \\
 &+ \mathbf{E} \left( \int_G (-\Delta u_2(x) \Phi_2(x) + a u_2(x) \Phi_2(x) + b u_1(x) \Phi_2(x)) dx \right) \\
 &+ \mathbf{E} \left( \int_\gamma (t[u_1(x)][\Phi_1(x)] + t[u_2(x)][\Phi_2(x)]) d\gamma \right)
 \end{aligned}$$

Applying Green's formula, gives

$$\begin{aligned}
 b(u, u) &= \mathbf{E} \left( \int_G (|\nabla u_1(x)|^2 + a(u_1(x))^2 - b u_2(x) u_1(x)) dx \right) \\
 &+ \mathbf{E} \left( \int_G (|\nabla u_2(x)|^2 + b u_1(x) u_2(x) + a(u_2(x))^2) dx \right) \\
 &+ \mathbf{E} \left( \int_\gamma (t[u_1(x)]^2 + t[u_2(x)]^2) d\gamma \right) \\
 &\geq \mathbf{P} u_1 \mathbf{P}_{(H_0^1(\Omega, \mathbf{F}, P; G))^2}^2 + \mathbf{P} u_2 \mathbf{P}_{(H_0^1(\Omega, \mathbf{F}, P; G))^2}^2
 \end{aligned}$$

Therefore,

$$b(u, u) \geq c \mathbf{P} u \mathbf{P}_{[H_0^1(\Omega, \mathbf{F}, P; G)]^2}, \quad (\text{Stochastic Ellipticity}) \quad (2.6)$$

where  $c = \text{const}$  and  $u = (u_1, u_2)$ . Since the bilinear form satisfies the continuity condition and the linear form is also continuous on  $[H_0^1(\Omega, \mathbf{F}, P; G)]^2$ , then by Lax-Milgram lemma, there exists a unique solution  $u = (u_1, u_2) \in [H_0^1(\Omega, \mathbf{F}, P; G)]^2$ , such that  $b(u, \Phi) = L(\Phi), \forall \Phi \in [H_0^1(\Omega, \mathbf{F}, P; G)]^2$ . Conversely, when  $b(u, \Phi) = L(\Phi), \forall \Phi \in [H_0^1(\Omega, \mathbf{F}, P; G)]^2$  and  $u = (u_1, u_2) \in [H_0^1(\Omega, \mathbf{F}, P; G)]^2$ , we have

$$\begin{aligned}
 &\mathbf{E} \left( \int_G (\nabla u_1(x) \nabla \Phi_1(x) + a u_1(x) \Phi_1(x) - b u_2(x) \Phi_1(x)) dx \right) \\
 &+ \mathbf{E} \left( \int_G (\nabla u_2(x) \nabla \Phi_2(x) + b u_1(x) \Phi_2(x) + a u_2(x) \Phi_2(x)) dx \right)
 \end{aligned}$$



$$\begin{aligned}
 &+ \mathbf{E} \left( \int_{\gamma} (t[u_1(x)][\Phi_1(x)] + t[u_2(x)][\Phi_2(x)]) d\gamma \right) \\
 &+ \mathbf{E} \left( \int_{\partial G} (F(u_1(x), \Phi_1(x)) + H(u_2(x), \Phi_2(x))) d\partial G \right) \\
 &- \mathbf{E} \left( \int_{\gamma} \left( \frac{\partial u_1(x)}{\partial V_A} \Phi_1(x) + \frac{\partial u_2(x)}{\partial V_A} \Phi_2(x) \right) d\gamma \right) \\
 &= \mathbf{E} \left( \int_G (W_1(x) \Phi_1(x) + W_2(x) \Phi_2(x)) dx \right),
 \end{aligned}$$

$$F(u_1(x), \Phi_1(x)) = H(u_2(x), \Phi_2(x)) = 0$$

(In Dirichlet problem  $u(x) = 0$ .) By comparison of two sides, we get the system (2.1), with constraints (2.2). This completes the proof. W

Now, we formulate the control problem with adding the control and we determine the cost functional.

