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A characterization of the existence of generalized stable sets

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ABSTRACT

The generalized stable sets solution introduced by van Deemen (1991) as a generalization of the von Neumann and Morgenstern stable sets solution for abstract systems. If such a solution concept exists, then it is equivalent to the admissible set appeared in game theory literature by Kalai and Schmeidler (1977). The purpose of this note is to provide a characterization for the existence of the generalized stable sets solution. (Minimal) undominated element, (Generalized) Stable Set, Admissible set.

Keywords: (Minimal) undominated element; (Generalized) Stable Set; Admissible set.
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1 Introduction

Von Neumann and Morgenstern in their classical work *Theory of Games and Economic Behavior* (1947) introduced the *Theory of Solutions and Standards of Behavior*. This theory specifies that a set F of elements (imputations) is a *Von Neumann-Morgenstern solution* when it possesses two properties: (a) No element inside F is dominated[1] by an element inside F and (b) Every element outside F is dominated by some element inside F (see [5, Page 40]). Later, the term *Von Neumann and Morgenstern's stable set solution* has been used for F by many authors to avoid confusion with other solution concepts. Von Neumann and Morgenstern give an interpretation of stable sets: A stable set is a characterization of what may be acceptable or established as a" standard of behavior" in society. The idea being that all the imputations in any given stable set correspond to some mode of behavior while imputations in different stable sets correspond to different modes of behavior. The core is contained in each Von Neumann-Morgenstern stable set. The Von Neumann-Morgenstern stable sets will be empty in the case of odd cycles. To avoid this particular problem, Van Deemen introduced the notion of the generalized stable set which is able to produce a solution for every possible cyclic binary relation.

Kalai and Schmeidler in [2] introduced the concept of the admissible set. The admissible set concept can be applied to a host of game-theoretic situations, ranging from non-cooperative games, where a coalition consists of an individual player, to fully cooperative games, where any coalition can be allowed to form. An equivalent definition of the concept of admissible set was introduced by Schwartz in [3]. Andrikopoulos in [1, Theorem 22] showed that if a generalized stable set solution exists, the union of all generalized stable sets of (X,R) is equivalent to the admissible set.

In this note, we prove that a feasible set[2] has a generalized stable set with respect to a dominance relation, if and only if every undominated set has a minimal undominated subset with respect to this relation. This is done in a general framework on which " dominance relation" means arbitrary binary relation defined in infinite set of alternatives. The approach that we take consists of associating each game or social activity with an abstract system (i.e., an abstract set endowed with a binary relation). For instance, Kalai and Schmeidler in [2] associate the mixed extension of a normal form game with an abstract system using a binary relation that only accounts for profitable single deviations.

2 The Main Result

We consider a dominance relation, denoted by R, and a (finite or infinite) non-empty set of alternatives X. An *abstract system* is a pair (X,R) where X is a set of alternatives and R is a dominance relation on X. We sometimes abbreviate $(x,y)\in R$ as xRy. The *asymmetric part* of R is defined as the binary relation $P(R)$ on X with $xP(R)y$ if and only if xRy but not yRx. A subset $D\subseteq X$ is R- *undominated* if and only if for no $x\in D$ there is a $y\in X\setminus D$ such that yRx. This was also defined by Kalai and Schmeidler under the name of R- *closed* set (see [2, Page 404]). An R-undominated set is *minimal* if none of its proper subsets has this property. The *transitive closure* of R is $\overline{R}=\{(x,y)|$ there exist $K\in \mathbb{N}$ and $x_0,...,x_K\in X$ such that $[x=x_0$ and $(x_{k-1},x_k)\in R$ for all $k\in\{1,...,K\}$ and $x_K=y]\}$. The sequence $x_1,x_2,...,x_{K-1}$ is known as the R*-path from* x *to* y. The transitive closure of R sometimes referred to as the *path dominance relation* of R. A subset F of X is called a *generalized stable set* with respect to R (see [4, Page 257]) if (i) for no element in F, an R-path starts toward another element in F, and (ii) for every x outside F, an R-path starts from some y in F terminating at x. The first property is called *internal stability of domination* and the second property *external stability of domination*. In fact, F is a generalized stable set of X if it is a stable set of X with respect to the R-path dominance relation. The *generalized stable sets solution* for an abstract system (X,R) is the collection of all its generalized stable sets. The *admissible set* for an abstract system (X,R) is the set $\mathcal{A}(X,R)=\{x\in X\,|\,y\in X$ and $(y,x)\in\overline{R}$ implies $(x,y)\in\overline{R}\}$ (see [2, Definition in Page 403]). The admissible set of (X,R) is equivalent to the union of all minimal R-undominated subsets of X (see [2, Theorem 5]).

