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Rational Contractions in b-Metric Spaces

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ABSTRACT

In this paper, we prove fixed point theorems for contractions and generalized weak contractions satisfying rational expressions in complete b-metric spaces. Our results generalize several well-known comparable results in the literature.

Keywords:

Fixed point; rational type; generalized weak contraction; b-metric spaces.

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1. INTRODUCTION

The Banach contraction principle is a very popular tool for solving problems in nonlinear analysis. Generalizations of this principle have been obtained in several directions. The following is an example of such generalizations. Jaggi in [7] proved the following fixed point theorem satisfying a contractive condition of rational type.

Theorem 1.1 Let *T* be a continuous self-map defined on a complete metric space (X,d). Suppose that *T* satisfies the following contractive condition:

$$d(T(x),T(y)) \le \alpha \frac{d(y,T(y))d(x,T(x))}{d(x,y)} + \beta d(x,y), \forall x,y \in X, x \ne y \quad (1.1)$$

where $\alpha, \beta \in [0,1)$, such that $\alpha + \beta < 1$. Then *T* has a unique fixed point.

Also, in 1975, Dass & Gupta [17] proved that every continuous self-map on the metric spaces (X,d) which satisfies the following contraction conditions:

$$d(T(x),T(y)) \le \alpha \frac{d(y,T(y))[1+d(x,T(x))]}{1+d(x,y)} + \beta d(x,y), \forall x,y \in X \quad (1.2)$$

where $\alpha, \beta \in [0,1)$ and $\alpha + \beta < 1$, have an unique fixed point.

Remark 1.2 By a simple calculation we have that every contraction which satisfies the condition (1.2) satisfies also the condition (1.1) .

Another generalization of the contraction principle was suggested by Alber and Guerre-Delabriere [10] in Hilbert spaces. Rhoades [11] has shown that their result is still valid in complete metric spaces.

Definition 1.3 Let (X,d) be a metric space. A mapping $T: X \to X$ is said to be a φ-weak contraction if

$$d(Tx,Ty) \le d(x,y) - \varphi(d(x,y))$$

For all $x, y \in X$, where $\varphi:[0,\infty) \to [0,\infty)$ is a continuous and non-decreasing function with $\varphi(t)=0$ if and only if $t=0$.

Theorem 1.4 [11] Let (X,d) be a complete metric space and T be a φ-weak contraction on X. Then, T has a unique fixed point.

There exists a large number of extensions of Theorem 1.4 in literature.[19,20,21].

The concept of *b*-metric space as a generalization of metric spaces was introduced by Czerwik in [2]. After that, several interesting result about the existence of a fixed point for single-valued and multi-valued operators in b-metric space have been obtained (see [2,3,4,5,6,8,9,12,13,14])

In this paper we prove a fixed point theorems for contractions and generalized weak contractions satisfying rational expressions in complete b-metric spaces.

2. PRELIMINARIES

Definition 2.1 [1,14] Let X be a set and let $s \ge 1$ be given real number. A function $d: X \times X \to \mathbb{R}^+$ is said to be a b-metric if and only if for all $x, y, z \in X$ the following conditions are satisfied:

1. $d(x,y)=0$ if and only if $x=y$;
2. $d(x,y)=d(y,x)$;
3. $d(x,z) \le s[d(x,y)+d(y,z)]$

The pair (X,d) is called a b-metric space with parameter s.

There exists more examples in the literature [1, 2, 3, 17] showing that the class of b-metrics is effectively larger than that of metric spaces, since a b-metric is a metric when $s=1$ in the above condition 3.

Example 1.[18] Let (X,d) be a metric space and $\rho(x,y)=(d(x,y))^p$, where $p>1$ is a real number. Then ρ is a b-metric with $s=2^{p-1}$. However, (X,ρ) is not necessarily a metric space.

For example, let X be the set of real numbers and let $d(x,y)=|x-y|$ be the usual Euclidian metric. Then $\rho(x,y)=(x-y)^2$ is a b-metric on $\mathbb{R}$ with $s=2$, but is not a metric on $\mathbb{R}$.

Example 2. [4] The space $l_p(0<p<1)$,

$$l_p=\{(x_n)\subset\mathbb{R}:\sum_{n=1}^{\infty}|x_n|^p<\infty\}$$

together with the function $d:l_p\times l_p\rightarrow\mathbb{R}$,

$$d(x,y)=(\sum_{n=1}^{\infty}|x_n-y_n|^p)^{\frac{1}{p}}$$

where $x=(x_n),y=(y_n)\in l_p$ is a b-metric space with parameter $s=2^{\frac{1}{p}}>1$.

