Complexity Measure in Simple Type Food Chain System

[1]L. M. Saha and [2]Rajni Sharma

[1]IIIMIT, Shiv Nadar University ,Village Chithera, Tehsil Dadri, District-GautamBudh Nagar,
Pin Code: 203207, U P, India
lmsaha.msf@gmail.com
[2]Department of Mathematics, Chitkara University, H. P., India
srajni1@gmail.com

ABSTRACT

Study of complexities arising during evolution of a food chain system has been investigated. Regular and chaotic motions have been observed for certain sets of values of a parameter of the system. For some detailed further study, the continuous model of food chain has been transformed into discrete model by using Euler's method. Various measurable quantities for emergence of chaos, like Lyapunov exponents, topological entropies, correlation dimensions, have been numerically calculated and represented through plots. Finally, the chaos indicator, named Dynamic Lyapunov Indicator (DLI), has been used to identify clearly chaotic and regular motion.

Keywords: Food Chain System; Topological Entropy; Correlation Dimension; Dynamic Lyapunov Indicator.

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Academic Discipline and Sub-Disciplines: Dynamical Systems: Complexity and Chaos

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1. INTRODUCTION

Equations dealing with natural system, particularly the biological system, are nonlinear of nature. Parameters involved in the system define its nonlinear properties. Often the variations of these parameters during evolution show complexity and chaos which are quite natural.

In Ecological systems, food webs or food chains are constituted of several layers such that the consumers which eat from the bottom resource layer are the prey of another predator. This is due to interdependent of species within the system. So, the evolutionary dynamics of food chain webs are of complex of nature and their dynamics is highly interesting. Small variations of parameter of the system show very divergent results. Models describing food chain can be obtained in the form of a set of ordinary differential equations or as a set of discrete form of equations. These models are obtained by observing the group behavior of involving species, e.g. functional group behavior. Because of dissipative dynamical forms continuous models are complicated and distinct dynamical consequences may occur.

Three species food chain system was described in articles [1–3] and then followed by many researchers with variations of ideas, e.g. see ref. [4-17], and many others. A detailed review of food chain research articles can be obtained from the article by Deng, [13] and by Elsadany [18].

The objective of this work is to study complexity dynamics of the food chain system introduced by Deng, [13], in more detailed. The study has been carried on the original model of Deng and a discrete form of it by discretizing it by using Euler method. This later part is to obtain some complexity measures like topological entropies, correlation dimensions etc. We have obtained bifurcations for the model as well as its regular and chaotic attractors. Numerical calculations are further extended to obtain plots for Lyapunov exponents, topological entropies, correlation data curve and DLI plots for regular and chaotic motions.

2. THE FOOD CHAIN MODEL

A continuous model of food chain introduced by Deng (2006) can be written as

$$\frac{dX}{dt} = r\,X\left(1 - \frac{X}{K}\right) - \frac{p_1\,X}{H_1 + X}\,Y \tag{1}$$

$$\frac{dY}{dt} = Y\left(\frac{b_1\,p_1\,X}{H_1 + X} - d_1 - s_1\,Y\right) - \frac{p_1\,Y}{H_2 + Y}\,Z \tag{2}$$

$$\frac{dZ}{dt} = Z\left(\frac{b_2\,p_2\,Y}{H_2 + Y} - d_2 - s_2\,Z\right) \tag{3}$$

where, X, Y, Z denote the population densities, respectively for prey, predator, and top-predator populations. K is the carring capacity of the prey population in absence of the predator population and in such a case r denotes the maximum per capita growth of the prey population. The parameters b_1, b_2 are the birth- to –consumption ratios for predators; H_1, H_2 are the semisaturation constants; p_1, p_2 are the maximum per capita capture rate; d_1, d_2 are per capita minimum death rate for each predator and the products $s_1\,Y$, $s_2\,Z$ are additional per capita density dependent death rate of each predator. One may obtain a detailed explanation of the model in [13].

