



## Differential subordination and superordination for certain subclasses of $p$ -valent functions

Jamal M. Shenan

Department of mathematics, Alazhar University-Gaza, P. O. Box 1277, Gaza, Palestine.  
shenanjm@yahoo.com

Ahmed S. Galiz

Department of Mathematics, Alquds Open University-Gaza Branch, P. O. Box 124, Gaza, Palestine.  
Ahmad2911971@hotmail.com

### ABSTRACT

In this paper, we study applications of the differential subordination and superordination of analytic  $p$ -valent functions in the open unit disc associated with certain operator defined by the Wright generalized hypergeometric function. Sandwich-type result involving this operator is also derived.

**Keywords:** Analytic function;  $p$ -valent function; the Wright generalized hypergeometric function; differential subordination and superordination.

**2010 Mathematics Subject Classification:** 30C80 and 30C45



---

# Council for Innovative Research

Peer Review Research Publishing System

**Journal:** Journal of Advances in Mathematics

Vol 6, No. 3

[editor@cirworld.com](mailto:editor@cirworld.com)

[www.cirworld.com](http://www.cirworld.com), [member.cirworld.com](http://member.cirworld.com)



## 1 Introduction

Let  $H(U)$  be the class of functions analytic in  $U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$  and  $H[a, k]$  be the subclass of  $H(U)$  consisting of functions of the form

$$f(z) = a + a_p z^k + a_{p+1} z^{p+1} + \dots \quad (a \in \mathbb{C}, p \in \mathbb{N} = \{1, 2, \dots\}).$$

Let  $A_p$  denote the class of functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \quad (p \in \mathbb{N}, z \in U), \quad (1)$$

which are analytic in the open unit disk  $U$ , and set  $A \equiv A_1$ .

For two functions  $f(z)$  given by (1) and

$$g(z) = z^p + \sum_{k=1+p}^{\infty} b_k z^k, \quad (2)$$

the hadmard product (or convolution) of  $f$  and  $g$  is defined by

$$(f * g)(z) = z^p + \sum_{k=p+1}^{\infty} a_k b_k z^k = (g * f)(z) \quad (p \in \mathbb{N}, z \in U). \quad (3)$$

Let  $f$  and  $F$  be members of  $H(U)$ , the function  $f(z)$  is said to be subordinate to  $F(z)$ , or  $F(z)$  is said to be superordinate to  $f(z)$ , if there exists a function  $w(z)$  analytic in  $U$  with  $w(0) = 0$  and  $|w(z)| < 1$  ( $z \in U$ ), such that  $f(z) = F(w(z))$ . In such a case we write  $f(z) \prec F(z)$ . In particular, if  $F$  is univalent, then  $f(z) \prec F(z)$  if and only if  $f(0) = F(0)$  and  $f(U) \subset F(U)$  (see [1, 2]).

Suppose that  $p$  and  $h$  are two functions in  $U$ , let

$$\phi(r, s, t; z) : \mathbb{C}^3 \times U \rightarrow \mathbb{C}.$$

If  $p$  and  $\phi(p(z), zp'(z), z^2 p''(z); z)$  are univalent in  $U$  and if  $p$  is analytic in  $U$  and satisfies the first order differential superordination

$$h(z) \prec \phi(p(z), zp'(z), z^2 p''(z); z) \quad (z \in U), \quad (4)$$

then  $p$  is called a solution of the differential superordination (4).

The univalent function  $q$  is called a subordinant solutions of (4) if  $q \prec p$  for all  $p$  satisfying (4). A subordinant  $\tilde{q}$  that satisfies  $q \prec \tilde{q}$  for all subordinant  $q$  of (4) is said to be the best subordinant. (see the monograph by Miller and Mocanu [14], and [15]).

Recently, Miller and Mocanu [15] obtained sufficient conditions on the functions  $h, q$  and  $\phi$  for which the following implication holds:

$$h(z) \prec \phi(p(z), zp'(z), z^2 p''(z); z) \rightarrow q(z) \prec p(z)$$

Using these results, the second author considered certain classes of first-order differential subordinations [7], as well as superordination-preserving integral operators [6]. Ali et al. [1], using the results from [7], obtained sufficient conditions for certain normalized analytic functions  $f$  to satisfy

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z), \quad (5)$$

where  $q_1$  and  $q_2$  are given univalent normalized functions in  $U$ .

Very recently, Shanmugam et al. [20–22] obtained the such called sandwich results for certain classes of analytic functions. Further subordination results can be found in [16, 23, 24 and 28].

Let  $\alpha_1, A_1, \dots, \alpha_q, A_q$  and  $\beta_1, B_1, \dots, \beta_s, B_s$  ( $q, s \in \mathbb{N} = \{1, 2, \dots\}$ ) be positive real parameters such that

$$1 + \sum_{k=1}^s B_k - \sum_{k=1}^q A_k > 0. \quad (6)$$

The Wright generalized hypergeometric function (see [25], [26] and [27])

$${}_q\Psi_s \left[ (\alpha_1, A_1, \dots, \alpha_q, A_q); (\beta_1, B_1, \dots, \beta_s, B_s); z \right] = {}_q\Psi_s \left[ (\alpha_n, A_n)_{1,q}; (\beta_n, B_n)_{1,s}; z \right] \text{ is defined by}$$

$${}_q\Psi_s \left[ (\alpha_n, A_n)_{1,q}; (\beta_n, B_n)_{1,s}; z \right] = \sum_{k=0}^{\infty} \left\{ \prod_{n=1}^q \Gamma(\alpha_n + kA_n) \right\} \left\{ \prod_{n=1}^s \Gamma(\beta_n + kB_n) \right\}^{-1} \frac{z^k}{k!} \quad (z \in U). \quad (1.7)$$

If  $A_i = 1$  ( $i = 1, \dots, q$ ) and  $B_j = 1$  ( $j = 1, \dots, s$ ) we have

$$\Omega_q \Psi_s \left[ (\alpha_n, 1)_{1,q}; (\beta_n, 1)_{1,s}; z \right] = {}_qF_s (\alpha_1, \dots, \alpha_q, \beta_1, \dots, \beta_s, z), \quad (8)$$

which is the generalized hypergeometric function where

$$\Omega = \left( \prod_{n=1}^q \Gamma(\alpha_n) \right)^{-1} \left( \prod_{n=1}^s \Gamma(\beta_n) \right). \quad (9)$$