The following theorem gives a characterization of the generalized stable sets solution. Note that this result can be applied to any dominance relation.

Theorem. Let (X,R) be an abstract system. Then, (X,R) has a generalized stable set if and only if every R-undominated set has a minimal R-undominated subset.

[1] The notion of *Domination* or *Superiority* is defined in [5, Page 37]. It refers to any process of comparing entities in pairs in order to find which pair is preferred, or has a greater amount of some quantitative property.

[2] The *feasible set* is the set of all alternatives (commodity bundles or payoff outcomes) that are possible solutions.

Proof. To prove the necessity of the theorem, assume that (X,R) has a generalized stable set, say F. Let $\mathcal{D}$ be the family of all the R-undominated subsets of X. We first show that $\mathcal{D}\neq\emptyset$. Suppose to the contrary that this is not the case. Then, for each $x\in X$ the singleton set $\{x\}$ is not R-undominated. Therefore, there exists $y^*\in X$ satisfying y^*Rx. Without loss of generality we can assume that $x\in F$. Since F satisfies internal stability of domination we conclude that $y^*\notin F$. Hence, there exists $z\in F$ (because of external stability of domination) such that $(z,y^*)\in\overline{R}$. It follows that $(z,x)\in\overline{R}$. Since $z,x\in F$ this is impossible unless we have $z=x$. Therefore, $y^*\overline{R}x$ and $x\overline{R}y^*$. We define a set D^* as follows:

$$D^*=\{y\in X\mid y\overline{R}x \text{ and } x\overline{R}y\}.$$

This set is non-empty, since $y^*\in D^*$. We prove that D^* is an R-undominated subset of X. Indeed, assume that λRy for some $\lambda\in X\setminus D^*$ and some $y\in D^*$. It follows that $(\lambda,x)\in\overline{R}$. Since $x\in F$, as in the case of y^* above, we conclude that $(x,\lambda)\in\overline{R}$. Therefore, $\lambda\in D^*$ which is impossible. The last implication shows that D^* is an R-undominated subset of X and thus $\mathcal{D}\neq\emptyset$, a contradiction to the hypothesis that $\mathcal{D}=\emptyset$. Therefore, $\mathcal{D}\neq\emptyset$.

Let now $D\in\mathcal{D}$. We show that D has a minimal R-undominated subset D^{M}. First, we prove that $D\bigcap F\neq\emptyset$. Indeed, let $x\in D$. If $x\in F$, then this is evident. Otherwise, for suppose $x\notin F$, there exists $y\in F$ such that $(y,x)\in\overline{R}$. Therefore, there exists a natural number n and alternatives $x_1,x_2,\ldots,x_{n-1},x_n$ such that $yRx_1\ldots x_{n-1}Rx_nRx$. Since $x\in D$, if $x_n\in X\setminus D$ we cannot have x_nRx. It then follows that $x_n\in D$. Similarly, $x_{n-1}\in D$, and an induction argument based on this logic yields $y\in D$. Hence in any case $D\bigcap F\neq\emptyset$ must be true. Let now $z\in D\bigcap F$. If z is an R-undominated element, then $\{z\}=D^{\mathrm{M}}$ is a minimal R-undominated subset of D. Otherwise, there exists $w^*\in X$ such that $(w^*,z)\in R\subseteq\overline{R}$. Since F satisfies internal stability of domination we conclude that $w^*\notin F$. Hence, there exists $w'\in F$ such that $(w',w^*)\in\overline{R}$. It follows that $(w',z)\in\overline{R}$. Since $w',z\in F$, we must have $w'=z$. It follows that $(z,w^*)\in\overline{R}$. Let $D^{\mathrm{M}}=\{w\in X\mid (w,z)\in\overline{R} \text{ and } (z,w)\in\overline{R}\}$. Since $w^*\in D^{\mathrm{M}}$, $D^{\mathrm{M}}\neq\emptyset$. We prove that D^{M} is a minimal R-undominated subset of X such that $D^{\mathrm{M}}\subseteq D$. To show that D^{M} is R-undominated, suppose that $(s,w)\in R\subseteq\overline{R}$ for some $s\in X\setminus D^{\mathrm{M}}$ and $w\in D^{\mathrm{M}}$; to deduce a contradiction. It follows that $(s,z)\in\overline{R}$ which implies that $s\notin F$. Therefore, there exists $s'\in F$ such that $(s',s)\in\overline{R}$. Hence, $(s',z)\in\overline{R}$. Since $s',z\in F$ this is impossible unless we have $s'=z$. It follows that $(z,s)\in\overline{R}$ which jointly to $(s,z)\in\overline{R}$ leads to a contradiction with $s\in X\setminus D^{\mathrm{M}}$. This contradiction implies that D^{M} is R-undominated. Since for each $w\in D^{\mathrm{M}}$, the set $D^{\mathrm{M}}\setminus\{w\}$ is not R-undominated, it follows that D^{M} is a minimal R-undominated subset of X. It remains to prove that $D^{\mathrm{M}}\subseteq D$. To see this, let $w\in D^{\mathrm{M}}$. Then, it must be that $(w,z)\in\overline{R}$, i.e., there exist $z_1,z_2,\ldots,z_n$ such that $wRz_1\ldots z_nRz$. Since D is R-undominated and $z\in D$, it follows as above that $w\in D$.

