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Complexity Study and Chaos Control in a Prey-Predator System

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ABSTRACT

A prey-predator system has been investigated with the application of random shock. Since the fluctuations of populations are random, the applied shock is also assumed like a random noise. To study complexities during evolution, numerical simulations have been carried out for both cases, without shock and with shock. Stabilities of fixed points have been discussed for both the cases. Also, bifurcation diagrams for both the cases have been drawn by varying a parameter while keeping other parameters fixed. Numerical calculations have been extended to obtain plots of Lyapunov exponents and topological entropies as the measure of complexity in the system. It has been observed that the random shock has little impact to reduce the chaotic motion in the system. Then, certain periodic changes in a parameter have been allowed to some extent,this results in bringing the system from chaos to regularity. Such changes may happen naturally in a prey-predator system and so there exists the possibility of coexistence. The chaos indicator DLI has been used for clarity in detection of regular and chaotic motion. Finally,the correlation dimension for the chaotic set has also been calculated for certain set of parameter values.

Keywords: Random shock; Lyapunov exponents; topological entropy; bifurcation; correlation dimension.

Mathematics Subject Classification 2010: 03D15, 41A25,37L30, 34D08 , 37B40, 35B32, 37M20, 30D05, 32H50

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1. INTRODUCTION

Models of interacting species show unpredictability and random fluctuations due to ecological changes. For their very existence and search of resources, species compete, evolve and disperse to various locations. There are various models of prey-predator systems based on their specific settings, bio and ecosystems and interactions among the species. Such a system may take the form of parasite-host, plant-herbivore, susceptible-infectious interactions, tumor cells (virus)-immune system etc. Also, these models may predict extinction of one or both the species as well as the case of their coexistence completely depending on the eco environment.

Since the work of Lotka (1925) and Volterra (1931) on predator-prey evolution problem, many articles with various assumptions, modifications and generalizations on conditions of interactions of species appeared in literatures. Studies on ecological systems are not simple and so results obtained through various studies cannot be assumed absolutely perfect. Random population fluctuations may occur due to random environmental changes. Such changes lead to transformation of regular or periodic evolutions into very unpredictable and chaotic changes. A detailed review can be obtained from the recent work by Grafton and Silva-Echenique (1997). Prey-Predator models appearing in literature are either indiscrete or in continuous forms. A wide review in this context can be obtained in some recent articles by Grafton and Silva-Echenique. (1994), Collie and Spencer(1994), Budiansky(1995), Liu and Xiao (2007), Lynch (2007), Elsadany et al (2012), . Djellit et al (2013).

Complexity and chaotic evolutionary behavior, to some extent, can be measured by Lyapunov exponents, topological entropies and correlation dimensions. Measuring topological entropy, which is related to the growth rate of a material line, is again an effective tool of finding complexity of a system. Topological entropy describes the *rate of mixing* of a dynamical system. For aiterated system it represents the exponential growth rate of the number of distinguishable orbits of the iterates. The more complex system is, the more topological entropy it will have. For wide description of these tools, articles referred are those by Grassberger and Procaccia (1983), Sandri (1996), Martelli (1999), Nagashima and Baba (2005).

In their study Grafton and Silva-Echenique (1997) have used certain modified discrete type of model of LotkaVolterra, and have applied a random shock to study some management strategies. The purpose was to understand evolutionary behavior due to interaction of species in such a case and also, to see how it effects chaotic evolution of the system. Application of such random shock may be justified as the changes in the ecological systems are also random.

The objective of this investigation is to study in detail the complexity occurring during the evolution of prey-predator system which leads to non-periodic fluctuations in species. The non-periodic fluctuations result in complications of predictability of motion. The model dealt by Grafton and Silva-Echenique(1997) has been again applied here for studying chaos and aperiodic motions. The role of random shock studied by Grafton and Silva-Echenique, has again been carefully investigated to see if such application is beneficiary to control the unpredictable behavior. During the process of study, we draw bifurcation diagrams, Lyapunov exponents, topological entropies and correlation dimensions for cases of without shock and with shock to see change in behavior of the system. Finally, plots of chaos indicator “dynamic Lyapunov exponents (DLI)” for regular and chaotic cases have been obtained for more justification of the results. DLI was recently introduced by Saha and Budhraja (2006) and successfully used in articles byYuasa and Saha (2008), Saha and Tehri (2010), Deleanu (2011), Sahni et al (2013).

2. Prey-Predator Model and Application of Random Shock

Prey-Predator interaction models are considered as the most important applications of mathematics to biology. In this regard, Lotka-Volterra predator-prey model is the simplest one proposed by Lotka (1925) and Volterra (1926).