## 2.2 Formulation of the Optimal Control Problem

Here, we formulate the problem and establish the necessary and sufficient conditions for the optimal control of distributed type. The space  $[L^2(\Omega, F, P; G)]^2$  being the space of controls. For a control  $y = (y_1, y_2) \in [L^2(\Omega, F, P; G)]^2$ , the state process of the system  $u = \{u_1(y), u_2(y)\}$  is given by the solution of the following system:

$$\begin{cases}
 -\Delta u_1(y) + au_1(y) - bu_2(y) = W_1 + y_1 & \text{in } G_1 \\
 u_1(y) = 0, u_2(y) = 0 & \text{on } \partial G \\
 -\Delta u_2(y) + bu_1(y) + au_2(y) = W_2 + y_2 & \text{in } G_2,
 \end{cases} \quad (2.7)$$

with conditions (2.2). The observation equation is given by  $\chi(y) = \{\chi_1(y), \chi_2(y)\} \equiv u(y)$ , the cost functional is given by:

$$C(y) = \mathbf{E} \left( \int_G ((u_1(y) - \chi_{1d})^2 + (u_2(y) - \chi_{2d})^2) dx \right) + \int_{\Omega} \left( \int_G (M(y_1^2 + y_2^2)) dx \right) dp, \quad (2.8)$$

where  $\chi_d = \{\chi_{1d}, \chi_{2d}\}$  in  $[L^2(\Omega, F, P; G)]^2$ .

Then, the control problem is defined by:

$$\begin{cases}
 y \in Y_{ad} & \text{such that} \\
 C(z) = \inf C(y) \quad \forall z \in Y_{ad},
 \end{cases}$$

where  $Y_{ad}$  is a closed convex subset from  $[L^2(\Omega, F, P; G)]^2$ .

Since the cost functional (2.8) can be written as:

$$\begin{aligned}
 C(y) &= \mathbf{E} \left( \int_G (((u_1(y) - u_1(0) + (u_1(0) - \chi_{1d}))^2) dx \right) \\
 &+ \mathbf{E} \left( \int_G (((u_2(y) - u_2(0) + (u_2(0) - \chi_{2d}))^2) dx \right)
 \end{aligned}$$



$$+ \int_{\Omega} \left( \int_G M(z_1^2 + z_2^2) dx \right) dp,$$

where

$$\Pi(y, z) = \mathbf{E} \left( \int_G ((u_1(y) - u_1(0))^2 + (u_1(z) - u_1(0))^2) dx \right) + \int_{\Omega} \int_G (M(z_1^2 + z_2^2)) dx dp, \quad (2.9)$$

$M > 0$  is a positive constant, then

$$L(z) = \mathbf{E} \left( \int_G (-u_1(0) + \chi_{1d})(u_1(i) - u_1(0)) + (-u_2(0) + \chi_{2d})(u_2(i) - u_2(0)) dx \right), \quad (2.10)$$

and  $\Pi(y, y)$  is a stochastic coercive on  $[L^2(\Omega, \mathbf{F}, P; G)]^2$ . Since  $L(z)$  is continuous on  $[L^2(\Omega, \mathbf{F}, P; G)]^2$ , then there exists a unique optimal control from the general theory in [8]. Moreover, we have the following theorem which gives the characterization of the optimal control.

**Theorem 2.2.** If the state  $u(y)$  is given by (2.1), (2.2) and if the cost functional is given by (2.8), then there exists a unique optimal control  $y = (y_1, y_2) \in Y_{ad}$  such that  $C(y) \leq C(z) \forall z \in Y_{ad}$ ; Moreover, it is characterized by:

$$\left\{ \begin{array}{ll} -\Delta h_1(y) + ah_1(y) + bh_2(y) = u_1(y) - \chi_{1d} & \text{in } G_1 \\ h_1(y) = 0, h_2(y) = 0 & \text{on } \partial G \\ -\Delta h_2(y) - bh_1(y) + ah_2(y) = u_2(y) - \chi_{2d} & \text{in } G_2, \end{array} \right.$$

with constraints

$$\left\{ \begin{array}{ll} \left\{ \frac{\partial h_1(y)}{\partial V_A^*} \right\}^{\pm} = t[h_1(y)], \left\{ \frac{\partial h_2(y)}{\partial V_A^*} \right\}^{\pm} = t[h_2(y)] & \text{on } \gamma \\ \left[ \frac{\partial h_1(y)}{\partial V_A^*} \right] = 0, \left[ \frac{\partial h_2(y)}{\partial V_A^*} \right] = 0 & \text{on } \gamma \\ [h_1(y)] = [h_2(y)] = 0 & \text{on } \gamma, \end{array} \right.$$

where  $h(y) = \{h_1(y), h_2(y)\}$  is the adjoint state process.