For the sufficiency of the theorem, we suppose that every R-undominated set has a minimal R-undominated subset. First, we prove that under this assumption (X,R) has at least one minimal R-undominated subset. There are two cases to consider: (i) There exists $x_0\in X$ such that for each $y\in X$, $(y,x_0)\notin P(\overline{R})$; (ii) For every $x\in X$ there exists $y\in X$ such that $yP(\overline{R})x$. In case (i), if x_0 is an R-undominated element then $\{x_0\}$ is a minimal R-undominated subset of X. Otherwise, there exists $y^* \in X$ such that $(y^*, x_0) \in R \subseteq \overline{R}$. Since $(y^*, x_0) \notin P(\overline{R})$ we conclude that $(x_0, y^*) \in \overline{R}$. Let $\overset{\frown}{D} = \{y \in X \mid (y, x_0) \in \overline{R} \text{ and } (x_0, y) \in \overline{R}\}$. Clearly, $\overset{\frown}{D} \neq \emptyset$. We now show that $\overset{\frown}{D}$ is a minimal R-undominated set. Indeed, suppose to the contrary, that $(z, y) \in R$ for some $z \in X \setminus \overset{\frown}{D}$ and $y \in \overset{\frown}{D}$. It follows that $(z, x_0) \in \overline{R}$. Since $(z, x_0) \notin P(\overline{R})$, we conclude that $(x_0, z) \in \overline{R}$. Hence, $z \in \overset{\frown}{D}$, a contradiction. $\overset{\frown}{D}$ is also minimal since none of its proper subsets is R-undominated. In case (ii), let $x \in X$. We define a set A_x as follows:

$$A_x = \{y \in X \mid yP(\overline{R})x\}.$$

It easy to check that $A_x \subset X \setminus \{x\} \subset X$ is an R-undominated set in X. By the assumption, there exists a minimal R-undominated set $\overset{\frown}{D}$ such that $\overset{\frown}{D} \subseteq A_x$. In both cases, therefore, (X, R) has a minimal R-undominated set. To prove that (X, R) has a generalized stable set, let $\mathrm{D} = \{\widetilde{D_i} \mid i \in I\}$ be the family of all the minimal R-undominated subsets of X. Since $\widetilde{D} \in \mathrm{D}$, this set is non empty. For each $i \in I$, we choose a $\widetilde{d_i} \in \widetilde{D_i}$. We show that $\widetilde{\mathrm{D}} = \{\widetilde{d_i} \mid i \in I\}$ is a generalized stable set of (X, R). We first show that D satisfies internal stability of domination. Indeed, suppose to the contrary that $(\widetilde{d_i}, \widetilde{d_j}) \in \overline{R}$ for some $i, j \in I$. Therefore, there exists a natural number n and alternatives $\delta_1, \delta_2, \dots, \delta_{n-1}, \delta_n$ such that $\widetilde{d_i} R\delta_1 \dots \delta_{n-1} R\delta_n R\widetilde{d_j}$. Then, since $\widetilde{D_j}$ is R-undominated, following the above reasoning, as $x_1, \dots, x_n$, we get $\widetilde{d_i} \in \widetilde{D_j}$, a contradiction since $\widetilde{D_i} \cap \widetilde{D_j} = \emptyset$ ($\widetilde{D_i}$ and $\widetilde{D_j}$ are minimal R-undominated subsets of X). Hence, $(\widetilde{d_i}, \widetilde{d_j}) \notin \overline{R}$. To complete the proof it remains to prove that D satisfies external stability of domination. Let $w \in X \setminus \mathrm{D}$. We prove that $(\widetilde{d_k}, w) \in \overline{R}$ for some $k \in I$. There are two cases to consider: (a_1) $w \in \widetilde{D_k}$ for some $k \in I$; (a_2) $w \notin \bigcup_{i \in I} \widetilde{D_i}$.