A general form of Prey-Predator model can be written in the form

$$x_{n+1} = x_n f(x_n, y_n)$$
$$y_{n+1} = y_n g(x_n, y_n) \qquad (2.1)$$

where, x_n and y_n are respectively the prey and predator populations at time n. The functions f and g be such that $\frac{\partial f}{\partial y} \le 0$ and $\frac{\partial g}{\partial x} \ge 0$. Keeping in mind this criteria, one may assume a predator-prey model as, Abd-Elalim et al (2012),

$$x_{n+1} = a_1 x_n (1 - x_n) - b_1 x_n y_n$$
$$y_{n+1} = a_2 y_n (1 - y_n) + b_2 x_n y_n \qquad (2.2)$$

where, all a_1 , b_1, a_2, b_2 are positive parameters. We wish to study the effect of random shock in the prey-predator system.

So, assuming only density dependence in the prey population, above model can be modified and written in more simplified form as

$$x_{n+1} = a x_n (1 - x_n) - b x_n y_n$$
$$y_{n+1} = b x_n y_n \qquad (2.3)$$

Fixed points of this system are $P_1^*(0, 0)$, $P_2^*(\frac{a-1}{a}, 0)$ and P_3^* $(\frac{1}{b}, \frac{ab-a-b}{b^2})$ and the Jacobian matrix is $J = \begin{pmatrix} a(1-x)-ax-by & -bx \\ by & bx \end{pmatrix}$. Any of the above fixed points be stable if the absolute value of determinant $|J|$ evaluated at that point be less than unity. So, performing stability analysis, one finds the fixed point P_1^*, i.e. the origin, is always stable. The fixed point P_2^* be stable if $(3b - 2\frac{b}{a} - ab) < 1$ and that of P_3^* be stable if $(a - 2\frac{a}{b}) < 1$; otherwise, these two are unstable. To proceed further, we have fixed b as b = 3.5 and restrict a up to the value a = 4.0. Then, through stability calculations we obtain P_2^* be initially unstable for $0 < a \le 0.806806$, then stable in $0.806808 \le a \le 2.478906$ and then unstable for $2.478907 \le a \le 4.0$. Proceeding similarly, we find P_3^* becomes initially stable in $0 < a \le 2.3333$ then unstable when a is further increased from this value, i.e. $2.3334 \le a \le 4.0$. These results, also, clearly reflect in the bifurcation diagram for cycle one.

Taking b = 3.5, and varying a, one obtains the bifurcation diagrams with respect to x and y directions as shown in Fig. 1. The right hand figures show the appearance of periodic windows of period 6 and period 7 after chaos within $3.3 \le a \le 3.7$. Such emergence of periodic windows is, of course, subject to parameter value b.

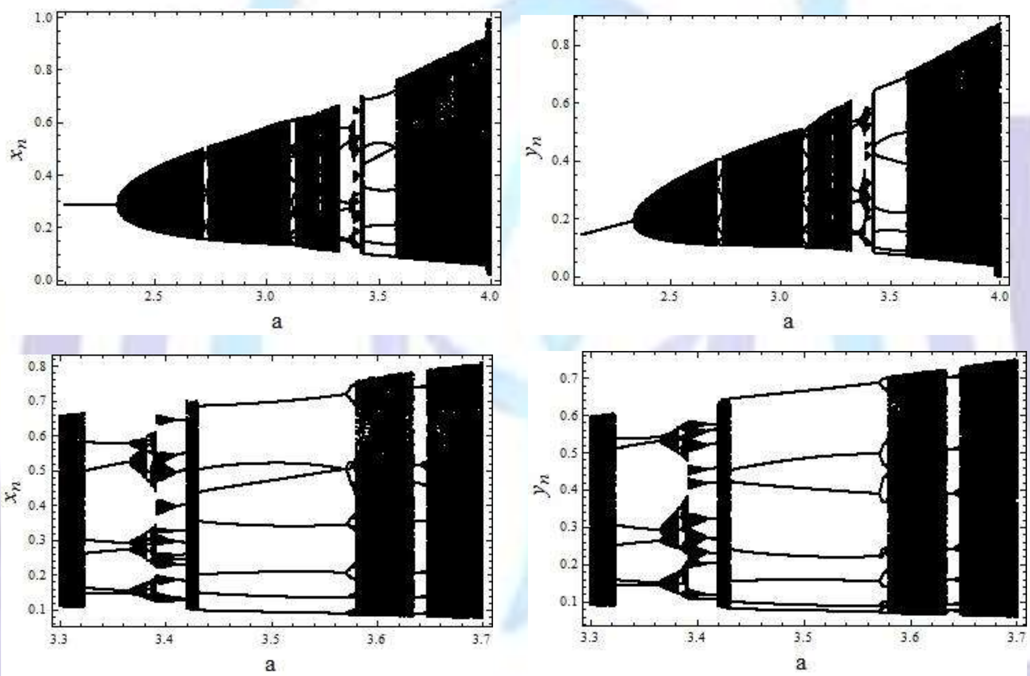

Fig.1: Bifurcation diagram of the unperturbed map (2.3) when b = 3.5, and initial conditions , and initial conditions $x_0 = 0.1$ and $y_0 = 0.1$.