Let

$$\begin{aligned}
 \theta_{p,q,s} [\alpha_1, \beta_1; A_1, B_1; z] &= \Omega z^p {}_q\Psi_s \left[ (\alpha_n, A_n)_{1,q}; (\beta_n, B_n)_{1,s}; z \right] \\
 &= z^p + \sum_{k=1}^{\infty} \frac{\prod_{n=1}^s \Gamma(\beta_n) \prod_{n=1}^q \Gamma(\alpha_n + kA_n)}{\prod_{n=1}^q \Gamma(\alpha_n) \prod_{n=1}^s \Gamma(\beta_n + kB_n) k!} z^{k+p}. \quad (10)
 \end{aligned}$$

Using the Wright hypergeometric function, we introduce the following linear operator

$$\phi_{p,q,s}^{m,l,\lambda} [\alpha_1, \beta_1; A_1, B_1] f : A_p \rightarrow A_p$$

which is defined by the following convolution

$$\phi_{p,q,s}^{0,l,\lambda} [\alpha_1, \beta_1; A_1, B_1] f (z) = f (z) * \theta_{p,q,s} [\alpha_1, \beta_1; A_1, B_1; z];$$

$$\begin{aligned}
 \phi_{p,q,s}^{1,l,\lambda} [\alpha_1, \beta_1; A_1, B_1] f (z) &= (1-\lambda) (f (z) * \theta_{p,q,s} [\alpha_1, \beta_1; A_1, B_1; z]) \\
 &\quad + \frac{\lambda}{(p+l)^{l-1}} \left( z^l f (z) * \theta_{p,q,s} [\alpha_1, \beta_1; A_1, B_1; z] \right)';
 \end{aligned}$$

and

$$\phi_{p,q,s}^{m,l,\lambda} [\alpha_1, \beta_1; A_1, B_1] f (z) = \phi_{p,q,s}^{m,l,\lambda} \left( \phi_{p,q,s}^{m-1,l,\lambda} [\alpha_1, \beta_1; A_1, B_1] f (z) \right). \quad (11)$$

If  $f \in A_p$ , then from (1) and (11), we can easily see that

$$\phi_{p,q,s}^{m,l,\lambda} [\alpha_1, \beta_1; A_1, B_1] f (z) = z^p + \sum_{k=p+1}^{\infty} \left[ \frac{p+l+\lambda(k-p)}{p+l} \right]^m \frac{\prod_{n=1}^s \Gamma(\beta_n) \prod_{n=1}^q \Gamma(\alpha_n + (k-p)A_n)}{\prod_{n=1}^q \Gamma(\alpha_n) \prod_{n=1}^s \Gamma(\beta_n + (k-p)B_n) (k-p)!} a_k z^k, \quad (12)$$

where  $m \in N_0 = N \cup \{0\}$ ,  $l \geq 0$ ,  $\lambda \geq 0$ , and  $p \in N$ .

We note that when  $A_i = 1$  ( $i = 1, \dots, q$ ) and  $B_j = 1$  ( $j = 1, \dots, s$ ), the operator

$$\phi_{p,q,s}^{m,l,\lambda} [\alpha_1, \beta_1; 1, 1] f (z) = L_{p,q,s,\lambda}^{m,l} (\alpha_1, \beta_1) f (z) \text{ was studied by El-Ashwah_ and Aouf [11], also when}$$

$A_i = 1$  ( $i = 1, \dots, q$ ),  $B_j = 1$  ( $j = 1, \dots, s$ ),  $p = 1$ , and  $l = 0$ , the operator

$$\phi_{1,q,s}^{m,0,\lambda} [\alpha_1, \beta_1; 1, 1] f (z) = D_{\lambda}^m (\alpha_1, \beta_1) f (z) \text{ was studied by Selvaraj and Karthikeyan [19], and for}$$

$A_i = 1$  ( $i = 1, \dots, q$ ) and  $B_j = 1$  ( $j = 1, \dots, s$ ), and  $m = 0$ , the operator

$$\phi_{p,q,s}^{0,l,\lambda} [\alpha_1, \beta_1; 1, 1] f (z) = H_p^{p,q} [\alpha_1] f (z) \text{ is the Dziok-Srivastava operator [10]. Moreover by specializing the}$$

parameters  $m, l, \lambda, p, q, s, \alpha_i, A_i$  ( $i = 1, \dots, q$ ) and  $\beta_j, B_j$  ( $j = 1, \dots, s$ ), we obtain various new operators from

the operator  $\phi_{p,q,s}^{m,l,\lambda} [\alpha_1, \beta_1; A_1, B_1] f (z)$  studied by several authors such as Catas [9], Kamali, and Orhan [12], Kumar et al. [13], Salagean [18], Al-Oboudi [2] and others.

It is easily verified from (12) that

$$z \left( \phi_{p,q,s}^{m,l,\lambda} [\alpha_1, \beta_1; A_1, B_1] f(z) \right)' = \frac{\alpha_1}{A_1} \phi_{p,q,s}^{m,l,\lambda} [\alpha_1 + 1, \beta_1; A_1, B_1] f(z) - \left( \frac{\alpha_1}{A_1} - p \right) \phi_{p,q,s}^{m,l,\lambda} [\alpha_1, \beta_1; A_1, B_1] f(z) \quad (A_1 > 0), \quad (13)$$

$$\lambda z \left( \phi_{p,q,s}^{m,l,\lambda} [\alpha_1, \beta_1; A_1, B_1] f(z) \right)' = (p+l) \phi_{p,q,s}^{m+l,\lambda} [\alpha_1, \beta_1; A_1, B_1] f(z) - [p(1-\lambda) + l] \phi_{p,q,s}^{m,l,\lambda} [\alpha_1, \beta_1; A_1, B_1] f(z) \quad (\lambda > 0). \quad (14)$$

To prove our results, we need the following definitions and lemmas.

**Definition 1** ([14]). Denote by  $Q$  the set of all functions  $q(z)$  that are analytic and injective on  $\bar{U} / E(q)$  where

$$E(q) = \{ \zeta \in \partial U : \lim_{z \rightarrow \zeta} q(z) = \infty \},$$

and are such that  $q'(\zeta) \neq 0$  for  $\zeta \in \partial U / E(q)$ . Further let the subclass of  $Q$  for which  $q(0) = a$  be denoted by  $Q(a)$ ,  $Q(0) \equiv Q_0$  and  $Q(1) \equiv Q_1$ .

**Lemma 1** ([14]). Let  $q(z)$  be univalent function in the unit disc  $U$  and let  $\theta$  and  $\phi$  be analytic in a domain  $D$  containing  $q(U)$  with  $\phi(w) \neq 0$  when  $w \in q(U)$ . Set  $q(z) = zq'(z)\phi(q(z))$ ,  $h(z) = \theta(q(z)) + Q(z)$  and suppose that

- i)  $Q$  is a starlike function in  $U$ ,
- ii)  $\operatorname{Re} z h'(z) / Q(z) > 0, z \in U$ .