Above equations can be transformed into dimensionless forms by using following transformation:

$$t = b_1\,p_1\,T \quad \Rightarrow \quad \frac{d}{dt} = b_1\,p_1\,\frac{d}{dT}$$

and then applying changes

$$x = \frac{X}{K},\ y = \frac{Y}{Y_0},\ z = \frac{Z}{Z_0},\ \beta_1 = \frac{H_1}{K},\ \beta_2 = \frac{H_2}{K},\ Y_0 = \frac{r\,K}{p_1},\ Z_0 = \frac{b_1\,r\,K}{p_2},\ \delta_1 = \frac{d_1}{b_1\,p_1},\ \delta_2 = \frac{d_2}{b_2\,p_2},$$

$$\sigma_1 = \frac{s_1\,Y_0}{b_1\,p_1},\ \sigma_2 = \frac{s_2\,Z_0}{b_2\,p_2},\ \zeta = \frac{b_1\,p_1}{r}\ \text{and}\ \varepsilon = \frac{b_1\,p_1}{b_2\,p_2} \tag{4}$$
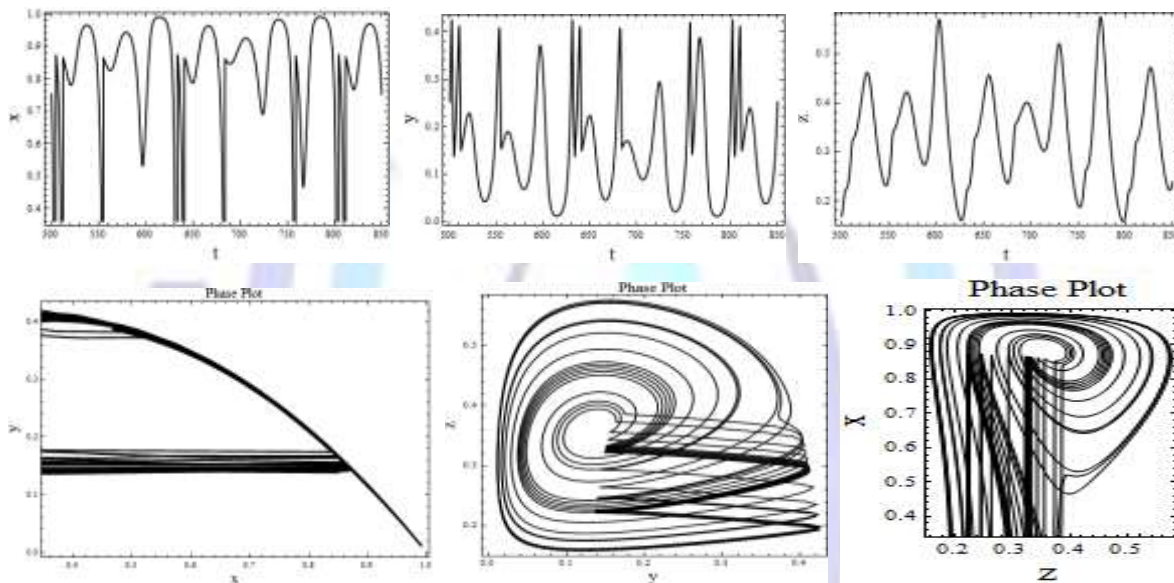
Then, finally the dimensionless form of above model takes the form

$$\zeta \dot{x} = x\left(1 - x - \frac{y}{\beta_1 + x}\right) \qquad (5)$$

$$\dot{y} = y\left(\frac{x}{\beta_1 + x} - \delta_1 - \sigma_1 y - \frac{z}{\beta_2 + y}\right) \qquad (6)$$

$$\dot{z} = \varepsilon z\left(\frac{y}{\beta_2 + y} - \delta_2 - \sigma_2 z\right) \qquad (7)$$