Numerical simulation is extended to obtain the regular and chaotic attractors of the system (2.3).

In Fig. 2, we have shown regular limit cycle attractor and a chaotic attractor for two different values of parameter a when b = 3.5.

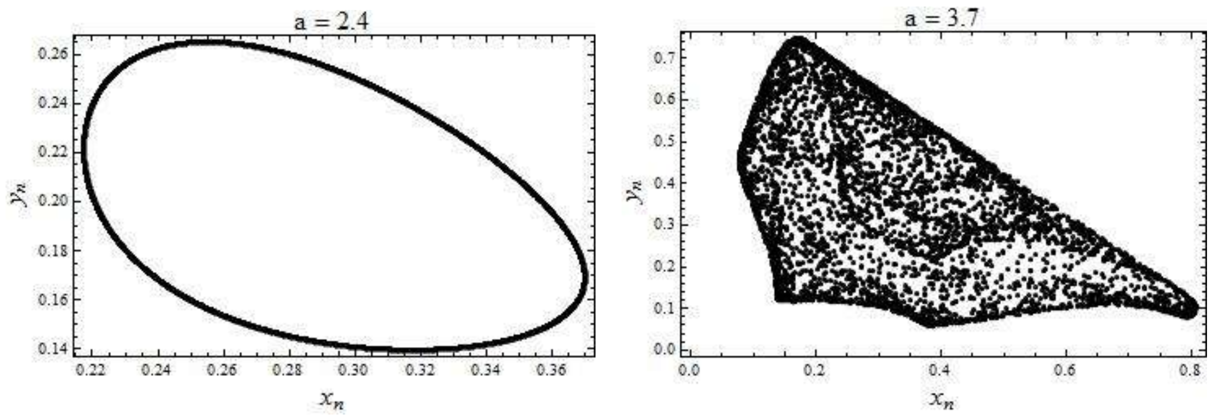

Fig. 2: Regular and chaotic attractor of system (2.3) when b = 3.5.

In their work, to see the effect of random shock, Grafton and Echenique (1997) have suggested and additional term in the first equation of (2.4) and the new model is

$$x_{n+1} = ax_n(1 - x_n) + k\zeta$$
$$y_{n+1} = b\, x_n\, y_n \qquad (2.4)$$

where ζ is a random number between 0 and 1 and k may be defined as the intensity of random shock. The motivation was to observe whether the application of random shock to the system can qualitatively mimic chaos or reduce chaos. Extending the numerical calculations for k = 0.025, 0.03, 0.04, it has been observed that chaotic nature of the evolution diminishes but only for such lower value of k. One may observe the results shown in Fig. 3. Thus, the application of random shock does not control the chaotic motion completely. Also, same can be observed if we draw the time series graphs, phase plots etc.