If  $p$  is analytic in  $U$  with  $p(0) = q(0)$ ,  $p(U) \subseteq D$  and

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)), \quad (15)$$

then  $p(z) \prec q(z)$ , and  $q$  is the best dominant of (15).

**Lemma 2** ([21]). Let  $q(z)$  be a convex univalent function in  $U$  and let  $\alpha \in \mathbb{C}$ ,  $\eta \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$  with

$$\Re \left\{ 1 + \frac{z q''(z)}{q'(z)} \right\} > \max \left\{ 0, -\Re \left( \frac{\sigma}{\eta} \right) \right\}.$$

If the function  $g(z)$  is analytic in  $U$  and

$$\sigma g(z) + \eta z g'(z) \prec \sigma q'(z) + \eta z q'(z),$$

then  $g(z) \prec q(z)$  and  $q(z)$  is the best dominant.

**Lemma 3** ([8]). Let  $q(z)$  be univalent function in the unit disc  $U$  and let  $\theta$  and  $\phi$  be analytic in a domain  $D$  containing  $q(U)$ . Suppose that

- i)  $\operatorname{Re} \theta(q(z)) / \phi(q(z)) > 0, z \in U$ ,
- ii)  $h(z) = zq'(z)\phi(q(z))$  is starlike in  $U$ .

If  $p \in H[q(0), 1] \cap Q$  with  $p(U) \subseteq D$ ,  $\theta(p(z)) + zp'(z)\phi(p(z))$  is univalent  $U$ , and

$$\theta(q(z)) + zq'(z)\phi(q(z)) \prec \theta(p(z)) + zp'(z)\phi(p(z)), \quad (16)$$

then  $q(z) \prec p(z)$ , and  $q$  is the best dominant of (16).

**Lemma 4** ([15]). Let  $q(z)$  be convex function in  $U$  and let  $\gamma \in \mathbb{C}$ , with  $\operatorname{Re} \gamma > 0$ . If  $p \in H[q(0), 1] \cap Q$  and  $p(z) + \gamma zp'(z)$  is univalent in  $U$ , then

$$q(z) + \gamma zq'(z) \prec p(z) + \gamma zp'(z),$$

implies  $q(z) \prec p(z)$ , and  $q$  is the best dominant.

**Lemma 5** ([17]). The function  $q(z) = (1-z)^{-2ab}$  is univalent in  $U$  if and only if

$$|2ab - 1| \leq 1 \text{ or } |2ab + 1| \leq 1.$$

Unless otherwise mentioned, we assume throughout the following sections that  $\alpha_1, A_1, \dots, \alpha_q, A_q$  and  $\beta_1, B_1, \dots, \beta_s, B_s$  ( $q, s \in \mathbb{N} = \{1, 2, \dots\}$ ) are positive real parameters such that  $1 + \sum_{k=1}^s B_k - \sum_{k=1}^q A_k > 0$ ,  $m \in \mathbb{N}_0$ ,  $l \geq 0$ , and  $\lambda > 0$ .

## 2. Subordination results for analytic functions.

**Theorem 1** Let  $\alpha \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$  and  $q(z)$  be a univalent function in  $U$ , with  $q(0) = 1$ , and suppose that

$$\Re \left\{ 1 + \frac{zq''(z)}{q'(z)} \right\} > \max \left\{ 0; -\frac{p\alpha_1}{A_1} \operatorname{Re} \frac{1}{\alpha} \right\}, \quad (z \in U; p \in \mathbb{N}) \quad (17)$$

If  $f \in A_p$  satisfies the subordination

$$\frac{\alpha}{p} \left( \frac{\phi_{p,q,s}^{m,l,\lambda} [\alpha_1 + 1, \beta_1; A_1, B_1] f(z)}{z^p} \right) + \frac{p - \alpha}{p} \left( \frac{\phi_{p,q,s}^{m,l,\lambda} [\alpha_1, \beta_1; A_1, B_1] f(z)}{z^p} \right) \prec q(z) + \frac{\alpha A_1 z q'(z)}{p\alpha_1}, \quad (18)$$

then

$$\frac{\phi_{p,q,s}^{m,l,\lambda} [\alpha_1, \beta_1; A_1, B_1] f(z)}{z^p} \prec q(z),$$

and the function  $q$  is the best dominant of (18).

**Proof.** If we consider the analytic function

$$h(z) = \frac{\phi_{p,q,s}^{m,l,\lambda} [\alpha_1, \beta_1; A_1, B_1] f(z)}{z^p},$$

by differentiating logarithmically with respect to  $z$ , we deduce that

$$\frac{zh'(z)}{h(z)} = \frac{z \left( \phi_{p,q,s}^{m,l,\lambda} [\alpha_1, \beta_1; A_1, B_1] f(z) \right)'}{\phi_{p,q,s}^{m,l,\lambda} [\alpha_1, \beta_1; A_1, B_1] f(z)} - p. \quad (19)$$

From (19), by using the identity (13), a simple computation shows that

$$\frac{\alpha}{p} \left( \frac{\phi_{p,q,s}^{m,l,\lambda} [\alpha_1 + 1, \beta_1; A_1, B_1] f(z)}{z^p} \right) + \frac{p - \alpha}{p} \left( \frac{\phi_{p,q,s}^{m,l,\lambda} [\alpha_1, \beta_1; A_1, B_1] f(z)}{z^p} \right) = h(z) + \frac{\alpha A_1 zh'(z)}{p\alpha_1}$$

hence the subordination (18) is equivalent to

$$h(z) + \frac{\alpha A_1 zh'(z)}{p\alpha_1} \prec q(z) + \frac{\alpha A_1 z q'(z)}{p\alpha_1}.$$

Combining the last relation together with Lemma 2 for the special case  $\eta = \alpha A_1 / p\alpha_1$  and  $\sigma = 1$ , we obtain our result.