Case (a_1). Since $w \in X \setminus \mathrm{D}$, it follows that $\widetilde{D_k} \neq \{w\}$. Put $B_w = \{s \in \widetilde{D_k} \mid (s, w) \in \overline{R}\}$. We have that $B_w \neq \emptyset$, because otherwise, for each $s \in \widetilde{D_k}$, $(s, w) \notin \overline{R}$ implies that $(s, w) \notin R$, and hence $\{w\} \subset \widetilde{D_k}$ is an R-undominated subset of X, a contradiction because of the minimal character of $\widetilde{D_k}$. Let $\overset{\frown}{D_k} = \widetilde{D_k} \setminus B_w$. We now show that $\overset{\frown}{D_k} = \emptyset$. We proceed by contradiction. Assume that $\overset{\frown}{D_k} \neq \emptyset$. Then, for each $\overset{\frown}{s} \in \overset{\frown}{D_k}$ and each $s \in B_w$ we have $(\overset{\frown}{s}, s) \notin R$ for suppose otherwise, $(\overset{\frown}{s}, s) \in R$ implies that $(\overset{\frown}{s}, w) \in \overline{R}$ contradicting $\overset{\frown}{s} \in \overset{\frown}{D_k}$. Therefore,

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$B_w \subset \widetilde{D_k}$ is an R-undominated subset of X, again a contradiction. Hence, $\widetilde{D_k} = \varnothing$. It follows that $\widetilde{D_k} = B_w$. But then, since $\widetilde{d_k} \in \widetilde{D_k}$ we conclude that $(\widetilde{d_k}, w) \in \overline{R}$.

Case (a_2). Let $w \notin \bigcup_{i \in I} \widetilde{D_i}$. We define

$A_w = \{r \in X \mid (r, w) \in \overline{R}\}$.

We show that A_w is non-empty. Indeed, suppose to the contrary that for each $r \in X$, $(r, w) \notin \overline{R}$. Then, $\{w\} = \widetilde{D_i}$ for some $i \in I$, a contradiction to $w \notin \bigcup_{i \in I} \widetilde{D_i}$. Hence, $A_w \neq \varnothing$. Since A_w is R-undominated, by the asumption, there exists a minimal R-undominated subset $\widetilde{D_k}$, $k \in I$, such that $\widetilde{D_k} \subseteq A_w$. It follows that $(\widetilde{d_k}, w) \in \overline{R}$. The last conclusion completes the proof.

The following result is an immediate corollary of the above theorem.

Corollary. [4, Theorem 3]. Let X be a nonempty finite set and R a non-empty asymmetric binary relation on X. Then there exists a generalized stable set F such that F is non empty.

Considering [2, Theorem 5] which shows that the admissible set is equivalent to the the union of all its R-undominated sets and [1, Theorem 22], we obtain that under the assumption that a generalized stable set exists, the union of all generalized stable sets coincides with the union of all R-undominated sets. In this case, the two general solution conceps are identical. But when the union of generalized stable sets is empty, that is not so, as the following example shows. Let $X = \{x_n \mid 1 \leq n \leq N, N \in \mathbb{N}\} \bigcup \{y_m \mid m \in \mathbb{N}\}$. Define the relation R as follows: $x_{n'} R x_n$ for each $n' > n$ and $y_{m'} R y_m$ for each $m' > m$. Then R is a partial order. Clearly, each subset of X violates external stability of domination. Thus, the family of generalized stable sets in X is empty. On the other hand, the admissible set is equal to the singleton $\{x_N\}$.

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