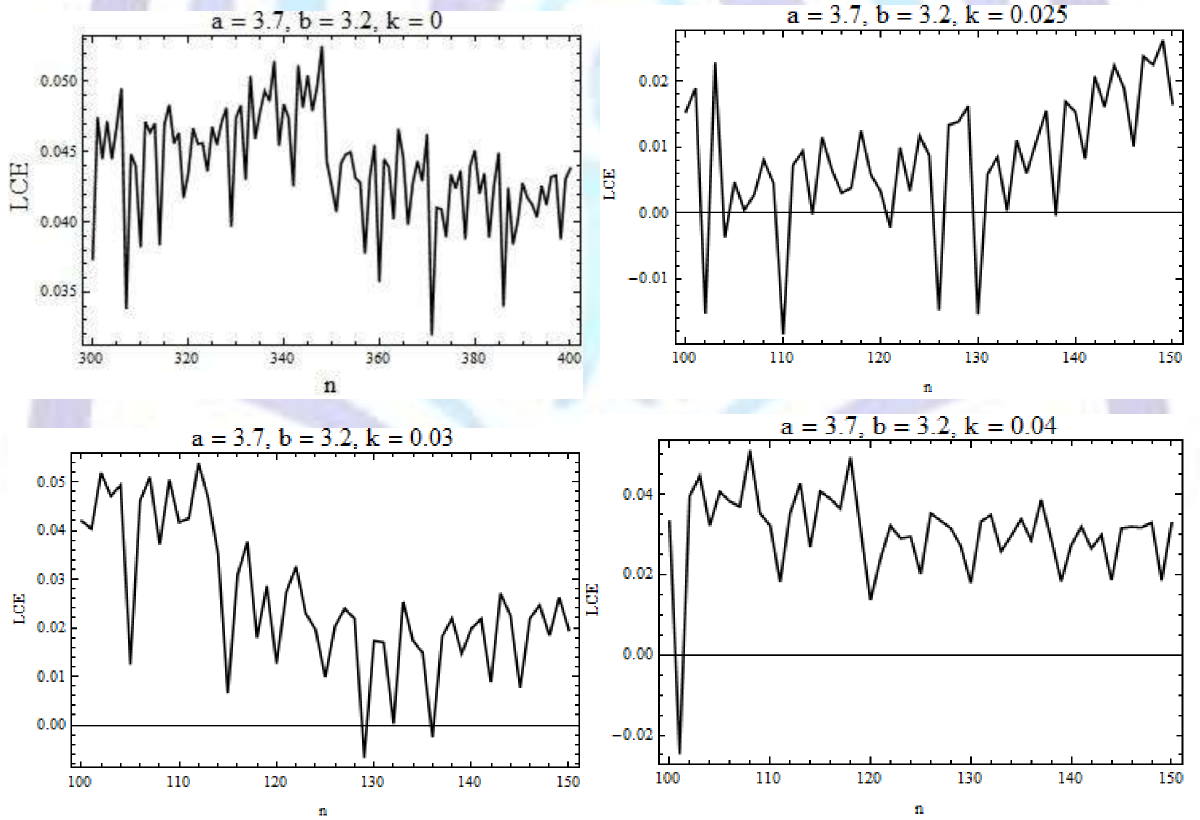

Fig. 3: Plots of Lyapunov exponents when random shock of intensity k = 0, 0.025, 0.03, 0.04 are applied for equation (2.4).

3. Controlling Chaos

It has been observed in some cases of discrete chaotic dynamical system, e.g. Saha and Tehri (2013), a slight periodic change in parameters may stabilize the chaotic system and make it to evolve in regular manner. In this article, again we have applied such changes in order to control chaotic evolutions. We have replaced the parameter a by a(1 – k cos x), i.e.

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a*ƒ* a(1 – k cos x). Then, our prey-predator system takes a slightly changed form. This we interpret as one in which the rate of change in species x_n will now be subject to certain periodic change in a. Our changed system becomes

$$x_{n+1} = a\,(1 - k \cos x_n)\, x_n\,(1 - x_n)$$
$$y_{n+1} = b\, x_n\, y_n \qquad (3.1)$$

where k is a small constant and should lie between 0 and 1.

Numerical simulation have been performed for this model for without shock and with shock. The results are given below:

For parameter values a = 3.7, b = 3.2 the above mentioned system (3.1) also evolves chaotically when k = 0. The plots of the chaotic attractor and corresponding time series are shown in Fig. 4.