Taking  $q(z) = \frac{1 + Az}{1 + Bz}$  in Theorem 1, where  $-1 \leq B < A \leq 1$ , the condition (17) becomes

$$\Re \left\{ \frac{1 - Bz}{1 + Bz} \right\} > \max \left\{ 0; -\frac{p\alpha_1}{A_1} \operatorname{Re} \frac{1}{\alpha} \right\}, \quad z \in U. \quad (20)$$

It is easy to check that the function  $\phi(\zeta) = \frac{1 - \zeta}{1 + \zeta}$ ,  $|\zeta| < |B|$ , is convex in  $U$  and since

$\phi(\bar{\zeta}) = \overline{\phi(\zeta)}$  for all  $|\zeta| < |B|$ , it follows that the image  $\phi(U)$  is a convex domain symmetric with respect to the real axis, hence

$$\inf \left\{ \Re \left( \frac{1 - Bz}{1 + Bz} \right); z \in U \right\} = \frac{1 - |B|}{1 + |B|} > 0. \quad (21)$$



Then, the inequality (20) is equivalent to

$$\frac{p\alpha_1}{A_1} \operatorname{Re} \frac{1}{\alpha} \geq \frac{1-|B|}{1+|B|},$$

hence we obtain the following result:

**Corollary 1** Let  $\alpha \in \mathbb{C}^*$  and  $-1 \leq B < A \leq 1$  with

$$\max \left\{ 0; -\frac{p\alpha_1}{A_1} \operatorname{Re} \frac{1}{\alpha} \right\} \leq \frac{1-|B|}{1+|B|}.$$

If  $f \in A_p$  satisfies the subordination

$$\begin{aligned} \frac{\alpha}{p} \left( \frac{\phi_{p,q,s}^{m,l,\lambda} [\alpha_1 + 1, \beta_1; A_1, B_1] f(z)}{z^p} \right) + \frac{p-\alpha}{p} \left( \frac{\phi_{p,q,s}^{m,l,\lambda} [\alpha_1, \beta_1; A_1, B_1] f(z)}{z^p} \right) \\ \prec \frac{1+Az}{1+Bz} + \frac{A_1\alpha(A-B)z}{p\alpha_1(1+Bz)^2}, \end{aligned} \quad (22)$$

then

$$\frac{\phi_{p,q,s}^{m,l,\lambda} [\alpha_1, \beta_1; A_1, B_1] f(z)}{z^p} \prec \frac{1+Az}{1+Bz},$$

and the function  $1+Az/1+Bz$  is the best dominant of (22).

For  $p=1$ ,  $A=1$  and  $B=-1$ , the above corollary reduces to:

**Corollary 2** Let  $\alpha \in \mathbb{C}^*$  such that

$$\frac{\alpha_1}{A_1} \operatorname{Re} \frac{1}{\alpha} \geq 0.$$

If  $f \in A_p$  satisfies the subordination

$$\begin{aligned} \alpha \left( \frac{\phi_{p,q,s}^{m,l,\lambda} [\alpha_1 + 1, \beta_1; A_1, B_1] f(z)}{z} \right) + (1-\alpha) \left( \frac{\phi_{p,q,s}^{m,l,\lambda} [\alpha_1, \beta_1; A_1, B_1] f(z)}{z} \right) \\ \prec \frac{1+Az}{1+Bz} + \frac{A_1\alpha(A-B)z}{\alpha_1(1+Bz)^2}, \end{aligned} \quad (23)$$

then

$$\frac{\phi_{p,q,s}^{m,l,\lambda} [\alpha_1, \beta_1; A_1, B_1] f(z)}{z} \prec \frac{1+z}{1-z},$$

and the function  $1+z/1-z$  is the best dominant of (23).

**Theorem 2** Let  $q(z)$  be a univalent function in  $U$ , with  $q(0)=1$  and  $q(z) \neq 0$  for all  $z \in U$ , and let  $\delta, \mu \in \mathbb{C}^*$  and  $\nu, \eta \in \mathbb{C}$  with  $\nu + \eta \neq 0$ , and suppose that  $f \in A_p$  and  $q$  satisfy the conditions:

$$\frac{\nu \phi_{p,q,s}^{m,l,\lambda} [\alpha_1 + 1, \beta_1; A_1, B_1] f(z) + \eta \phi_{p,q,s}^{m,l,\lambda} [\alpha_1, \beta_1; A_1, B_1] f(z)}{(\nu + \eta)z^p} \neq 0 \quad z \in U, \quad (24)$$

and

$$\Re \left\{ 1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} \right\} > 0 \quad z \in U. \quad (25)$$

If

$$1 + \delta \mu \left[ \frac{\nu z \left( \phi_{p,q,s}^{m,l,\lambda} [\alpha_1 + 1, \beta_1; A_1, B_1] f(z) \right)' + \eta z \left( \phi_{p,q,s}^{m,l,\lambda} [\alpha_1, \beta_1; A_1, B_1] f(z) \right)'}{\nu \phi_{p,q,s}^{m,l,\lambda} [\alpha_1 + 1, \beta_1; A_1, B_1] f(z) + \eta \phi_{p,q,s}^{m,l,\lambda} [\alpha_1, \beta_1; A_1, B_1] f(z)} - p \right] < 1 + \delta \frac{zq'(z)}{q(z)}, \quad (26)$$

then

$$\left[ \frac{\nu \phi_{p,q,s}^{m,l,\lambda} [\alpha_1 + 1, \beta_1; A_1, B_1] f(z) + \eta \phi_{p,q,s}^{m,l,\lambda} [\alpha_1, \beta_1; A_1, B_1] f(z)}{(\nu + \eta)z^p} \right]^\mu < q(z),$$

and the function  $q$  is the best dominant of (26). (the power is the principal one).

**Proof.** Let

$$h(z) = \left[ \frac{\nu \phi_{p,q,s}^{m,l,\lambda} [\alpha_1 + 1, \beta_1; A_1, B_1] f(z) + \eta \phi_{p,q,s}^{m,l,\lambda} [\alpha_1, \beta_1; A_1, B_1] f(z)}{(\nu + \eta)z^p} \right]^\mu, \quad z \in U. \quad (27)$$

According to (24) the function  $h$  is analytic in  $U$ , differentiating (27) logarithmically with respect to  $z$  we get

$$\frac{zh'(z)}{h(z)} = \mu \left[ \frac{\nu z \left( \phi_{p,q,s}^{m,l,\lambda} [\alpha_1 + 1, \beta_1; A_1, B_1] f(z) \right)' + \eta z \left( \phi_{p,q,s}^{m,l,\lambda} [\alpha_1, \beta_1; A_1, B_1] f(z) \right)'}{\nu \phi_{p,q,s}^{m,l,\lambda} [\alpha_1 + 1, \beta_1; A_1, B_1] f(z) + \eta \phi_{p,q,s}^{m,l,\lambda} [\alpha_1, \beta_1; A_1, B_1] f(z)} - p \right]. \quad (28)$$

In order to prove our result we will use Lemma 1. Considering in this lemma

$$\theta(w) = 1 \text{ and } \phi(w) = \frac{\delta}{w},$$

then  $\theta$  is analytic in  $\mathbb{C}$  and  $\phi(w) \neq 0$  is analytic in  $\mathbb{C}^*$ . Also, if we let

$$Q(z) = zq'(z) = \phi(q(z)) = \delta \frac{zq'(z)}{q(z)},$$

and

$$g(z) = \theta(q(z)) + Q(z) = 1 + \delta \frac{zq'(z)}{q(z)},$$

then, since  $Q(0) = 1$  and  $Q'(0) \neq 0$ , the assumption (2.9) yields that  $Q$  is a starlike function in  $U$ . From (25) we also have

$$\Re \frac{zq'(z)}{Q(z)} = \Re \left\{ 1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} \right\} > 0 \quad z \in U,$$

and then, by using Lemma 1 we deduce that the subordination (26) implies  $h(z) < q(z)$  and the function  $q$  is the best dominant of (26).

