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An n-order (F,α,ρ,d)- Convex Function and Duality Problem

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Abstract

A class of n-order (F,α,ρ,d)-convex function and their generalization on functions is introduced. Using the assumption on the functions involved, weak, strong, and converse duality theorems are established for the n-order dual problem

Keywords: n-order (F,α,ρ,d)-convex function; generalization convexity; weak; strong; converse duality theorems.

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1 Introducation

The importance of convex function is well known in optimization theory. But for many mathematical models used in decision science, economic, management science, applied mathematics and engineering. The notion of convexity does no longer suffice. So it is possible to generalize the notion of convexity and to extend the validity of result to larger classes of optimization problems. Consequently, various generalizations of convex functions have been introduced in the literature. More specifically, the concept of (F,ρ)-convexity was introduce by Preda[8] an extension of F - convexity defined by Hanson and Mond[7], generalized concavity and duality, in generalized concavity in Optimization and economics is presnted by [6] and ρ-convexity given by Vial[9] Gualti and Islam [4], the generalized convexity and duality for multiobjective programming and the proper efficiency and duality for vector valued optimization problem were introduced by Weir and Mond [10, 11]. Ahmed [2] established optimality conditions and duality results for multiobjective programming problems involving F - convexity and (F,ρ) -convexity assumptions respectively. And also, Ahmed and Husain discussed in [1] the second order (F,α,ρ,d)-convexity and duality in multiobjective programming.

In this paper we will define n-order (F,α,ρ,d)-convexity and duality in nonlinear programming. These concepts are then used to develop weak, strong, and strict converse duality theorem for n-order dual problem.

Let M be a nonempty subset of R^n and let $f:M\to R, g_i:M\to R^m, i=1,2,...,m$ are assumed to be n-differentiable functions over M . Consider the following nonlinear programming problem P

$$\left.\begin{aligned} &Minimize\ f(x)\\ &subject\ to\ g_i(x)\leqq 0, x\in M, i=1,2,...,m\end{aligned}\right\} \qquad (1.1)$$

Definition 1 *A functional* $F:M\times M\times R^n\to R$ *is said to be sublinear in its third and component if for all* $x,\bar{x}\in M$

$$(i)F(x,\bar{x},a+b)\le F(x,\bar{x},a)+F(x,\bar{x},b)\forall a,b\in R^n$$

$$(ii)F(x,\bar{x},\beta a)=\beta F(x,\bar{x},a),\forall\beta\in R,\beta\ge 0 \quad and \quad \forall a\in R^n$$

Definition 2 *An n-differentiable function* f , $f:M\to R$ *is said to be n- order* (F,α,ρ,d) *- convex function at* $\bar{x}$ *on* M *if for all* $x\in M$ *, then there exists a vector* $P\in R^n$ *, a real valued function* $\alpha:M\times M\to R^+-\{0\}$*,a real valued function* $d:M\times M\to R$ *and a real number* ρ *such that*

$$f(x)-f(\bar{x})+\frac{1}{n!}(P\cdot\nabla)^n f(\bar{x})\ge F\left(x,\bar{x};\alpha(x,\bar{x})\left\{\sum_{r=1}^{n}\frac{1}{r!}(P\cdot\nabla)^r f(\bar{x})\right\}\right)$$
$$+\rho d^2(x,\bar{x}) \qquad (1.2)$$

where $(P\cdot\nabla)^r=(p_1\frac{\partial}{\partial x_1}+p_2\frac{\partial}{\partial x_2}+....+p_n\frac{\partial}{\partial x_n})^r$

Definition 3 *An n-differentiable function* f , $f:M\to R$ *is said to be n- order* (F,α,ρ,d) *- pseudoconvex function at* $\bar{x}$ *on* M *if for all* $x\in M$ *, then there exists a vector* $P\in R^n$ *, a real valued function* $\alpha:M\times M\to R^+-\{0\}$*,a real valued function* $d:M\times M\to R$ *and a real number* ρ *such that*

$$f(x)<f(\bar{x})-\frac{1}{n!}(P\cdot\nabla)^n f(\bar{x}) \qquad (1.3)$$

$$\Rightarrow F\left(x,\bar{x};\alpha(x,\bar{x})\left\{\sum_{r=1}^{n}\frac{1}{r!}(P\cdot\nabla)^r f(\bar{x})\right\}\right)<-\rho d^2(x,\bar{x})$$