The time series and phase plot chaotic attractors for the continuous system (5) – (7) be given in Figure 1 for parameter values , $\zeta = 0.02$, $\beta_1 = 0.26$, $\beta_2 = 0.5$, $\delta_1 = 0.2$, $\delta_2 = 0.18$, $\sigma_1 = 0$, $\sigma_1 = 0.1$ and $\varepsilon = 0.375$.
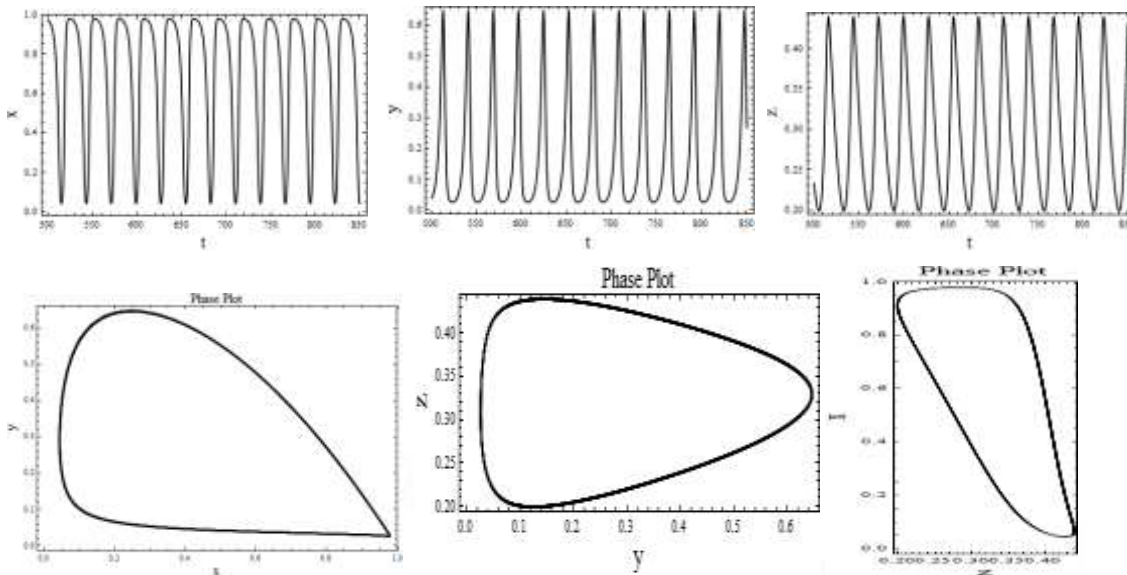

Figure 1: Time series and phase plots of continuous system for chaotic motion.

Extending the numerical simulations further one obtains the plots for Lyapunov exponents (LCE) for the chaotic motion as shown in Figure 3.
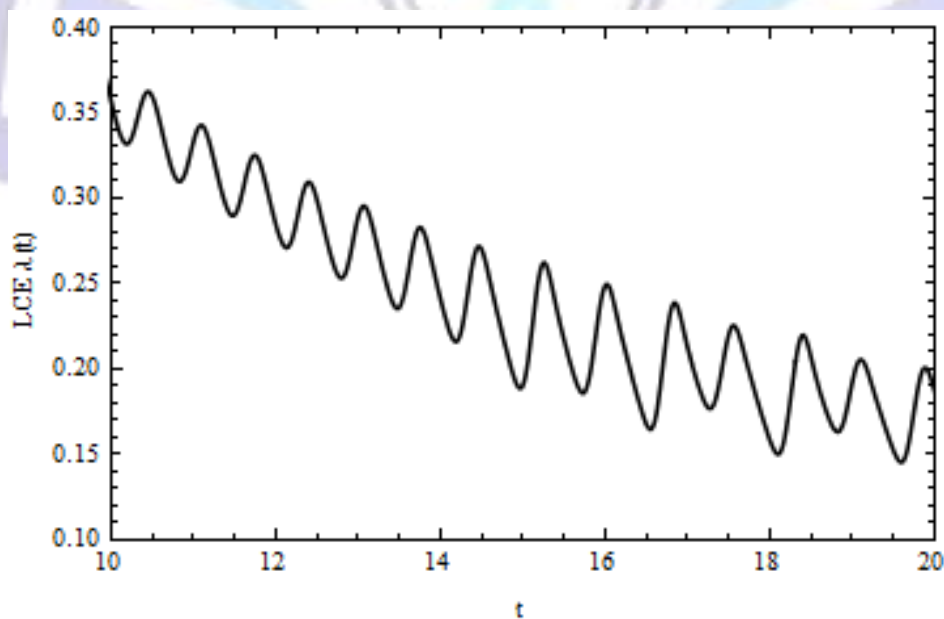
Figure 2: Plot of Lyapunov exponents for the chaotic motion.