Taking  $\nu = 0$ ,  $\eta = \delta = 1$  and  $q(z) = \frac{1 + Az}{1 + Bz}$  in Theorem 2, it is easy to check that the assumption (25) holds whenever  $-1 \leq B < A \leq 1$ , hence we obtain the next results.

**Corollary 3** Let  $-1 \leq B < A \leq 1$  and  $\mu \in \mathbb{C}^*$ . Let  $f \in A_p$  and suppose that

$$\frac{\phi_{p,q,s}^{m,l,\lambda} [\alpha_1, \beta_1; A_1, B_1] f(z)}{z^p} \neq 0 \quad z \in U.$$

If

$$1 + \mu \left[ \frac{z \left( \phi_{p,q,s}^{m,l,\lambda} [\alpha_1, \beta_1; A_1, B_1] f(z) \right)'}{\phi_{p,q,s}^{m,l,\lambda} [\alpha_1, \beta_1; A_1, B_1] f(z)} - p \right] < 1 + \frac{(A - B)z}{(1 + Az)(1 + Bz)}, \quad (29)$$

then

$$\left[ \frac{\phi_{p,q,s}^{m,l,\lambda} [\alpha_1, \beta_1; A_1, B_1] f(z)}{z^p} \right]^\mu \prec \frac{1+Az}{1+Bz},$$

and the function  $1+Az/1+Bz$  is the best dominant of (29). (the power is the principal one).

### Remarks

- 1) Putting  $A_i = 1$  ( $i = 1, \dots, q$ ) and  $B_j = 1$  ( $j = 1, \dots, s$ ) in Theorem 2 we obtain the corresponding result due to El-Ashwah and Aouf [11, Theorem 2].
- 2) Putting  $\nu = 0$ ,  $\eta = p = 1$ ,  $m = 0$ ,  $q = s + 1$ ,  $\alpha_i = A_i = 1$  ( $i = 1, \dots, s + 1$ ),  $\beta_j = B_j = 1$  ( $j = 1, \dots, s$ ),  $\delta = 1/ab$  ( $a, b \in \mathbb{C}^*$ ),  $\mu = a$ , and  $q(z) = (1-z)^{-2ab}$  in Theorem 2, then combining this together with Lemma 5 we obtain the corresponding result due to Obradović et al. [16, Theorem 1], see also Aouf and Bulboacă [4, Corollary 3.3].
- 3) For  $\nu = 0$ ,  $\eta = p = 1$ ,  $m = 0$ ,  $q = s + 1$ ,  $\alpha_i = A_i = 1$  ( $i = 1, \dots, s + 1$ ),  $\beta_j = B_j = 1$  ( $j = 1, \dots, s$ ),  $\delta = 1/b$  ( $b \in \mathbb{C}^*$ ),  $\mu = 1$  and  $q(z) = (1-z)^{-2b}$ , Theorem 2 reduces to the recent result of Srivastava and Lashin [24].
- 4) Putting  $\nu = 0$ ,  $\eta = p = \delta = 1$ ,  $m = 0$ ,  $q = s + 1$ ,  $\alpha_i = A_i = 1$  ( $i = 1, \dots, s + 1$ ),  $\beta_j = B_j = 1$  ( $j = 1, \dots, s$ ) and  $q(z) = (1+Bz)^{\mu(A-B)/B}$  ( $-1 \leq A < B \leq 1$ ,  $B \neq 0$ ) in Theorem 2, and using Lemma 5 we get the corresponding result due to Aouf and Bulboacă [4, Corollary 3.4].
- 5) Putting  $\nu = 0$ ,  $\eta = p = 1$ ,  $m = 0$ ,  $q = s + 1$ ,  $\alpha_i = A_i = 1$  ( $i = 1, \dots, s + 1$ ),  $\beta_j = B_j = 1$  ( $j = 1, \dots, s$ ),  $\delta = e^{i\lambda}/ab \cos \gamma$  ( $a, b \in \mathbb{C}^*$ ;  $|\gamma| < \pi/2$ ),  $\mu = a$  and  $q(z) = (1-z)^{-2a \cos \gamma e^{-i\gamma}}$  in Theorem 2, we obtain the corresponding result due to Aouf et al. [5, Theorem 1], see also Aouf and Bulboacă [4, Corollary 3.5].

**Theorem 3** Let  $q(z)$  be a univalent function in  $U$ , with  $q(0) = 1$ , and Let  $\gamma, \mu \in \mathbb{C}^*$  and  $\nu, \eta, \delta, \Omega \in \mathbb{C}$  with  $\nu + \eta \neq 0$ , and suppose that  $f \in A_p$  and  $q$  satisfy the conditions:

$$\frac{\nu \phi_{p,q,s}^{m,l,\lambda} [\alpha_1 + 1, \beta_1; A_1, B_1] f(z) + \eta \phi_{p,q,s}^{m,l,\lambda} [\alpha_1, \beta_1; A_1, B_1] f(z)}{(\nu + \eta) z^p} \neq 0 \quad z \in U, \quad (30)$$

and

$$\Re \left\{ 1 + \frac{z q''(z)}{q'(z)} \right\} > \max \left\{ 0; -\operatorname{Re} \frac{\delta}{\gamma} \right\}, \quad z \in U, \quad (31)$$

if

$$\psi(z) = \left[ \frac{\nu \phi_{p,q,s}^{m,l,\lambda} [\alpha_1 + 1, \beta_1; A_1, B_1] f(z) + \eta \phi_{p,q,s}^{m,l,\lambda} [\alpha_1, \beta_1; A_1, B_1] f(z)}{(\nu + \eta) z^p} \right]^\mu$$

$$\times \left[ \delta + \mu \gamma \left( \frac{\nu z \left( \phi_{p,q,s}^{m,l,\lambda} [\alpha_1 + 1, \beta_1; A_1, B_1] f(z) \right)' + \eta z \left( \phi_{p,q,s}^{m,l,\lambda} [\alpha_1, \beta_1; A_1, B_1] f(z) \right)'}{\nu \phi_{p,q,s}^{m,l,\lambda} [\alpha_1 + 1, \beta_1; A_1, B_1] f(z) + \eta \phi_{p,q,s}^{m,l,\lambda} [\alpha_1, \beta_1; A_1, B_1] f(z)} - p \right) \right] + \Omega, \quad (32)$$

and

$$\psi(z) \prec \delta q(z) + \gamma z q'(z) + \Omega, \quad (33)$$

then

$$\left[ \frac{\nu \phi_{p,q,s}^{m,l,\lambda} [\alpha_1 + 1, \beta_1; A_1, B_1] f(z) + \eta \phi_{p,q,s}^{m,l,\lambda} [\alpha_1, \beta_1; A_1, B_1] f(z)}{(\nu + \eta) z^p} \right]^\mu \prec q(z),$$

and the function  $q$  is the best dominant of (33) (all the power are the principal ones).