Example 3. [4] The space $L_p(0<p<1)$ of all real functions $x(t),t\in[0,1]$ such that:

$$\int_o^1|x(t)|^p dt<\infty$$

is a b-metric space if we take

$$d(x,y)=(\int_0^1|x(t)-y(t)|^p dt)^{\frac{1}{p}} \text{ for each } x,y\in L_p.$$

The parameter $s=2^{\frac{1}{p}}>1$

We need the following definitions.

Definition 2.2. Let (X,d) be a b-metric space. Then a sequence $(x_n)_{n\in\mathbb{N}}$ is called:

(a) b-convergent if there exists $x\in X$ such that $d(x_n,x)\rightarrow 0$ as $n\rightarrow\infty$. In this case, we write $\lim_{n\rightarrow\infty}x_n=x$.

(b) b-Cauchy if $d(x_n,x_m)\rightarrow 0$, as $n,m\rightarrow\infty$.

Proposition 2.3. In a b-metric space (X,d), the following assertions hold:

(p_1) A b-convergent sequence has a unique limit.

(p_2) Each b-convergent sequence is b-Cauchy.

(p_3) In general, a b-metric is not continuous as the following example shows.

Example 4. [see Example 3 in 16] Let $X=\mathbb{N}\cup\{0\}$ and let $d:X\times X\rightarrow\mathbb{R}$ be defined by:

$$d(m,n)=\begin{cases}0 & if\ m=n\\ \left|\frac{1}{m}-\frac{1}{n}\right| & if\ m\, and\, n\, are\, even\, or\, mn=\infty\\ 5 & if\ m\, and\, n\, are\, odd\, and\, m\neq n\\ 2 & otherwise\end{cases}$$

It is easy to see that for all $m,n,p\in X$, we have:

$$d(m,n)\leq 3(d(m,p)+d(p,n))$$

Thus, (X,d) is a b-metric space with $s=3$.

Let $x_n=2n$ for each $n\in\mathbb{N}$.

Then

$$d(2n,\infty)=\frac{1}{2n}\to 0 \text{ as } n\to\infty$$

That is, $x_n\to\infty$, but $d(x_n,1)=2 \not\to d(\infty,1)$ as $n\to\infty$.

Aghajani *et al*. [1] proved the following simple lemma about the b-convergent sequences.

Lemma 2.4 [1] Let (X,d) be a b-metric space with $s\geq 1$, and suppose that $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ b-converge to x,y respectively. Then, we have

$$\frac{1}{s^2}d(x,y)\leq \liminf_{n\to\infty} d(x_n,y_n)\leq \limsup_{n\to\infty} d(x_n,y_n)\leq s^2 d(x,y)$$

In particular, if $x=y$, then, $\lim_{n\to\infty} d(x_n,y_n)=0$. Moreover, for each $z\in X$ we have

$$\frac{1}{s}d(x,y)\leq \liminf_{n\to\infty} d(x_n,z)\leq \limsup_{n\to\infty} d(x_n,z)\leq sd(x,z)$$

Lemma 2.5 [5] Let (X,d) be a b-metric space with parameter s and $(x_n)_{n\in\mathbb{N}}$ a sequence in X such that:

$$d(x_{n+1},x_{n+2})\leq qd(x_n,x_{n+1}), \quad \text{for all } n\in\mathbb{N}$$

where $0\leq q<1$. Then the sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy sequence in X provided that $sq<1$.

3. MAIN RESULTS

Theorem 3.1 Let (X,d) be a complete b-metric space with parameter s and with continuous b-metric in each variable, $T:X\to X$ be a continuous mapping such that:

$$d(T(x),T(y))\leq \alpha\frac{d(y,T(y))d(x,T(x))}{d(x,y)}+\beta d(x,y), \forall x,y\in X, x\neq y \qquad (3.1)$$

where α,β are positive real constants such that $s\beta+\alpha<1$. Then T has a unique fixed point.

Proof. For a arbitrary point $x_0\in X$ we construct the sequence $(x_n)_{n\in\mathbb{N}}$ such that $x_{n+1}=f(x_n)$ for all $n\in\mathbb{N}$.

So

$$d(x_1,x_2)=d(Tx_0,Tx_1)\le\alpha\frac{d(x_0,Tx_0)d(x_1,Tx_1)}{d(x_0,x_1)}+\beta d(x_0,x_1)$$
$$=\alpha\frac{d(x_0,x_1)d(x_1,x_2)}{d(x_0,x_1)}+\beta d(x_0,x_1)$$

Equivalently $d(x_1,x_2)\le\frac{\beta}{1-\alpha}d(x_0,x_1)$ where $\frac{\beta}{1-\alpha}=k<1$.