**Proof.** Since  $C(y)$  is differentiable and  $Y_{ad}$  is bounded, then the optimal control  $z$  is characterized (see e.g [8,9]). Using equations (2.9), (2.10), we get

$$\Pi(y, z - y) \geq L(z - y), \quad (2.11)$$

and

$$\begin{aligned} & \Pi(y, z - y) - L(z - y) \\ &= \mathbf{E} \left( \int_G ((u_1(y) - u_1(0))(u_1(z - y) - u_1(0))) dx \right) \\ &+ \mathbf{E} \left( \int_G ((u_2(y) - u_2(0))(u_2(z - y) - u_2(0))) dx \right) \\ &- \mathbf{E} \left( \int_G (((u_1(0) - \chi_{1d})(u_1(z - y) - u_1(0)))) dx \right) \end{aligned}$$



$$\begin{aligned}
 &+ \mathbb{E} \left( \int_G ((u_2(0) - \chi_{2d})(u_2(z-y) - u_2(0))) dx \right) \\
 &+ \int_{\Omega} \left( \int_G My_1(z_1 - y_1) + My_2(z_2 - y_2) dx \right) dp \\
 &= \int_{\Omega} \left( \int_G My_1(z_1 - y_1) + My_2(z_2 - y_2) dx \right) dp \\
 &+ \mathbb{E} \left( \int_G ((u_1(y) - \chi_{1d})(u_1(z) - u_1(y))) dx \right) \\
 &+ \mathbb{E} \left( \int_G ((u_2(y) - \chi_{2d})(u_2(z) - u_2(y))) dx \right) \geq 0,
 \end{aligned}$$

with  $(B^*h(y), u(y)) = (h(y), Bu(y))$ , and  $B$  is defined by:

$$B\Phi = B\{u_1(y), u_2(y)\} = (-\Delta u_1(y) + au_1(y) - bu_2(y), -\Delta u_2(y) + bu_1(y) + au_2(y)).$$

Applying the derivative in the sense of distribution, we get

$$B^*h(y) + M^T h(y) = u(y) - \chi_d,$$

where  $B = -\Delta$ ,  $M = a_{ij}$  and  $j = 1, 2$ . So,

$$\begin{aligned}
 &\Pi(y, z-y) - L(z-y) \\
 &= \mathbb{E} \left( \int_G ((ah_1 - bh_2)(u_1(z) - u_1(y)) + (bh_1 + ah_2)(u_2(z) - u_2(y))) dx \right) \\
 &+ \int_{\Omega} \left( \int_G (My_1, z_1 - y_1) + (My_2, z_2 - y_2) dx \right) dp \\
 &+ \mathbb{E} \left( \int_G (h_1(-\Delta u_1(z) + \Delta u_1(y)) + (h_2)(-\Delta u_2(z) + \Delta u_2(y))) dx \right) \geq 0
 \end{aligned}$$

Hence, from (2.11) we obtain

$$\mathbb{E} \left( \int_G ((h_1 + My_1)(z_1 - y_1) + (h_2 + My_2)(z_2 - y_2)) dx \right) \geq 0 \quad W$$

### 2.3 $N \times N$ Stochastic Elliptic Systems

In this section, we generalize the problem (2.1).

$$\begin{cases} Bu_i(x) = -Nu_i(x) + W_i(x) & \text{in } G, \\ u_i(x) = 0 & \text{on } \partial G, \end{cases} \quad (2.12)$$

with constraints

$$\begin{cases} [u_i(x)] = 0 & \text{on } \gamma \\ \left\{ \frac{\partial u_i(x)}{\partial V_A} \right\}^{\pm} = t[u_i(x)] & \text{on } \gamma \\ = 0 & \text{on } \gamma, i = 1, 2, \dots, n, \end{cases} \quad (2.13)$$



where  $G$  is an open subset of  $\mathbf{R}^n$  with smooth boundary,  $B$  is an  $n \times n$  diagonal matrix of Laplace operators ( $-\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ ), and  $N$  is an  $n \times n$  matrix with coefficients defined by:

$$a_{ij} = \begin{cases} -1 & i < j, \\ 1 & i \geq j. \end{cases}$$

The model  $B$  of the system (2.12) is given by

$$Bu = B\{u_1, u_2, u_3, \dots, u_n\} = \{-\Delta u_1 + \sum_{j=1}^n a_{1j} u_j, \dots, -\Delta u_n + \sum_{j=1}^n a_{nj} u_j\}.$$