**Proof.** Let  $h(z)$  be defined by (27), then we have from (28)



$$zh'(z) = \mu h(z) \left[ \frac{\nu z \left( \phi_{p,q,s}^{m,l,\lambda} [\alpha_1 + 1, \beta_1; A_1, B_1] f(z) \right)' + \eta z \left( \phi_{p,q,s}^{m,l,\lambda} [\alpha_1, \beta_1; A_1, B_1] f(z) \right)'}{\nu \phi_{p,q,s}^{m,l,\lambda} [\alpha_1 + 1, \beta_1; A_1, B_1] f(z) + \eta \phi_{p,q,s}^{m,l,\lambda} [\alpha_1, \beta_1; A_1, B_1] f(z)} - p \right].$$

Let us consider the following functions:

$$\theta(w) = \delta w + \Omega, \text{ and } \phi(w) = \gamma, w \in \mathbb{C},$$

$$Q(z) = zq'(z) = \phi(q(z)) = \gamma \frac{zq'(z)}{q(z)}, z \in U,$$

and

$$g(z) = \theta(q(z)) + Q(z) = \delta q(z) + \gamma zq'(z) + \Omega, z \in U.,$$

From the assumption we see that  $Q$  is starlike in  $U$  and, that

$$\Re \frac{zg'(z)}{Q(z)} = \Re \left\{ \frac{\delta}{\gamma} + 1 + \frac{zq''(z)}{q'(z)} \right\} > 0, z \in U,$$

thus, by applying Lemma 1 the proof is completed.

Taking  $q(z) = \frac{1+Az}{1+Bz}$  in Theorem 3, where  $-1 \leq B < A \leq 1$ , and according to (2.5), the condition (2.15) becomes

$$\max \left\{ 0; -\Re \frac{\delta}{\gamma} \right\} \leq \frac{1-|B|}{1+|B|}.$$

Hence, for the special case  $\nu = \gamma = 0, \eta = 0$ , we obtain the following result:

**Corollary 4** Let  $-1 \leq B < A \leq 1, \mu \in \mathbb{C}^*$  and  $\delta \in \mathbb{C}$  with

$$\max \left\{ 0; -\Re(\delta) \right\} \leq \frac{1-|B|}{1+|B|}.$$

Let  $f \in A_p$  and suppose that

$$\frac{\phi_{p,q,s}^{m,l,\lambda} [\alpha_1, \beta_1; A_1, B_1] f(z)}{z^p} \neq 0, z \in U,$$

and

$$\left( \frac{\phi_{p,q,s}^{m,l,\lambda} [\alpha_1 + 1, \beta_1; A_1, B_1] f(z)}{z^p} \right)^\mu \left[ \delta + \mu \left( \frac{z \left( \phi_{p,q,s}^{m,l,\lambda} [\alpha_1, \beta_1; A_1, B_1] f(z) \right)'}{\phi_{p,q,s}^{m,l,\lambda} [\alpha_1, \beta_1; A_1, B_1] f(z)} - p \right) \right] + \Omega < \delta \frac{1+Az}{1+Bz} + \Omega + \frac{(A-B)z}{(1+Bz)^2}, \quad (34)$$

then

$$\left[ \frac{\phi_{p,q,s}^{m,l,\lambda} [\alpha_1, \beta_1; A_1, B_1] f(z)}{z^p} \right]^\mu < \frac{1+Az}{1+Bz},$$

and the function  $1+Az/1+Bz$  is the best dominant of (32) (all the powers are the principal ones).

**Remark:** Taking  $\nu = 0, \eta = \gamma = p = 1, \alpha = \beta = 0$  and  $q(z) = \frac{1+z}{1-z}$  in Theorem 3 we obtain the corresponding result due to Aouf and Bulboacă [4, Corollary 3.7].

### 3 Superordination and sandwich results

**Theorem 4.** Let  $q(z)$  be convex function in  $U$  with  $q(0) = 1$  and let  $\alpha \in \mathbb{C}^*$  with  $\frac{\alpha_1}{A_1} \operatorname{Re} \alpha > 0$ . Let  $f \in A_p$  and

suppose that  $\phi_{p,q,s}^{m,l,\lambda} [\alpha_1, \beta_1; A_1, B_1] f(z) / z^p \in H[q(0), 1] \cap \mathcal{Q}$ . If the function

$$\frac{\alpha}{p} \left( \frac{\phi_{p,q,s}^{m,l,\lambda} [\alpha_1 + 1, \beta_1; A_1, B_1] f(z)}{z^p} \right) + \frac{p - \alpha}{p} \left( \frac{\phi_{p,q,s}^{m,l,\lambda} [\alpha_1, \beta_1; A_1, B_1] f(z)}{z^p} \right),$$

is univalent in  $U$ , and

$$q(z) + \frac{\alpha A_1 z q'(z)}{p \alpha_1} \prec \frac{\alpha}{p} \left( \frac{\phi_{p,q,s}^{m,l,\lambda} [\alpha_1 + 1, \beta_1; A_1, B_1] f(z)}{z^p} \right) + \frac{p - \alpha}{p} \left( \frac{\phi_{p,q,s}^{m,l,\lambda} [\alpha_1, \beta_1; A_1, B_1] f(z)}{z^p} \right), \quad (35)$$

then

$$q(z) \prec \frac{\phi_{p,q,s}^{m,l,\lambda} [\alpha_1, \beta_1; A_1, B_1] f(z)}{z^p},$$

and  $q$  is the best subordinate of (34).