Definition 4 *An n-differentiable function* f , $f : M \to R$ *is said to be strictly n-order* (F,α,ρ,d) *- pseudoconvex function at* $\bar{x}$ *on* M *if for all* $x \in M$ *, then there exists a vector* $P \in R^n$ *, a real valued function* $\alpha : M \times M \to R^+ - \{0\}$*,a real valued function* $d : M \times M \to R$ *and a real number* ρ *such that*

$$F\left(x,\bar{x};\alpha(x,\bar{x})\left\{\sum_{r=1}^{n}\frac{1}{r!}(P\cdot\nabla)^r f(\bar{x})\right\}\right) \geq -\rho d^2(x,\bar{x}) \tag{1.4}$$

$$\Rightarrow f(x) > f(\bar{x}) - \frac{1}{n!}(P\cdot\nabla)^n f(\bar{x})$$

or equivalently

$$f(x) \leq f(\bar{x}) - \frac{1}{n!}(P\cdot\nabla)^n f(\bar{x}) \tag{1.5}$$

$$\Rightarrow F\left(x,\bar{x};\alpha(x,\bar{x})\left\{\sum_{r=1}^{n}\frac{1}{r!}(P\cdot\nabla)^r f(\bar{x})\right\}\right) < -\rho d^2(x,\bar{x})$$

Definition 5 *An n-differentiable function* f , $f : M \to R$ *is said to be n- order* (F,α,ρ,d) *- quasiconvex function at* $\bar{x}$ *on* M *if for all* $x \in M$ *, then there exists a vector* $P \in R^n$ *, a real valued function* $\alpha : M \times M \to R^+ - \{0\}$*,a real valued function* $d : M \times M \to R$ *and a real number* ρ *such that*

$$f(x) \leq f(\bar{x}) - \frac{1}{n!}(P\cdot\nabla)^n f(\bar{x}) \tag{1.6}$$

$$\Rightarrow F\left(x,\bar{x};\alpha(x,\bar{x})\left\{\sum_{r=1}^{n}\frac{1}{r!}(P\cdot\nabla)^r f(\bar{x})\right\}\right) \leq -\rho d^2(x,\bar{x})$$

Definition 6 *An n-differentiable function* f , $f : M \to R$ *is said to be strongly n-order* (F,α,ρ,d) *- pseudoconvex function at* $\bar{x}$ *on* M *if for all* $x \in M$ *, then there exists a vector* $P \in R^n$ *, a real valued function* $\alpha : M \times M \to R^+ - \{0\}$*,a real valued function* $d : M \times M \to R$ *and a real number* ρ *such that*

$$f(x) \leq f(\bar{x}) - \frac{1}{n!}(P\cdot\nabla)^n f(\bar{x}) \tag{1.7}$$

$$\Rightarrow F\left(x,\bar{x};\alpha(x,\bar{x})\left\{\sum_{r=1}^{n}\frac{1}{r!}(P\cdot\nabla)^r f(\bar{x})\right\}\right) \leq -\rho d^2(x,\bar{x})$$

Example 7 *Consider the function* $f : M(=R_+) \to R$ *such that* $f(x) = x^5 - 3x^4$*. If we define the function*

$$F(x,\bar{x};\alpha) = \alpha(x-\bar{x}) - 4x, \quad d(x,\bar{x}) = x - \bar{x}$$

$$\alpha(x,\bar{x}) = \frac{x+\bar{x}+1}{2}, \qquad P = x - \bar{x}$$

then for $\rho = 0, f$ is 5-order (F,α,ρ,d) -convex at $\bar{x} = 0$ with respect to the vector $P = x - \bar{x}$

Theorem 8 *(Kuhn-Tucker necessary conditions)[3, 5] Assume that* x^* *is an optimal solution of the problem 1.1 at which the Kuhn-Tucker constraints qualification is satisfied . Then there exists* $0 \leq \lambda \in R^m$ *for which* $\lambda \neq 0$ *such that*

$$\nabla f(\bar{x}) + \sum_{i=1}^{n}\lambda_i \nabla g_i(\bar{x}) = 0$$

$$\lambda_i g_i(\bar{x}) = 0, i = 1,2,...,m$$

2 n-Order duality Nonlinear Programming Problems

In this section, we generalize the Mond-Weir type second order dual problem to n-order dual problem associated with the problem (P) and establish weak,strong and strict converse duality theorem under generalized n-order (F,α,ρ,d) -convexity (MD)