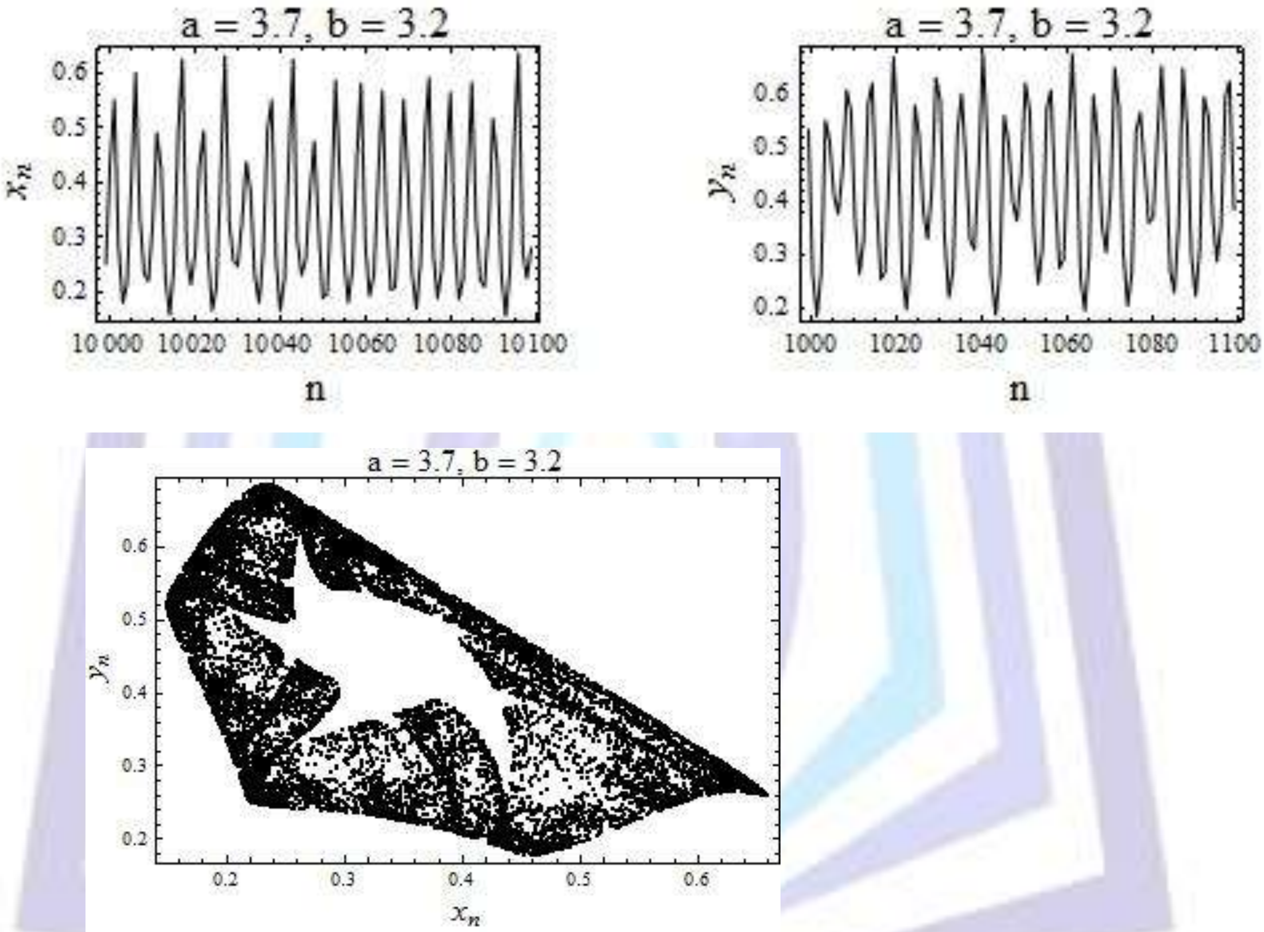

Fig. 4: Chaotic evolution of the prey-predator system when k = 0.

Now, let us replace the parameter a by a (1 – k cos x) i.e. a*ƒ*a (1 – k cos x). Such a replacement is justified because the rate of change specified by a may not remain constant forever. The fluctuations in species become subject of changes. There could be periodic addition or subtraction in the rate of changes. With such a replacement of parameter a and substituting small values for parameter k, (e. g. k = 0.1 to k = 0.5), we find the prey-predator system is no more chaotic and instead shows regularity as shown in Fig. 5. The system either show periodic behavior with finite period or quasi-periodic with limit cycle.

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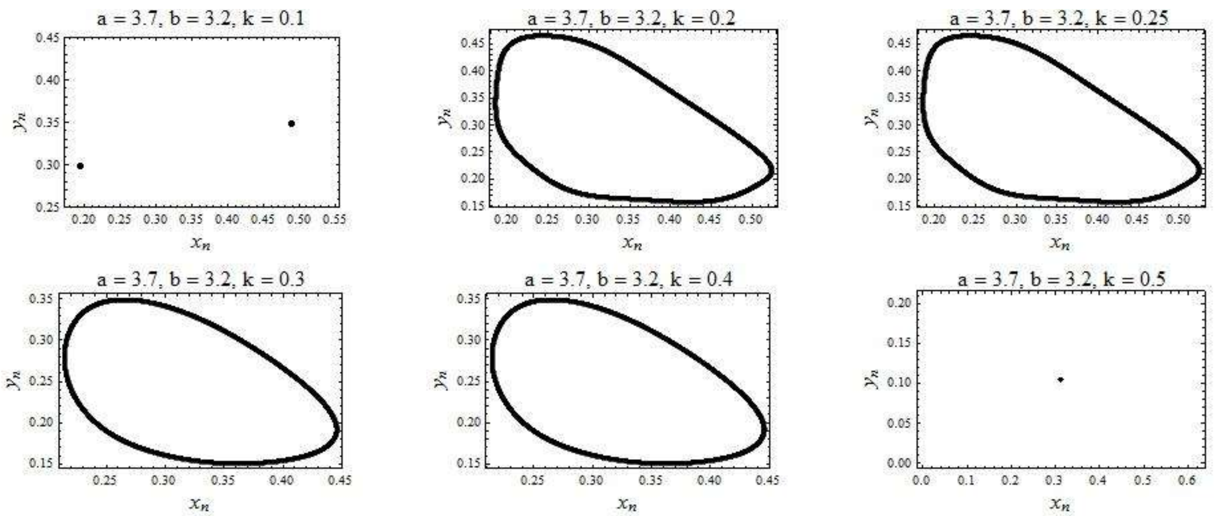

Fig. 5: Phase plots of system (2.5) for values of k between k = 0.1 to k = 0.5.