On the Sobolev space

$$[H_0^1(\Omega, \mathbf{F}, P; G)]^n = \prod_{m=1}^n (H_0^1(\Omega, \mathbf{F}, P; G))^m,$$

we define the bilinear and linear forms:

$$b(u, \Phi) = \sum_{i=1}^n \mathbf{E} \left( \int_G ((-\Delta u_i) \Phi_i) dx + \int_\gamma (t[u_i]) [\Phi_i] d\gamma \right) + \sum_{i,j=1}^n \mathbf{E} \left( \int_G a_{ij} \Phi_i u_j dx \right),$$

$$L(\Phi) = \sum_{i=1}^n \mathbf{E} \left( \int_G W_i \Phi_i dx \right), \quad (2.14)$$

respectively. From (2.3) and (2.14), we have

$$\begin{aligned} b(u, u) &= \sum_{i=1}^n \mathbf{E} \left( \int_G |\nabla u_i|^2 dx + \int_\gamma (t[u_i]^2) d\gamma + \sum_{i=j}^n \int_G |u_j|^2 dx \right) \\ &\geq \sum_{i=1}^n \mathbf{E} \left( \int_G |\nabla u_i|^2 dx + \sum_{i=j}^n \int_G |u_j|^2 dx \right) \end{aligned}$$

i.e.,  $b(u, u)$  is stochastic coercive on  $[H_0^1(\Omega, \mathbf{F}, P; G)]^n$ . Then, by Lax-Milgram lemma, there exists a unique solution  $u = \{u_1, u_2, \dots, u_n\} \in [H_0^1(\Omega, \mathbf{F}, P; G)]^n$  such that  $b(u, \Phi) = L(\Phi)$ , which proves the existence of the state process for the system (2.10)-(2.12). So, we can formulate the corresponding control problem: The space  $(L^2(\Omega, \mathbf{F}, P; G))^n$  is the space of controls. For a given control  $y = (y_i)_{i=1}^n \in (L^2(\Omega, \mathbf{F}, P; G))^n$ , the state process  $u(y) = \{u_i(y)\}_{i=1}^n$  of the system is given by

$$\begin{cases} -\Delta u_i(y) = -\sum_{j=1}^n a_{ij} u_j(y) + W_i + y_i & \text{in } G_i \\ u_i(y) = 0 & \text{on } \partial G, \end{cases}$$

where  $i = 1, 2, \dots, n$ , with constraints (2.11). The cost functional is given by:

$$C(y) = \mathbf{E} \left( \sum_{i=1}^n \int_G ((u_i(y) - \chi_{id})^2) dx \right) + \sum_{i=1}^n \int_\Omega \left( \int_G ((M(y_i))^2) dx \right) dp,$$





where

$$\Pi(y, z) = \mathbf{E} \left( \sum_{i=1}^n \int_G ((u_i(y) - u_i(0))(u_i(z) - u_i(0))) dx \right) + \sum_{i=1}^n \int_{\Omega} \left( \int_G (M(u_i)^2) dx \right) dp,$$

$$L(z) = \mathbf{E} \left( \sum_{i=1}^n \int_G ((-u_i(0) + \chi_{id})(u_i(z) - u_i(0))) dx \right),$$

where  $\chi_d = \{\chi_{1d}, \chi_{2d}, \dots, \chi_{nd}\}$  in  $[L^2(\Omega, \mathbf{F}, P; G)]^n$ .

In this case, the necessary and sufficient condition for  $y$  to be an optimal control is that the following equations and inequalities be satisfied.

$$\begin{cases} Bh(y) + N^T h(y) = u(y) - \chi_d & \text{in } G \\ h_i(y) = 0 & \text{on } \partial G \\ \left\{ \frac{\partial h_i(y)}{\partial V_A^*} \right\}^{\pm} = t[h_i(y)], \left[ \frac{\partial h_i(y)}{\partial V_A^*} \right] = 0 & \text{on } \gamma, \end{cases}$$

where  $N^T$  is the transpose of  $N$  with variational inequality

$$\sum_{j=1}^n \mathbf{E} \left( \int_G (h_j + My_j)(z_j - y_j) dx \right) \geq 0, \text{ where } \frac{\partial h(y)}{\partial V_A^*} = \sum_{i,k=1}^n \cos(v, x_k) \frac{\partial h_i(y)}{\partial x_k}.$$

### Remark 2.1

1- If constraints are absent, i.e. when  $Y_{ad} = Y$ , then  $h(z) + My = 0, z_j \neq y_j$  or  $y = -\frac{h(z)}{M}$  the differential problem of finding the vector-function satisfies the the following relations.