The continuous system's regular motion can be observed when we increase parameter ζ keeping other parameters intact. For $\zeta = 0.7$, the above type of plots show regular motion. The time series and phase plots in this regard are given in Figure 3.

Figure 3 : Time series and phase plots of continuous system for regular motion.

For extending our numerical calculations we transformed the continuous model shown by equations (5) – (7). A discrete form of above continuous model can be obtained by Euler's method and written as

$$x_{n+1} = x_n + \frac{h}{\zeta} x_n (1 - x_n - \frac{y_n}{\beta_1 + x_n})$$

$$y_{n+1} = y_n + h y_n (\frac{x_n}{\beta_1 + x_n} - \delta_1 - \sigma_1 y_n - \frac{z_n}{\beta_2 + y_n})$$

$$z_{n+1} = z_n + h \varepsilon z_n (\frac{y_n}{\beta_2 + y_n} - \delta_2 - \sigma_2 z_n)$$

(8)

where h must be taken as very small positive number, i.e. 0 < h << 1. This h would be considered as a parameter of system (8).

3. BIFURCATION ANALYSIS AND ATTRACTORS:

Bifurcation diagrams of discrete food - chain model (8), obtained by varying ε, are obtained along all three coordinate planes and shown in Figure 4. Parameters are chosen as h=0.001, $\zeta = 0.01$, $\beta_1 = 0.26$, $\beta_2 = 0.5$, $\delta_1 = 0.2$, $\delta_2 = 0.18$,
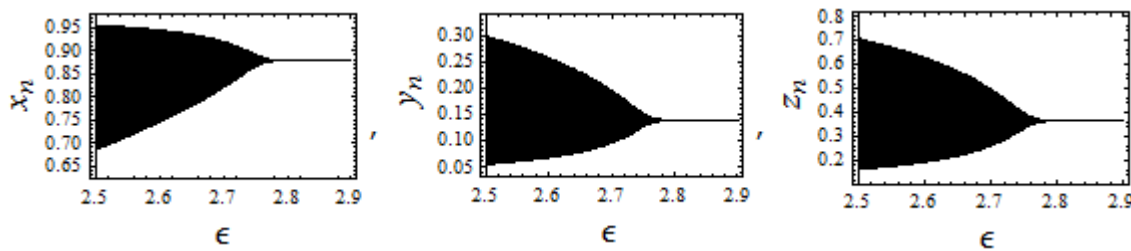
$\sigma_1 = 0$, $\sigma_1 = 0.1$ and varying ε such that $2.5 \leq \varepsilon \leq 3.5$.
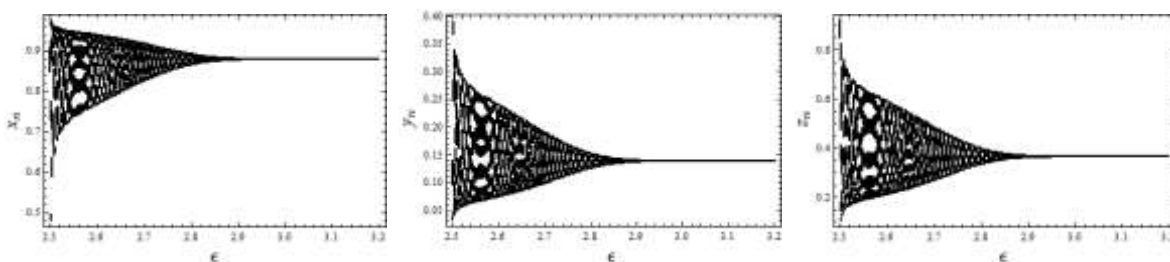
**Figure 4: Bifurcation diagrams of map (8) when h = 0.001.**