$$\text{Maximize } f(u) - \frac{1}{n!}(P\cdot\nabla)^n f(u) \tag{2.1}$$

$$\text{Subject to } \sum_{r=1}^{n}\frac{1}{r!}(P\cdot\nabla)^r f(u) + \sum_{i=1}^{m}\sum_{r=1}^{n}\lambda_i \frac{1}{r!}(P\cdot\nabla)^r g_i(u) = 0 \tag{2.2}$$

$$\sum_{i=1}^{m}\lambda_i g_i(u) - \sum_{i=1}^{m}\sum_{r=1}^{n}\lambda_i \frac{1}{r!}(P\cdot\nabla)^r g_i(u) \geq 0 \tag{2.3}$$

$$\lambda_i \geq 0 . i = 1,2,...,m \tag{2.4}$$

Theorem 9 *(Weak duality). Suppose that for all feasible in* (P) *and all feasible* $(u,\lambda_1,...,\lambda_m,P)$ *in* (MD)

(i) $\sum_{i=1}^{m}\lambda_i g_i(.)$ is n-order (F,α,ρ,d) -quasiconvex at $u,$ and assume that any

one of the following conditions holds:

(ii) $\lambda_i > 0, i = 1,2,...,m.$ and $f(.)$ is strong n-order $(F,\alpha_1,\rho_1,d)-$ pseudoconvex

at u with $\alpha^{-1}\rho + \alpha_1^{-1}\rho_1 \geq 0$

(iii) $f(.)$ is strictly n-order $(F,\alpha_2,\rho_2,d)-$ pseudoconvex at u with

$\alpha^{-1}\rho + \alpha_2^{-1}\rho_2 \geq 0.$

Then the following cannot hold

$$f(x) \leq f(u) - \frac{1}{n!}(P\cdot\nabla)^n f(u) \tag{2.5}$$

Proof. Let x be any feasible solution in (P) and $(u,\lambda_1,...,\lambda_m,p)$ be any feasible solution in $(MD).$ Then we have

$$\sum_{i=1}^{m}\lambda_i g_i(x) \leq 0 \leq \sum_{i=1}^{m}\lambda_i g_i(u) - \sum_{i=1}^{m}\sum_{r=1}^{n}\lambda_i \frac{1}{r!}(P\cdot\nabla)^r g_i(u) \tag{2.6}$$

using n-order (F,α,ρ,d) -quasiconvexity of $\sum_{i=1}^{m}\lambda_i g_i(.)$ at $u,$ we get

$$F\left(x,u;\alpha(x,u)\left\{\sum_{i=1}^{m}\sum_{r=1}^{n}\frac{\lambda_i}{r!}(P\cdot\nabla)^r g(u)\right\}\right) \leq -\rho d^2(x,u) \tag{2.7}$$

Since $\alpha(x,u) > 0$ the inequality 2.7 with the sublinearity of F yields

$$F\left(x,u;\left\{\sum_{i=1}^{m}\sum_{r=1}^{n}\frac{\lambda_i}{r!}(P\cdot\nabla)^r g_i(u)\right\}\right) \leq -\alpha^{-1}(x,u)\rho d^2(x,u) \tag{2.8}$$

The first dual constraint and the sublinearity of F give

$$F(x,u,\sum_{r=1}^{n}\frac{1}{r!}(P\cdot\nabla)^{r} f(u)) \geq -F(x,u;\sum_{r=1}^{n}\frac{1}{r!}(P\cdot\nabla)^{r} f(u)) \quad (2.9)$$

The inequality 2.8 and 2.9

$$F(x,u;\sum_{r=1}^{n}\frac{1}{r!}(P\cdot\nabla)^{r} f(u)) \geq \alpha^{-1}(x,u)\rho d^{2}(x,u) \quad (2.10)$$

Now suppose contrary to the result that 2.5 holds ,i.e.

$$f(x) \leq f(u) - \frac{1}{n!}(P\cdot\nabla)^{n} f(u) \quad (2.11)$$

which by virtue of (ii), leads to

$$F\left(x,u;\alpha_1(x,u)\left\{\sum_{r=1}^{n}\frac{1}{r!}(P\cdot\nabla)^{r} f(u)\right\}\right) \leq -\rho_1 d^{2}(x,u) \quad (2.12)$$

From the inequality 2.12 and using the sublinearity of F with $\alpha_1(x,u) > 0,$ we obtain

$$F\left(x,u;\sum_{r=1}^{n}\frac{1}{r!}(P\cdot\nabla)^{r} f(u)\right) \leq -\alpha_1^{-1}(x,u)\rho_1\lambda d^{2}(x,u) \leq \alpha^{-1}(x,u)\rho d^{2}(x,u) \quad (2.13)$$

which contradicts 2.10. Hence 2.5 cannot hold.