For the state process

$$\begin{cases} Bu + Nu + \frac{h(z)}{M} = W & \text{in } G \\ u = 0 & \text{on } \partial G \\ \left\{ \frac{\partial u(y)}{\partial V_A} \right\}^{\pm} = t[u(y)], \left[ \frac{\partial u(y)}{\partial V_A} \right] = 0 & \text{on } \gamma. \end{cases}$$

For the adjoint state process



$$\left\{ \begin{array}{l} Bh(y) + N^T h(y) - u(y) = -\chi_d \quad \text{in } G \\ h(y) = 0 \quad \text{on } \partial G \\ \left\{ \frac{\partial h(y)}{\partial V_A^*} \right\}^\pm = t[h(y)], \left[ \frac{\partial h(y)}{\partial V_A^*} \right] = 0 \quad \text{on } \gamma. \end{array} \right.$$

2- We can find the last relations when  $N = 2$ , and get  $(2 \times 2)$  stochastic elliptic systems with constraints and distributed Dirichlet conditions.

### 3 $2 \times 2$ Neumann Stochastic Elliptic Systems with Constraints

In this section, we study the optimal control problem for non cooperative  $2 \times 2$  stochastic elliptic system with Neumann conditions.

$$\left\{ \begin{array}{l} -\Delta u_1(x) = W_1(x) - au_1(x) + bu_2(x) \quad \text{in } G_1 \\ \frac{\partial u_1(x)}{\partial V_A} = g_1, \frac{\partial u_2(x)}{\partial V_A} = g_2 \quad \text{on } \partial G \\ \Delta u_2(x) = W_2(x) - bu_1(x) - au_2(x) \quad \text{in } G_2, \end{array} \right. \quad (3.1)$$

under constraints

$$\left\{ \begin{array}{l} \left\{ \frac{\partial u_1(x)}{\partial V_A} \right\}^\pm = t[u_1(x)], \left[ \frac{\partial u_1(x)}{\partial V_A} \right] = 0 \quad \text{on } \gamma \\ \left\{ \frac{\partial u_2(x)}{\partial V_A} \right\}^\pm = t[u_2(x)], \left[ \frac{\partial u_2(x)}{\partial V_A} \right] = 0 \quad \text{on } \gamma, \end{array} \right.$$

where  $g \in H^{\frac{1}{2}}(\Omega, F, P; G)$ ,  $k = 1, 2$ ,  $u = \{u_1, u_2\}$ .

### 3.1 Existence and Uniqueness of Solution

In this subsection, we study the existence and uniqueness of solutions for  $2 \times 2$  non cooperative stochastic systems governed by Neumann problems. Since

$$[H_0^1(\Omega, F, P; G)]^2 \subseteq [H^1(\Omega, F, P; G)]^2,$$

then

$$PuP_{[H_0^1(\Omega, F, P; G)]^2}^2 \subseteq PuP_{[H^1(\Omega, F, P; G)]^2}^2,$$

which proves the coerciveness of bilinear form  $a(u, u)$  on  $[H^1(\Omega, F, P; G)]^2$

$$b(u, u) \geq cPuP_{[H^1(\Omega, F, P; G)]^2}^2 \quad (\text{Stochasticcoerciveness}) \quad (3.2)$$

### Theorem 3.1.

Assume that (3.2) holds, and then there exists a unique solution  $u = \{u_1, u_2\}$  of system (3.1).