However, by choosing h a greater value, h = 0.01 and keeping other parameters same, the above bifurcation diagram have certain changes and shown in Figure 5.

Figure 5: Bifurcation diagrams of map (8) when h = 0.01.

We see from above figures that the motion seems to be chaotic or irregular within $2.5 \leq \varepsilon \leq 2.7$. To obtain regular attractor of the discrete map (8), we have used h=0.01, $\zeta = 0.01$, $\beta_1 = 0.26$, $\beta_2 = 0.5$, $\delta_1 = 0.2$, $\delta_2 = 0.18$, $\sigma_1 = 0$,

σ_1 = 0.1and ε = 0.05. Figure 6 below is for time series and phase plot in x-y plane. These shows clearly the motion is quasi-periodic and so regular. We proceed our numerical calculations further to obtain the set of parameters for which the evolution of the map (8) be chaotic.
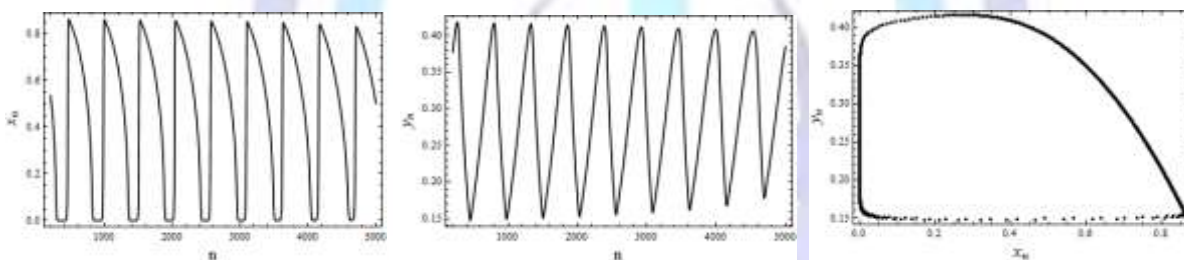
Figure 6: Chaotic evolution of map (8). The extreme right plot shows that the motion is quasi-periodic type.

4. NUMERICAL SIMULATIONS FOR COMPLEXITY: CALCULATIONS OF LYAPUNOV EXPONENTS, TOPOLOGICAL ENTROPIS AND CORELATION DIMENSIONS

The complexity of a physical system or a dynamical process expresses the degree to which components engage in organized structured interactions. High complexity is achieved in systems that exhibit a mixture of order and disorder (randomness and regularity) and that have a high capacity to generate emergent phenomena. Complexity in a deterministic dynamical system means certain features that strongly related to the nonlinearity of the system. In this regard mathematics has largest role in contribution to the study of complex systems leading to the discovery of chaos in deterministic systems during long term evolution. Such systems comprise many interacting parts and can generate a new quality of collective behavior through self-organization, e.g. the spontaneous formation of temporal, spatial or functional structures. These systems are often characterized by extreme sensitivity to initial conditions as well as emergent behavior that are not readily predictable or even completely deterministic. These concepts can be viewed through the calculations for Lyapunov exponents, topological entropies and correlation dimensions.