It has been observed from phase plots, Fig. 5, that for cases when k = 0.1, 0.15, 0.2, 0.25, 0.3, 0.35, 0.4, 0.45 the system is no more chaotic. For all these cases, either the motion is periodic of finite period or quasi-periodic (limit cycle). One can check that the time series plots for these cases provide the same results.

4. Calculations of Lyapunov Exponenets, Topologocal Entropies and Correlation dimensions for Controlled System

(a) Lyapunov Exponents:

Plots of lyapunov exponents for different values of k as discussed above for system (3.1) are given below in Fig. 6. One observes for $0 < k \leq 0.2$, Lyapunov exponents are negative and so, one can definitely say the original chaotic system shown in Fig.5 when a = 3.7, b = 3.2 and k = 0 is no more chaotic within above range of values of k. In other words chaos is controlled by assuming changes in parameter a.

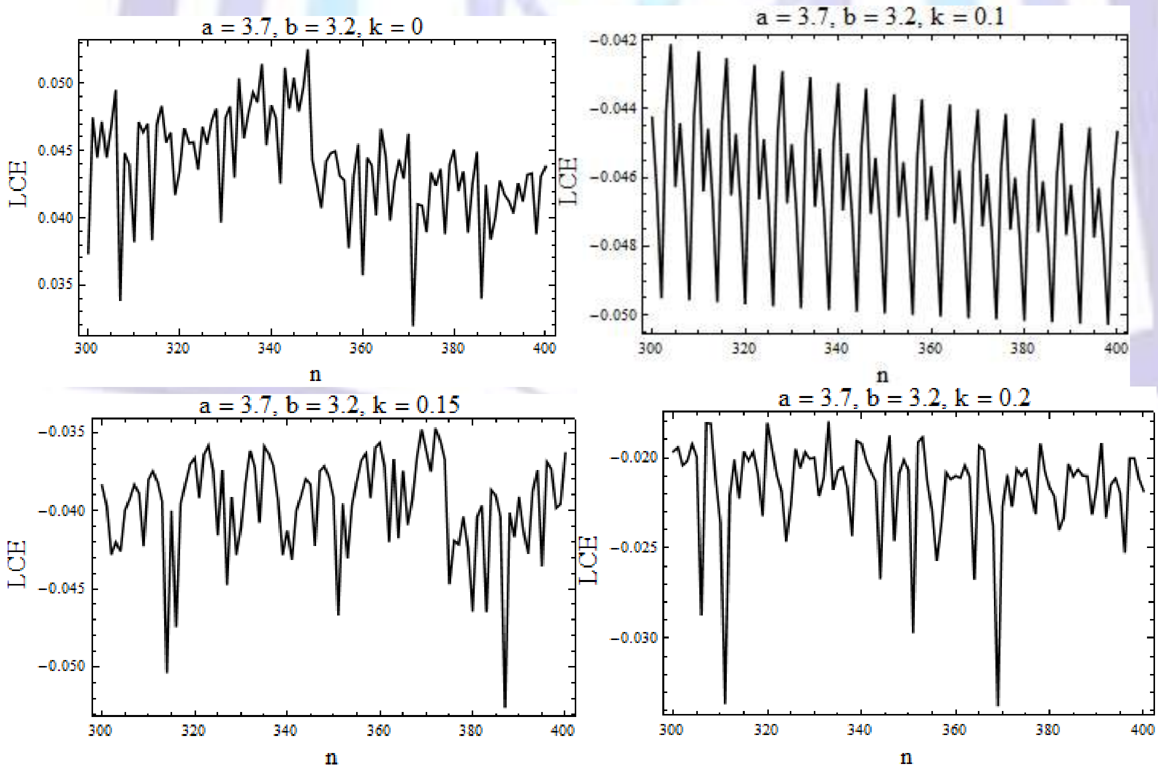

Fig.6 : Plots of Lyapunov exponents for uncontrolled case k = 0 and three controlled cases.

Above plots indicate that for k = 0.1, 0.15, 0.2 Lyapunov exponents are negative.