**Proof.** Let us define the function  $g$  by

$$g(z) = \frac{\phi_{p,q,s}^{m,l,\lambda} [\alpha_1, \beta_1; A_1, B_1] f(z)}{z^p}, \quad z \in U.$$

From the assumption of the theorem, the function  $g$  is analytic in  $U$ , by differentiating logarithmically with respect to  $z$  the function  $g$ , we deduce that

$$\frac{z g'(z)}{g(z)} = \frac{z \left( \phi_{p,q,s}^{m,l,\lambda} [\alpha_1, \beta_1; A_1, B_1] f(z) \right)'}{\phi_{p,q,s}^{m,l,\lambda} [\alpha_1, \beta_1; A_1, B_1] f(z)} - p. \quad (36)$$

After some computations, and using the identity (1.13), from (3.2) we get

$$g(z) + \frac{\alpha A_1 z g'(z)}{\alpha_1 p (p - \beta)} = \frac{\alpha}{p} \left( \frac{\phi_{p,q,s}^{m,l,\lambda} [\alpha_1 + 1, \beta_1; A_1, B_1] f(z)}{z^p} \right) + \frac{p - \alpha}{p} \left( \frac{\phi_{p,q,s}^{m,l,\lambda} [\alpha_1, \beta_1; A_1, B_1] f(z)}{z^p} \right),$$

and now, by using Lemma 4 we get the desired result.

Taking  $q(z) = \frac{1 + Az}{1 + Bz}$  in Theorem 4, where  $-1 \leq B < A \leq 1$ , hence we obtain the next results.

**Corollary 5** Let  $-1 \leq B < A \leq 1$  and  $f \in A_p$ . Suppose that  $\phi_{p,q,s}^{m,l,\lambda} [\alpha_1, \beta_1; A_1, B_1] f(z) / z^p \in H[q(0), 1]$ . If the function

$$\frac{\alpha}{p} \left( \frac{\phi_{p,q,s}^{m,l,\lambda} [\alpha_1 + 1, \beta_1; A_1, B_1] f(z)}{z^p} \right) + \frac{p - \alpha}{p} \left( \frac{\phi_{p,q,s}^{m,l,\lambda} [\alpha_1, \beta_1; A_1, B_1] f(z)}{z^p} \right),$$

is univalent in  $U$ , and

$$\frac{1 + Az}{1 + Bz} + \frac{\alpha A_1 (A - B) z}{p \alpha_1 (1 + Bz)^2} \prec \frac{\alpha}{p} \left( \frac{\phi_{p,q,s}^{m,l,\lambda} [\alpha_1 + 1, \beta_1; A_1, B_1] f(z)}{z^p} \right) + \frac{p - \alpha}{p} \left( \frac{\phi_{p,q,s}^{m,l,\lambda} [\alpha_1, \beta_1; A_1, B_1] f(z)}{z^p} \right), \quad (37)$$

then

$$\frac{1 + Az}{1 + Bz} \prec \frac{\phi_{p,q,s}^{m,l,\lambda} [\alpha_1, \beta_1; A_1, B_1] f(z)}{z^p},$$

and  $1 + Az / 1 + Bz$  is the best subordinate of (37).

Using arguments similar to those of the proof of Theorem 3, and then by applying Lemma 3 we obtain the following result.

**Theorem 5** Let  $q(z)$  be convex function in  $U$ , with  $q(0) = 1$ . Let  $\gamma, \mu \in \mathbb{C}^*$  and  $\nu, \eta, \delta, \Omega \in \mathbb{C}$  with  $\nu + \eta \neq 0$   $\text{Re}(\delta/\gamma) > 0$ . Let  $f \in A_p$  and suppose that  $f$  satisfies the conditions:

$$\frac{\nu \phi_{p,q,s}^{m,l,\lambda} [\alpha_1 + 1, \beta_1; A_1, B_1] f(z) + \eta \phi_{p,q,s}^{m,l,\lambda} [\alpha_1, \beta_1; A_1, B_1] f(z)}{(\nu + \eta) z^p} \neq 0, \quad z \in U,$$

and

$$\left[ \frac{\nu \phi_{p,q,s}^{m,l,\lambda} [\alpha_1, \beta_1; A_1, B_1] f(z) + \eta \phi_{p,q,s}^{m,l,\lambda} [\alpha_1, \beta_1; A_1, B_1] f(z)}{(\nu + \eta) z^p} \right]^\mu \in H[q(0), 1] \cap \mathcal{Q}$$

If the function  $\psi$  given by (2.16) is univalent in  $U$ , and

$$\delta q(z) + \gamma z q'(z) + \Omega \prec \psi(z), \quad (38)$$

then

$$q(z) \prec \left[ \frac{\nu \phi_{p,q,s}^{m,l,\lambda} [\alpha_1 + 1, \beta_1; A_1, B_1] f(z) + \eta \phi_{p,q,s}^{m,l,\lambda} [\alpha_1, \beta_1; A_1, B_1] f(z)}{(\nu + \eta) z^p} \right]^\mu,$$

and the function  $q$  is the best subordinate of (38). (all the power are the principal ones).

Combining Theorem 1 with Theorem 4 and Theorem 3 with Theorem 5, we obtain, respectively, the following two sandwich results:

**Theorem 6** Let  $q_1$  and  $q_2$  be two convex function in  $U$ , with  $q_1(0) = q_2(0) = 1$ . Let  $\alpha \in \mathbb{C}^*$  with  $\frac{\alpha_1}{A_1} \text{Re} \alpha > 0$ .

Let  $f \in A_p$  and suppose that  $\phi_{p,q,s}^{m,l,\lambda} [\alpha_1, \beta_1; A_1, B_1] f(z) / z^p \in H[q(0), 1] \cap \mathcal{Q}$ . If the function

$$\frac{\alpha \left( \frac{\phi_{p,q,s}^{m,l,\lambda} [\alpha_1 + 1, \beta_1; A_1, B_1] f(z)}{z^p} \right) + \frac{p - \alpha \left( \frac{\phi_{p,q,s}^{m,l,\lambda} [\alpha_1, \beta_1; A_1, B_1] f(z)}{z^p} \right)}{p},$$

is univalent in  $U$ , and

$$q_1(z) + \frac{\alpha A_1 z q_1'(z)}{p \alpha_1} \prec \frac{\alpha \left( \frac{\phi_{p,q,s}^{m,l,\lambda} [\alpha_1 + 1, \beta_1; A_1, B_1] f(z)}{z^p} \right) + \frac{p - \alpha \left( \frac{\phi_{p,q,s}^{m,l,\lambda} [\alpha_1, \beta_1; A_1, B_1] f(z)}{z^p} \right)}{p} \prec q_2(z) + \frac{\alpha A_1 z q_2'(z)}{p \alpha_1}, \quad (39)$$

then

$$q_1(z) \prec \frac{\phi_{p,q,s}^{m,l,\lambda} [\alpha_1, \beta_1; A_1, B_1] f(z)}{z^p} \prec q_2(z),$$

and  $q_1$  and  $q_2$  are, respectively, the best subordinate and the best dominant of (39).