On the other hand , when hypothesis (iii) holds, the inequality 2.11 implies

$$F\left(x,u;\alpha_2(x,u)\left\{\sum_{r=1}^{n}\frac{1}{r!}(P\cdot\nabla)^{r} f(u)\right\}\right) \leq -\rho_2 d^{2}(x,u) \quad (2.14)$$

Since F is sublinear and $\alpha_2(x,u) > 0,$ it follows from 2.14 that

$$F\left(x,u;\sum_{r=1}^{n}\frac{1}{r!}(P\cdot\nabla)^{r} f(u)\right) \leq -\alpha_2^{-1}(x,u)\rho_1\lambda d^{2}(x,u) \leq \alpha^{-1}(x,u)\rho d^{2}(x,u)$$

a contradiction to 2.10. hence 2.5 can not hold

Theorem 10 *(Strong duality) let $\bar{x}$ is an optimal solution of (P) at which the Kuhn-Tucker constraint qualification is satisfied. Then there exists $\bar{\lambda} \in R^{m}$ such that $(\bar{x},\bar{\lambda},P=0)$ is feasible for (MD) and the corresponding values of (P) and (MD) are equal.*

Proof. Since $\bar{x}$ an optimal solution of (P) at which the Kuhn-Tucker constraints qualification is satisfied,then by Theorem 1,there exists $\bar{\lambda} \in R^{m}$ such that

$$\nabla f(\bar{x}) + \sum_{i=1}^{k}\bar{\lambda}_i \nabla g_i(\bar{x}) = 0$$

$$\bar{\lambda}_i g_i(\bar{x}) = 0, i = 1,2,...,k$$

$$\bar{\lambda}, \geq 0,$$

Therefore $(\bar{x},\bar{\lambda},P=0)$ is feasible for (MD) and the corresponding values of (P) and (MD) are equal. From the weak duality theorem $(\bar{x},\bar{\lambda},P=0)$ is an optimal solution of the problem (MD).

Theorem 11 *(Strict converse duality Thorem) Let $\bar{x}$ and $(u,\lambda_1,...,\lambda_m,p)$ are optimal solution of (P) and (MD),* *respectively,such that*

$$f(\bar{x}) = f(\bar{u}) - \frac{1}{n!}(P \cdot \nabla)^n f(u) \tag{2.15}$$

Suppose that any one of the following conditions is satisfied

(i) $\sum_{i=1}^{m} \lambda_i g_i(.)$ is n-order (F, α, ρ, d)-quasiconvex at $\bar{u}$ and $f(.)$ is strictly n-order (F, α_1, ρ_1, d)- pseudoconvex at $\bar{u}$ with $\alpha^{-1}\rho + \alpha_1^{-1}\rho_1 \geq 0.$

(ii) $\sum_{i=1}^{m} \lambda_i g_i(.)$ is strictly n-order (F, α, ρ, d)-pseudoconvex at $\bar{u}$ and $f(.)$ is n-order (F, α_1, ρ_1, d)-quasiconvex at $\bar{u}$ with $\alpha^{-1}\rho + \alpha_1^{-1}\rho_1 \geq 0.$

Then $\bar{x} = \bar{u},$ that is, $\bar{u}$ is an optimal solution of (P)

Proof. We assume that $\bar{x} \neq \bar{u}$ and reach a contradiction.Since $\bar{x}$ and $(\bar{u}, \lambda_1, \lambda_2, ..., \lambda_m, p)$ are, respectively, the feasible solution of (P) and (MD) ,we have

$$\sum_{i=1}^{m} \lambda_i g_i(\bar{x}) \leq 0 \leq \sum_{i=1}^{m} \lambda_i g_i(\bar{u}) - \sum_{i=1}^{m} \sum_{r=1}^{n} \lambda_i \frac{1}{r!}(\bar{P} \cdot \nabla)^r g_i(\bar{u}) \tag{2.16}$$

Using n-order (F, α, ρ, d)-quasiconvexity of $\sum_{i=1}^{m} \lambda_i g_i(.)$ at $\bar{u},$ we get

$$F\left(\bar{x}, \bar{u}; \alpha(\bar{x}, \bar{u})\left\{\sum_{i=1}^{m} \sum_{r=1}^{n} \lambda_i \frac{1}{r!}(\bar{P} \cdot \nabla)^r g_i(\bar{u})\right\}\right) \leq -\rho d^2(\bar{x}, \bar{u}) \tag{2.17}$$