**Proof.** Since the bilinear form  $b(u, \Psi)$  is continuous and stochastic coercive on  $[H^1(\Omega, F, P; G)]^2$ , then by Lax Milgram lemma there exist a unique solution of:

$$b(u, \Psi) = L(\Psi), \forall u \in [H^1(\Omega, F, P; G)]^2, (3.3)$$

where  $L(\Psi)$  is continuous linear form defined on  $[H^1(\Omega, F, P; G)]^2$  by using Green's formula, we obtain (3.1):

$$L(\Psi) = \mathbf{E} \left( \int_G (W_1 \Psi_1 + W_2 \Psi_2) dx + \int_{\partial G} (g_1 \Psi_1 + g_2 \Psi_2) d\partial G \right),$$

then (3.3) is equivalent to

$$\begin{aligned} b(u, \Psi) &= \mathbf{E} \left( \int_G (\nabla u_1 \nabla \Psi_1 + \nabla u_2 \nabla \Psi_2 + au_1 \Psi_1 - bu_2 \Psi_1 + bu_1 \Psi_2 + au_2 \Psi_2) dx \right) \\ &+ \mathbf{E} \left( \int_{\gamma} (t[u_1][\Psi_1] + t[u_2][\Psi_2]) d\gamma + \int_{\partial G} \left( \frac{\partial u_1(x)}{\partial V_A} \Psi_1 + \frac{\partial u_2(x)}{\partial V_A} \Psi_2 \right) d\partial G \right) \\ &= \mathbf{E} \left( \int_G (W_1 \Psi_1 + W_2 \Psi_2) dx + \int_{\partial G} (g_1 \Psi_1 + g_2 \Psi_2) d\partial G \right) \end{aligned}$$

Hence (3.3) is equivalent to (3.1) and there exists a unique solution of (3.1).

### 3.2 Formulation of the Optimal Control Problem with Neumann Conditions

Here, we formulate the problem and establish necessary and sufficient conditions for the optimal control of distributed type. The space  $[L^2(\Omega, F, P; G)]^2$  is the space of controls. For a control  $y \in [L^2(\Omega, F, P; G)]^2$ , the state  $u(y)$  of the system is given by the solution of

$$\left\{ \begin{array}{ll} -\Delta u_1(y) = -au_1(y) + bu_2(y) + W_1(y) + y_1 & \text{in } G_1 \\ \frac{\partial u_1(y)}{\partial V_A} = g_1, \frac{\partial u_2(y)}{\partial V_A} = g_2 & \text{on } \partial G \\ \left\{ \frac{\partial u_k(y)}{\partial V_A} \right\}^{\pm} = t[u_k(y)], \left[ \frac{\partial u_k(y)}{\partial V_A} \right] = 0, k = 1, 2 & \text{on } \gamma \\ \Delta u_2(y) = -bu_1(y) - au_2(y) + W_2(y) + y_2 & \text{in } G_2. \end{array} \right. (3.4)$$

The observation is given by  $\chi(y) = u(y)$ , the cost functional is given again by (3.4). The optimal control is characterize by the following theorem:

**Theorem 3.2.** Assume that (3.2) holds, if the cost functional is given by (2.7), then there exists an optimal control  $y = (y_1, y_2) \in [L^2(\Omega, F, P; G)]^2$ . Moreover, it is characterized by the following equations and inequalities:



$$\left\{ \begin{array}{l} -\Delta h_1(y) = -ah_1(y) - bh_2(y) + M \frac{\partial u_1(y)}{\partial V_A} - \chi_{1d} \quad \text{in } G_1 \\ \frac{\partial h_1(y)}{\partial V_A^*} = 0, \frac{\partial h_2(y)}{\partial V_A^*} = 0 \quad \text{on } \partial G \\ \Delta h_2(y) = bh_1(y) - ah_2(y) + M \frac{\partial u_2(y)}{\partial V_A} - \chi_{1d} \quad \text{in } G_2. \end{array} \right.$$

Under constraints

$$\left\{ \frac{\partial h_k(y)}{\partial V_A^*} \right\}^\pm = t[h_k(y)], \left[ \frac{\partial h_k(y)}{\partial V_A^*} \right] = 0 \quad \text{on } \gamma, k = 1, 2.$$

Together with (3.4), where  $p(u)$  is the adjoint state

$$\mathbb{E} \left( \int_G (Ny_1(z_1 - y_1) + Ny_2(z_2 - y_2) + h_1(z_1 - y_1) + h_2(z_2 - y_2)) dx \right) \geq 0. \quad (3.5)$$