(b) Topological Entropies

To investigate chaotic behavior in a wide variety of systems evolving with time, an alternate replacement of Lyapunov exponents which could be more reliable and acceptable as indicator is the topological entropy,Balmforth et al (1964), Adler et al (1965), Bowen (1970).

Topological entropy describes the *rate of mixing* of a dynamical system. It has a relationship to both Lyapunov exponents, through the dependence of rate, and to the ergodicity, because of the association of mixing. For a system having non-zero topological entropy, the rate of mixing must be exponential which is reminiscent of Lyapunov exponents. But such exponentially of mixing is not relative to time, but rather to the number of discrete steps through which the system has evolved. Positivity of Lyapunov exponent and topological entropy are characteristics of chaos and these provide certain measure of complexity.

For our system, the numerical process is extended and the plots of topological entropies for unperturbed and perturbed cases, (i.e. without shock and with shock), have been obtained within $3.5 \le a \le 4.0$. These are shown in Fig. 7 below for k = 0 & k = 0.3. The figure on right hand side clearly show no complexity occurring when k = 0.3

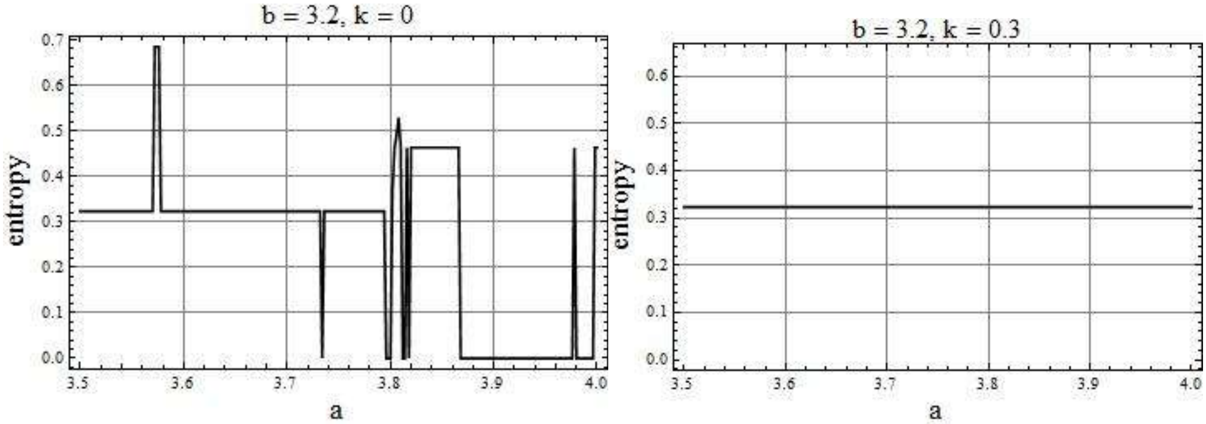

Fig. 7: Entropy plots for Prey-predator system for variation of parameter a. The right column figure shows changes within periodic window region due to change in shock strength.

(c) Correlation Dimensions

Correlation dimension provides the measures of dimensionality of the system at any stage. Such a dimension is referred as a fractal dimension. In case of our model (3.1), we follow the method of Martelli (1999) and obtain the data for the correlation integral for k = 0 and k = 0.3 and obtain the plots for the correlation curve shown in Fig. 8.

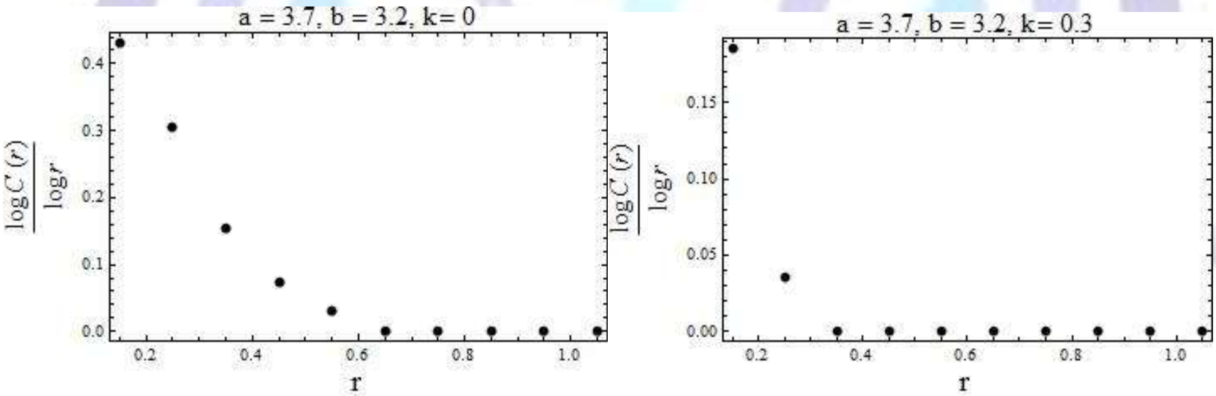

Fig. 8: Plots of correlation curve for k = 0 and k = 0.3.