Since $\alpha(\bar{x}, \bar{u}) > 0,$ the inequality 2.17 along with the sublinearity of F yields

$$F\left(\bar{x}, \bar{u}; \left\{\sum_{i=1}^{m} \sum_{r=1}^{n} \lambda_i \frac{1}{r!}(\bar{P} \cdot \nabla)^r g_i(\bar{u})\right\}\right) \leq -\alpha^{-1}(\bar{x}, \bar{u}) \rho d^2(\bar{x}, \bar{u}) \tag{2.18}$$

The first dual constraint and the sublinearity of F imply

$$F\left(\bar{x}, \bar{u}; \sum_{r=1}^{n} \frac{1}{r!}(P \cdot \nabla)^r f(\bar{u})\right) + F\left(\bar{x}, \bar{u}; \left\{\sum_{i=1}^{m} \sum_{r=1}^{n} \lambda_i \frac{1}{r!}(\bar{P} \cdot \nabla)^r g_i(\bar{u})\right\}\right)$$

$$\geq F\left(\bar{x}, \bar{u}; \sum_{r=1}^{n} \frac{1}{r!}(P \cdot \nabla)^r f(\bar{u}) + \sum_{i=1}^{m} \sum_{r=1}^{n} \lambda_i \frac{1}{r!}(\bar{P} \cdot \nabla)^r g_i(\bar{u})\right) = 0 \tag{2.19}$$

The inequality 2.18 ,7 and $\alpha^{-1}\rho + \alpha_1^{-1}\rho_1 \geq 0$ imply

$$F\left(\bar{x}, \bar{u}; \sum_{r=1}^{n} \frac{1}{r!}(P \cdot \nabla)^r f(\bar{u})\right) \geq -\alpha^{-1}(\bar{x}, \bar{u}) \rho d^2(\bar{x}, \bar{u}) \tag{2.20}$$

Using strict n-order (F, α_1, ρ_1, d)-pseudoconvexity of $f(.)$ with $\alpha(\bar{x}, \bar{u}) > 0$

$$f(\bar{x}) > f(\bar{u}) - \frac{1}{n!}(P \cdot \nabla)^n f(u)$$

contradicting 2.15

When the hypothesis (ii)holds ,it follows from 2.16 that

$$F\left((\bar{x},\bar{u};\alpha(\bar{x},\bar{u})\left\{ \sum_{i=1}^{m}\sum_{r=1}^{n}\lambda_i \frac{1}{r!}(\bar{P}\cdot\nabla)^r g_i(\bar{u}) \right\} \right) < -\rho d^2(\bar{x},\bar{u})$$

Since $\alpha(\bar{x},\bar{u}) > 0$,the above inequality with the sublinearity of F gives

$$F\left((\bar{x},\bar{u};\left\{ \sum_{i=1}^{m}\sum_{r=1}^{n}\lambda_i \frac{1}{r!}(\bar{P}\cdot\nabla)^r g_i(\bar{u}) \right\} \right) < -\alpha^{-1}(\bar{x},\bar{u})\rho d^2(\bar{x},\bar{u})$$

Which on using first dual constraint with the sublinearity of F implies

$$F\left(\bar{x},\bar{u};\sum_{r=1}^{n}\frac{1}{r!}(P\cdot\nabla)^r f(\bar{u}) \right) > \alpha^{-1}(\bar{x},\bar{u})\rho d^2(\bar{x},\bar{u}) \qquad (2.21)$$

As $\alpha^{-1}\rho + \alpha_1^{-1}\rho_1 \geq 0,$ we obtain

$$F\left(\bar{x},\bar{u};\sum_{r=1}^{n}\frac{1}{r!}(P\cdot\nabla)^r f(\bar{u}) \right) > -\alpha_1^{-1}(\bar{x},\bar{u})\rho_1 d^2(\bar{x},\bar{u}) \qquad (2.24)$$

the n-order (F,α_1,ρ_1,d)-quasiconvexity of $f(.)$ and 2.24 with $\alpha(\bar{x},\bar{u}) > 0$ yield

$$f(\bar{x}) > f(\bar{u}) - \frac{1}{n!}(\bar{P}\cdot\nabla)^n f(\bar{u})$$

again contradicting 2.15

Conclusion 12. *In this paper, n-order* (F,α,ρ,d)*-convexity and its generalization are introduced, which many other generalized convexity concept in mathematical programming as the special case. our concepts are suitable to discuss the weak,strong and strict converse duality theorems for the generalization of Mond-Wier type second order dual problem (n-order dual problem).*

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