### 3.3 Generalization $N \times N$ for Stochastic Neumann Elliptic Systems with Constraints

Here, we generalize the discussion to  $N \times N$  stochastic elliptic systems with Neumann condition

$$\left\{ \begin{array}{l} AU = -MU + W \quad \text{in } G \\ \frac{\partial U(x)}{\partial V_A} = g \quad \text{on } \partial G \\ \left\{ \frac{\partial U(x)}{\partial V_A} \right\}^\pm = t[U(x)], \left[ \frac{\partial U(x)}{\partial V_A} \right] = 0 \quad \text{on } \gamma, \end{array} \right.$$

where  $G$  is an open subset of  $\mathbb{R}^n$  with smooth boundary,  $A$  is an  $N \times N$  diagonal matrix of Laplace operator  $(-\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2})$ ,  $M$  is an  $N \times N$  matrix with coefficients defined by:

$$a_{ij} = \begin{cases} -1 & i < j, \\ 1 & i \geq j. \end{cases}$$

In this case, the bilinear and linear forms are given by:

$$\begin{aligned} b(u, \Psi) &= \sum_{i=1}^n \mathbb{E} \left( \int_G ((-\Delta u_i) \Psi_i) dx + \int_\gamma (t[u_i][\Psi_i]) d\gamma \right) \\ &+ \sum_{i,j=1}^n \mathbb{E} \left( \int_G a_{ij} \Psi_i y_j dx + \int_{\partial G} \frac{\partial y_i(u)}{\partial V_A} \Psi_i d\partial G \right) \end{aligned}$$



$$L(\Psi) = \sum_{i=1}^n \mathbb{E} \left( \int_G W_i \Psi_i dx + \int_{\partial G} g_i \Psi_i d\partial G \right)$$

The cost functional is given by:

$$C(y) = \mathbb{E} \left( \sum_{i=1}^n \int_G (u_i(y) - \chi_{id})^2 dx \right) + \sum_{i=1}^n \int_{\Omega} \int_G M(y_i)^2 dx dp,$$

where  $\chi_d = \{\chi_{1d}, \chi_{2d}, \dots, \chi_{nd}\}$  in  $[L^2(\Omega, \mathbf{F}, P; G)]^n$ . In this case the necessary and sufficient condition for  $y$  to be an optimal control is that the following equations and inequalities be satisfied.

$$\begin{cases} Ah(y) + M^T h(y) = M \frac{\partial h(y)}{\partial V_A^*} - \chi_d & \text{in } G \\ \frac{\partial h(u)}{\partial V_A^*} = 0 & \text{on } \partial G \\ \left\{ \frac{\partial h(u)}{\partial V_A^*} \right\}^\pm = t[h_i(y)], \left[ \frac{\partial h_i(y)}{\partial V_A^*} \right] = 0 & \text{on } \gamma, \end{cases}$$

where  $\frac{\partial h_i(y)}{\partial V_A^*} = \sum_{i,k=1}^n \cos(v, x_k) \frac{\partial h_i(y)}{\partial x_k}$ ,  $\sum_{j=1}^n \mathbb{E} \left( \int_G ((h_j + My_j)(z_j - y_j)) dx \right) \geq 0$

**Remark 3.1** 1- If constraints are absent, i.e. when  $Y_{ad} = Y$ , then  $h(y) + Ny = 0$  or  $y = -\frac{h(y)}{N}$  the differential problem of finding the vector-function satisfies the following relations:

For the state process

$$\begin{cases} AU + MU = W & \text{in } G \\ \frac{\partial U(y)}{\partial V_A} + \frac{h(u)}{N} = g & \text{on } \partial G \\ \left\{ \frac{\partial U(y)}{\partial V_A} \right\}^\pm = t[u(y)], \left[ \frac{\partial U(y)}{\partial V_A} \right] = 0 & \text{on } \gamma. \end{cases}$$

For the adjoint state process

$$\begin{cases} Ah(y) + M^T h(y) - M \frac{\partial h(y)}{\partial V_A^*} = -\chi_d & \text{in } G \\ \frac{\partial h(y)}{\partial V_A^*} = 0, & \text{on } \partial G \\ \left\{ \frac{\partial h(y)}{\partial V_A^*} \right\}^\pm = t[h(y)], \left[ \frac{\partial h_i(y)}{\partial V_A^*} \right] = 0 & \text{on } \gamma. \end{cases}$$



2- We can find the last relations when  $N = 2$ , and get  $2 \times 2$  stochastic elliptic systems with constraints and distributed Neumann conditions.

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