(a) **Lyapunov Exponents** (LCEs):

Lyapunov exponents are dynamical measure capable to characterize deterministic chaos in the system which features to the highly sensitive dependence on initial conditions. Actually it means the exponential divergence of orbits originated closely with very small difference in initial conditions. It is an important and effective element to identify regularity and chaos in the system and can be explained in the following ways:

From above definition, a clear interpretation for Lyapunov exponent is given as: it is the measure of loss of information during the process of iteration.

For a n dimensional system, Lyapunov exponents can be defined by the expression

$$\lambda(X_0, U_0) = \lim_{n \to \infty} \frac{1}{n} \log \left\| \coprod_{t=0}^{n-1} J(X_t) U_0 \right\|, \qquad (9)$$

where $X \in R^n$, $F: R^n \to R^n$, $U_0 = X_0 - Y_0$ and J is the Jacobian matrix of map F and

$$\left\| X_n - Y_n \right\| \approx e^{\lambda(X_0, U_0)n},$$

X_0 and Y_0 are initial points of two trajectories which are supposed to be very close to each other. Taking h = 0.01 and ζ = 0.02, β_1 = 0.26, β_2 = 0.5, δ_1 = 0.2, δ_2 = 0.18, σ_1 = 0.1, σ_2 = 0.05and then, different values of ϵ as ϵ = 0.05,0.1,0.2,0.3,0.5,0.6 we have calculated Lyapunov exponents. Plots of values of LCEs are shown in Figure 7.
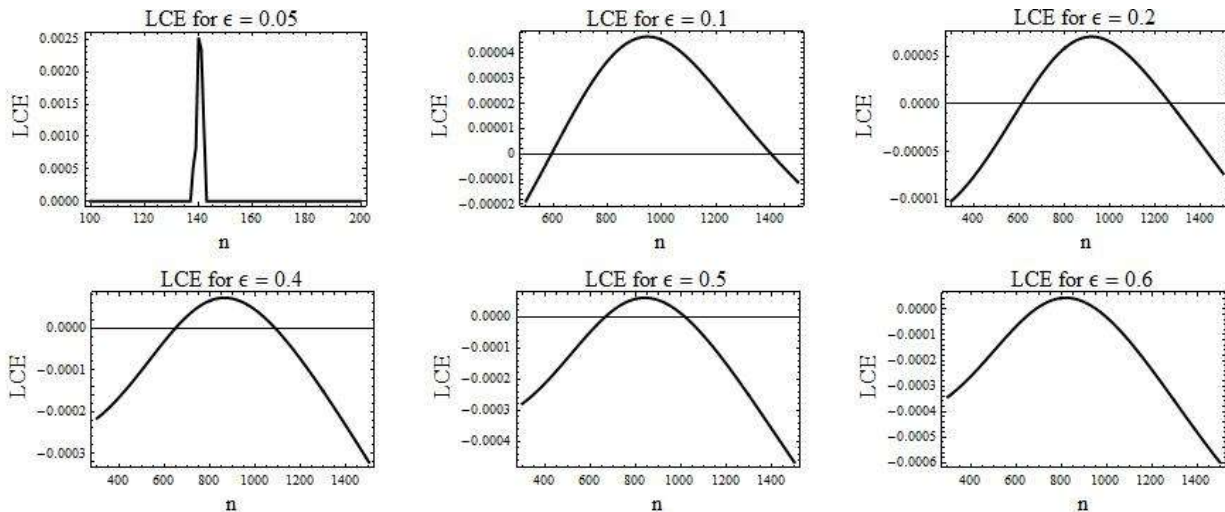

Figure7: Plots of Lyapunov exponents of system (9).

Study of above plots reveals the fact that for ϵ = 0.05, 0.1, 0.2, 0.3, 0.5 system evolution becomes chaotic but when ϵ further increases, e.g. ϵ = 0.6, LCEs are negative and system returns to regularity. This type of situation may follow when we chose other sets of parameters.