Similarly, $d(x_2,x_3)\le k(d(x_1,x_2))$

Inductively:

$$d(x_n,x_{n+1})=d(Tx_{n-1},Tx_n)\le\alpha\frac{d(x_{n-1},Tx_{n-1})d(x_n,Tx_n)}{d(x_{n-1},x_n)}+\beta(d(x_{n-1},x_n)$$
$$\le\frac{\beta}{1-\alpha}d(x_{n-1},x_n)=kd(x_{n-1},x_n) \tag{3.2}$$

By Lemma 2.5 the above sequence is Cauchy in complete b-metric space (X,d).

So there exists a $z\in X$ such that $\lim_{n\to\infty}x_n=z$.

By the continuity of T and d we have:

$$Tz=T(\lim_{n\to\infty}x_n)=\lim_{n\to\infty}Tx_n=\lim_{n\to\infty}x_{n+1}=z$$

and this prove that z is a fixed point.

If there exists another point $w\ne z$ in X such that $Tw=w$, then

$$d(w,z)=d(Tw,Tz)\le\frac{\alpha d(w,Tw)d(z,Tz)}{d(w,z)}+\beta d(w,z)$$
$$=\beta d(w,z)<d(w,z)$$

which is a contradiction. Hence, the fixed point is unique.

Theorem 3.2 Let (X,d) be a complete b-metric space with parameter s and with continuous b-metric in each variable, $T:X\to X$ be a continuous mapping such that:

$$d(F(x),F(y))\le\alpha\frac{d(y,F(y))[1+d(x,F(x))]}{1+d(x,y)}+\beta d(x,y),\forall x,y\in X \tag{3.3}$$

where α,β are positive real constants such that $s\beta+\alpha<1$. Then T has a unique fixed point.

Proof. The proof of this Theorem follows immediately by Remark 1.2 .

Theorem 3.3 Let (X,d) be a complete *b*-metric space with parameter s and with continuous *b*-metric in each variable, $T:X\to X$ be a continuous mapping such that:

$$sd(Tx,Ty)\le M(x,y)-\varphi(M(x,y)),\ \text{for all}\ x,y\in X \tag{3.4}$$

where $\varphi:[0,\infty)\to[0,\infty)$ is continuous and non-decreasing function with $\varphi(t)=0$ if and only if $t=0$ and

$$M(x,y)=\max\{\frac{d(Tx,Tx)d(y,Ty)}{d(x,y)},d(x,y)\} \tag{3.5}$$

Then T has a fixed point.

Proof. Let $x_0 \in X$ be such that $x_0 \neq Tx_0$. We construct the sequence $(x_n)_{n\in\mathbb{N}}$ in X as follows

$$x_{n+1} = T(x_n), \qquad n = 0,1,2,...$$

$$\begin{aligned} d(x_n, x_{n+1}) &\le sd(x_n, x_{n+1}) = sd(Tx_{n-1}, Tx_n) \\ &\le \max\{\frac{d(x_{n-1}, Tx_{n-1})d(x_n, Tx_n)}{d(x_{n-1}, x_n)}, d(x_{n-1}, x_n)\} \\ &\quad -\varphi(\max\{\frac{d(x_{n-1}, Tx_{n-1})d(x_n, Tx_n)}{d(x_{n-1}, x_n)}, d(x_{n-1}, x_n)\}) \\ &= \max\{d(x_n, x_{n+1}), d(x_{n-1}, x_n)\} - \varphi(\max\{d(x_n, x_{n+1}), d(x_{n-1}, x_n)\}) \end{aligned} \tag{3.6}$$

Suppose that there exists n_0 such that $d(x_{n_0}, x_{n_0+1}) > d(x_{n_0-1}, x_{n_0})$.

Then from (3.6)

$$\begin{aligned} d(x_{n_0}, x_{n_0+1}) &\le \max\{d(x_{n_0}, x_{n_0+1}), d(x_{n_0-1}, x_{n_0})\} \\ &\quad -\varphi(\max\{d(x_{n_0}, x_{n_0+1}), d(x_{n_0-1}, x_{n_0})\}) \\ &= d(x_{n_0}, x_{n_0+1}) - \varphi(d(x_{n_0-1}, x_{n_0})) < d(x_{n_0}, x_{n_0+1}) \end{aligned}$$

which is a contradiction. Hence, $d(x_n, x_{n+1}) \le d(x_{n-1}, x_n)$ for all $n \ge 1$ and so the $(d(x_n, x_{n+1}))_{n\in\mathbb{N}}$ is a non-increasing sequence of positive real numbers.

Then there exists $r \ge 0$ such that $\lim_{n\to\infty} d(x_n, x_{n+1}) = r$.