Then, we have used the least square linear fit to the data and obtained the equations of the straight line fitting the data. The value k = 0, corresponds to the equation of the straight line y= 0.356333 – 0.42763 x fitting approximately the correlation integral curve. Thus, the correlation dimension of the fractal set in this case is 0.356333 ≈ 0.36.

Similarly, k = 0.3, corresponds to the equation of the straight line y = 0.0920378 – 0.116485 x fitting approximately the correlation integral curve. Thus, the correlation dimension of the fractal set in this case is 0.0920378 ≈ 0.09. If we neglect few transient data i.e. the few initial data, then the fractal dimension for case k = 0.3 would be approximately zero. This is evident from the right hand plot of the integral curve in Fig. 8.

5. Plots of Dynamic Lyapunov Exponents(DLI):

Moving forward for further confirmation of chaos and regular motion, we extend our numerical calculations to get plots of DLI. Some recent articles, Saha and Budhraja (2007), Saha and Tehri (2007), Budhraja (2007), have shown that DLI plots give perfect identification for regular and chaotic motions. Here, we fix the parameters a = 3.7 and b = 3.2 and we extend our calculations to obtain plots for DLI for the original chaotic case when k = 0 and for the controlled regular case when k = 0.3. The plots are given in Fig. 9 below

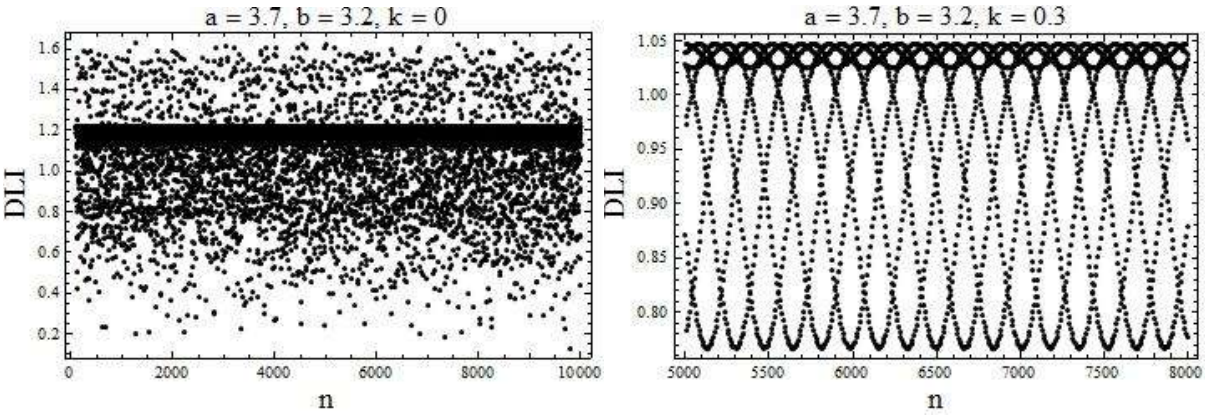

Fig. 9: DLI plots for chaotic (k = 0) and controlled regular system (k = 0.3).

From the above plots one can clearly observe that the left figure consists of randomly distributed points whereas the right figure e merges as a regular pattern, respectively indicating chaotic and regular evolutions in system (3.1).

6. Discussions:

Complexity in a prey-predator system exhibiting due to certain set of parameter values have been investigated through analytical and numerical simulations. In this regard plots of bifurcation diagrams, regular and chaotic attractors, Lyapunov exponents, topological entropies etc. have been obtained. Chaotic evolution may result in extinction of one or both the species and also, unpredictable fluctuations. For coexistence of species, it is necessary that the system be regular. Use of random shock, as suggested in an earlier article, has been applied to bring the system from chaos to regularity. But it failed to control chaotic motion. Then, certain periodic variation is applied and it has been found that the prey-predator system evolved from chaos to regularity. In this context it can be said that among the species if such periodic change occur in their population then coexistence is possible. In many natural system, often such situation occurs and the species survive with coexistence. For confirmation of chaos and regularity, plots of the recently discovered indicator DLI havebeen obtained. Also, plots of Lyapunov exponents, topological entropies and correlation integrals have been obtained as a measure of chaos and complexity. The last of these provide certain measure of dimensionality i. e. fractal dimension.

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