(b)**Topological Entropy**:

Chaotic attractors of a dynamical system composed of complex pattern and as the Lyapunov exponent has certain limitation, see ref. [19] – [24],to investigate chaotic behavior in a wide variety of system a better acceptable indicator is the topological entropy. Actually, the topological entropy indicates the complexity of the system. Definition and mathematical derivation of topological entropy be given in [25] and in the recent article by Saha and Kumra [17].

Keeping parameters h = 0.01, ζ = 0.02, β_1 = 0.26, $\beta2$ = 0.5, δ_1 = 0.2, δ_2 = 0.18, σ_1 = 0, σ_2 = 0.1, and varying ϵ one can observe easily the evolution is chaotic when ζ = 0.02 and regular when ζ = 5.5. Topological entropy calculations for these two cases have been performed numerically. The plots for chaotic and regular cases are shown, respectively, as plots (a) and (b) in Figure 7.
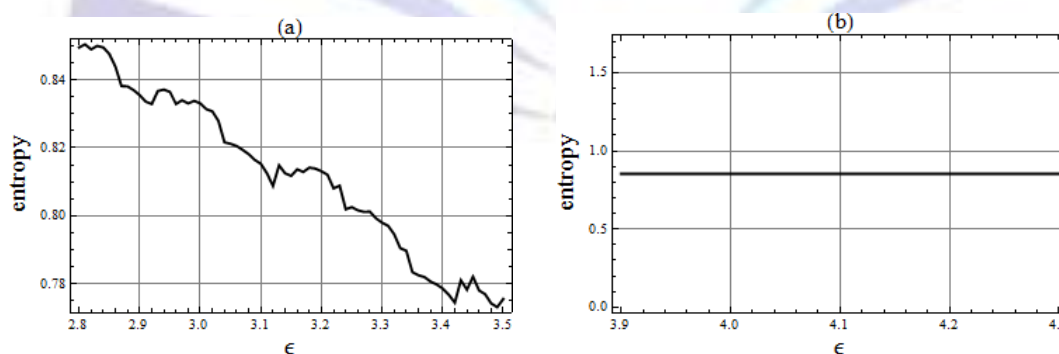

Figure 7: Topological entropy plots for chaotic evolution, plot (a), and for regular evolution, plot (b).

Above plots perfectly provide ideas of complexity in the chaotic system.

(c) Correlation Dimensions

Correlation dimension provides a *measure of dimensionality* of the chaotic attractor. . It is a very practical and efficient method then other methods, like box counting etc. Being one of the characteristic invariants of nonlinear system dynamics,

the correlation dimension also gives a measure of complexity for the underlying attractor of the system. To calculate correlation dimension one has use certain statistical approach. Here, we use the method described in [26].

Correlation dimension provide the dimensionality of the system. Here below we have two plots for correlation integral data for the system for a chaotic case when $\zeta = 0.03$ and for a regular case when

$\zeta = 5.5$ while as keeping other parameters fixed as h = 0.01, $\zeta = 0.02$, $\beta_1 = 0.26$, $\beta2 = 0.5$, $\delta_1 = 0.2$, $\delta_2 = 0.18$, $\sigma_1 = 0$, $\sigma_2 = 0.1$ and $\mathcal{E} = 4.4$, in Figure 8. The plot (b) of the regular case shows zero slope of the curve and zero intercept to y-axis; thus the correlation dimension is zero in this case. However, plot (a) of the chaotic case is different. By using least square linear fit, one obtains the equation of the straight line approximately fitting the curve as