Taking the limit as $n \to \infty$ in (3.6) and using the properties of the function φ and the continuity of the distance d we get

$$r \le r - \varphi(r)$$

Which satisfy if and only if $r = 0$, that is

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0 \tag{3.7}$$

Next, we show that $(x_n)_{n\in\mathbb{N}}$ is a *b*-Cauchy sequence in *X*. Suppose the contrary, that is, $(x_n)_{n\in\mathbb{N}}$ is a not a b-Cauchy sequence. Then there exists $\varepsilon > 0$ for which we can find two subsequences (x_{m_k}) and (x_{n_k}) of $(x_n)_{n\in\mathbb{N}}$ such that n_k is smallest index for which

$$n_k > m_k > k, \; d(x_{m_k}, x_{n_k}) \ge \varepsilon. \tag{3.8}$$

This means that

$$d(x_{m_k}, x_{n_i-1}) < \varepsilon \tag{3.9}$$

From (3.8) and (3.9) we have:

$$\begin{aligned} \varepsilon \le d(x_{m_k}, x_{n_k}) &\le sd(x_{m_k}, x_{m_k-1}) + sd(x_{m_k-1}, x_{n_k}) \\ &\le sd(x_{m_k}, x_{m_k-1}) + s^2 d(x_{m_k-1}, x_{n_k-1}) + s^2 d(x_{n_k-1}, x_{n_k}) \end{aligned} \tag{3.10}$$

Using (3.7) and taking the upper limit as $k \to \infty$, we get

$$\frac{\varepsilon}{s^2} \leq \limsup_{k\to\infty} d(x_{m_k-1}, x_{n_k-1}) \tag{3.11}$$

By triangular inequality, we have

$$d(x_{m_k-1}, x_{n_k-1}) \leq sd(x_{m_k-1}, x_{m_k}) + sd(x_{m_k}, x_{n_k-1})$$

Taking the upper limit as $k \to \infty$, we get

$$\limsup_{k\to\infty} d(x_{m_k-1}, x_{n_k-1}) \leq \varepsilon s \tag{3.12}$$

So, by (3.11) and (3.12) we have

$$\frac{\varepsilon}{s^2} \leq \limsup_{k\to\infty} d(x_{m_k-1}, x_{n_k-1}) \leq \varepsilon s \tag{3.13}$$

From (3.4) and (3.5), we have

$$\begin{aligned} sd(x_{m_k}, x_{n_k}) &= \\ sd(Tx_{m_k-1}, Tx_{n_k-1}) &\leq \max\{\frac{d(x_{m_k-1}, Tx_{m_k-1})d(x_{n_k-1}, Tx_{n_k-1})}{d(x_{m_k-1}, x_{n_k-1})}, d(x_{m_k-1}, x_{n_k-1})\} \\ &-\varphi(\max\{\frac{d(x_{m_k-1}, Tx_{m_k-1})d(x_{n_k-1}, Tx_{n_k-1})}{d(x_{m_k-1}, x_{n_k-1})}, d(x_{m_k-1}, x_{n_k-1})\}) \\ &= \max\{\frac{d(x_{m_k-1}, x_{m_k})d(x_{n_k-1}, x_{n_k})}{d(x_{m_k-1}, x_{n_k-1})}, d(x_{m_k-1}, x_{n_k-1})\} \\ &-\varphi(\max\{\frac{d(x_{m_k-1}, x_{m_k})d(x_{n_k-1}, x_{n_k})}{d(x_{m_k-1}, x_{n_k-1})}, d(x_{m_k-1}, x_{n_k-1})\} \end{aligned} \tag{3.14}$$

Taking the upper limit as $k \to \infty$ in (3.14) we get

$$\varepsilon s \leq \varepsilon s - \varphi(\varepsilon s) < \varepsilon s$$

which is contradiction. Therefore, $(x_n)_{n\in\mathbb{N}}$ is a *b*-Cauchy sequence in *X*. Since *X* is a b-complete metric space, there exists $z \in X$ such that $x_n \to z$ as $n \to \infty$ and

$$z = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} T(x_n)$$

Using the triangular inequality, we get

$$d(z, Tz) \leq sd(z, Tx_n) + sd(Tx_n, Tz)$$

Letting $n \to \infty$, we get

$$d(z, Tz) \leq s \lim_{n\to\infty} d(z, Tx_n) + s \lim_{n\to\infty} d(Tx_n, Tz) = 0$$

So, we have $Tz = z$. Thus, z is a fixed point of T .

Corollary 3.4 Let (X, d) be a complete *b*-metric space with parameter s and with continuous *b*-metric in each variable, $T : X \to X$ be a continuous mapping such that:

$$sd(Tx,Ty) \le k \max\{\frac{d(Tx,Tx)d(y,Ty)}{d(x,y)}, d(x,y)\} \text{, for all } x,y \in X \qquad (3.15)$$

where $k \in (0,1)$. Then T has a fixed point.

Proof. In Theorem 3.3, taking $\varphi(t) = (1-k)t$ for all $t \in [0,\infty)$, we get Corollary 3.4.

Remark 3.5 Since a b-metric is a metric when $s = 1$, so our results can be viewed as a generalization and extension of several other comparable results.

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