$$Y = 0.2954 - 0.3404 \, x$$

The intercept of this straight line the y-axis is equal to 0.2954 $\approx$ 0.3. Thus, the correlation dimension for chaotic attractor form in this case is approximately equal to 0.3.
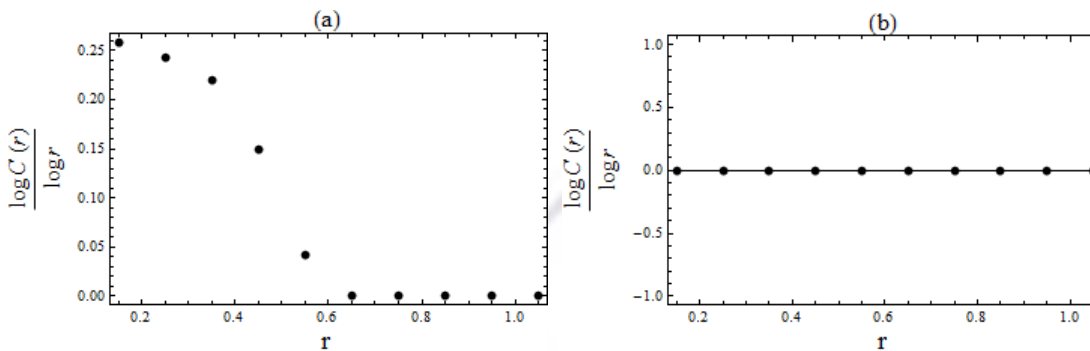

Figure 8: Plots of correlation integral curves; (a) when the evolution is chaotic & (b) when the evolution is regular.

5. APPLICATIONS OF INDICATOR DLI :

An indicator named as dynamic Lyapunov exponents (DLI) has recently been introduced by Saha and Budhraja,[27]. Its working ability to distinguish chaotic and regular motions is tested for various discrete systems [28] – [30]. When applied to discrete food chain model (8) and plotted after numerical simulations, it has been observed that DLI's, form a definite pattern for the motion is regular motion and for chaotic motion it shows randomly distributed points, (with no definite pattern).
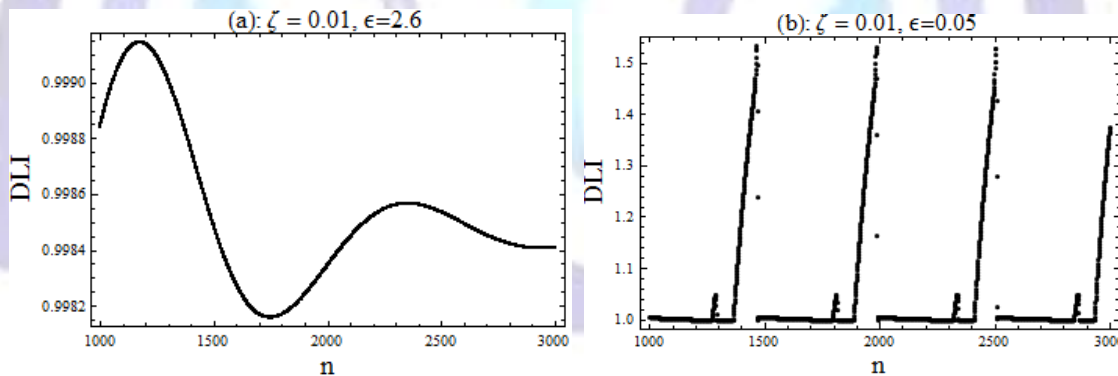

Figure 9: DLI plots for chaotic and regular evolutions.

6. DISCUSSIONS

The food chain system introduced by Bo Deng, [13], has been considered for the study of complexity arising in the system through some potential numerical simulations like obtaining plots like LCEs, topological entropies and calculating correlation dimensions for regular as well as for chaotic motions. LCEs provide the measure of exponential divergence, topological entropy that of complexity and correlation dimension a measure of dimensionality. This last one is nothing but fractal dimension for the chaotic set emerging during evolution. This type of study provides some light how complexity in the system because of its nonlinearity plays role. Such investigations in food chain system also throw lights on conditions that whether the participating species survive in coexistence or extinct. Regularity obtained for certain set of parameters indicate the possibility of survival. Chaotic situations indicates uncertainty which means the possibility of extinction. Finally, we have used an indicator, DLI, described recently for clear identification of regular and chaotic evolution.

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