**Theorem 7** Let  $q_1$  and  $q_2$  be two convex function in  $U$ , with  $q_1(0) = q_2(0) = 1$ . Let  $\gamma, \mu \in \mathbb{C}^*$  and  $\nu, \eta, \delta, \Omega \in \mathbb{C}$  with  $\nu + \eta \neq 0$   $\text{Re}(\delta/\gamma) > 0$ . Let  $f \in A_p$  and suppose that  $f$  satisfies the conditions:

$$\frac{\nu \phi_{p,q,s}^{m,l,\lambda} [\alpha_1 + 1, \beta_1; A_1, B_1] f(z) + \eta \phi_{p,q,s}^{m,l,\lambda} [\alpha_1, \beta_1; A_1, B_1] f(z)}{(\nu + \eta) z^p} \neq 0, \quad z \in U,$$

and

$$\left[ \frac{\nu \phi_{p,q,s}^{m,l,\lambda} [\alpha_1 + 1, \beta_1; A_1, B_1] f(z) + \eta \phi_{p,q,s}^{m,l,\lambda} [\alpha_1, \beta_1; A_1, B_1] f(z)}{(\nu + \eta) z^p} \right]^\mu \in H[q(0), 1] \cap \mathcal{Q}$$

If the function  $\psi$  given by (2.16) is univalent in  $U$ , and

$$\delta q_1(z) + \gamma z q_1'(z) + \Omega \prec \psi(z) \prec \delta q_2(z) + \gamma z q_2'(z) + \Omega, \quad (40)$$

then



$$q_1(z) \prec \left[ \frac{v\phi_{p,q,s}^{m,l,\lambda} [\alpha_1 + 1, \beta_1; A_1, B_1] f(z) + \eta\phi_{p,q,s}^{m,l,\lambda} [\alpha_1, \beta_1; A_1, B_1] f(z)}{(v + \eta)z^p} \right] \prec q_2(z),$$

and  $q_1$  and  $q_2$  are, respectively, the best subordinate and the best dominant of (40).  
(all the power are the principal ones).

## References

- [1] Ali, R. Ravichandran, M. V. Khan, M. H. and Subramaniam, K. G. Subramaniam, Differential sandwich theorems for certain analytic functions, *Far East J. Math. Sci.* 15 (1) (2004) 87–94.
- [2] Al-Oboudi, F. M. On univalent functions defined by a generalized Salagean operator, *Internat. J. Math. Math. Sci.* 27 (2004) 1429-1436.
- [3] Aghalary, R. R. Ali, M. Joshi, S. B. and Ravichandran, V. Ravichandran, Inequalities for analytic functions defined by certain liner operator, *Internat. J. Math. Sci.* 4 (2005), 267-274.
- [4] Aouf, M. K. and Bulboacă, T. Subordination and superordination properties of multivalent functions defined by certain integral operator, *J. Franklin Inst.* 347 (2010) 641-653.
- [5] M. K. Aouf, F. M. Al-Oboudi, M. M. Haidan, On some results for  $\lambda$ -spirallike and  $\lambda$ -Robertson functions of complex order, *Publ. Inst. Math. Belgrade* 77 (91) (2005) 93–98.
- [6] Bulboacă, T., A class of superordination preserving integral operators, *Indag-Math. (New Ser.)* 13 (3) (2002) 301–311.
- [7] Bulboacă, T. Classes of first-order differential subordinations, *Demonstratio Math.* 35 (2) (2002) 287–392.
- [8] Bulboacă, T. *Differential Subordinations and Superordinations. Recent Results*, House of Scientific Book Publ, Cluj-Napoca, 2005.
- [9] Catas, A. On certain classes of p-valent functions defined by multiplier transformations, in: *Proc. Book of the International Symposium on Geometric Function Theory and Applications*, Istanbul, Turkey, August 2007, pp. 241-250.
- [10] Dziok J. and Srivastava, H. M. Classes of analytic functions associated with the generalized hypergeometric function. *Applied Mathematics and Computation*, 103(1999), 1–13.
- [11] El-Ashwah, R. M. and Aouf M. K., Differential subordination and superordination for certain subclasses of p-valent functions, *Math. Comp. Mode.* 51 (2010) 349-360.
- [12] Kamali, M. and Orhan, H. On a subclass of certain starlike functions with negative coefficients, *Bull. Korean Math. Soc.* 41 (1) (2004) 53-71.
- [13] Kumar, S. S. Taneja, H.C. and Ravichandran, V. , Classes multivalent functions defined by Dziok\_Srivastava linear operator and multiplier transformations, *Kyungpook Math. J.* (46) (2006) 97109.
- [14] Miller, S. S. and Mocanu, P. T., *Differential Subordinations: Theory and Applications*, in: *Monographs and Textbooks in Pure and Applied Mathematics*, vol. 225, Marcel Dekker, New York, Basel, 2000.
- [15] Miller, S. S. and Mocanu, P. T., Subordinants of differential superordinations, *Complex Var. Theory Appl.* 48 (10) (2003) 815-826.
- [16] Obradović, M. and Aouf, M. K. S. Owa, On some results for starlike functions of complex order, *Publ. Inst. Math. Belgrade* 46 (60) (1989)79–85.
- [17] Royster, W. C. , On the univalence of a certain integral, *Michigan Math. J.* 12 (1965) 385–387.
- [18] Salagean, G. S., Subclasses of univalent functions, in: *Lecture Notes in Math.*, vol. 1013, Springer-Verlag, 1983, pp. 362-372.
- [19] Selvaraj, C. K. and Karthikeyan, R. Differential subordination and superordinations for certain subclasses of analytic functions, *Far East J. Math. Sci.* 29 (2)(2008) 419-430.
- [20] Shanmugam, T. N. Ravichandran, and Sivasubramanian, V. S., Differential sandwich theorems for some subclasses of analytic functions, *Austral. J. Math. Anal. Appl.* 3 (1) (2006) (e-journal) article 8.



- [21] Shanmugam, T. N., Sivasubramanian, S. and Silverman, H., On sandwich theorems for some classes of analytic functions, *Int. J. Math. Math. Sci.* 2006, Article ID 29684, 1–13.
- [22] Shanmugam, T. N. Ramachandran, C. Darus, M. and Sivasubbramanian, S., Differential sandwich theorems for some subclasses of analytic functions involving a linear operator, *Acta Math. Univ. Comenianae* 74 (2) (2007) 287–294.
- [23] Singh, V. On some criteria for univalence and starlikeness, *Indian J. Pure Appl. Math.* 34 (4) (2003) 569–577.
- [24] Srivastava, H. M. Lashin, A. Y. , Some applications of the Briot–Bouquet differential subordination, *J. Inequalities Pure Appl. Math.* 6 (2) (2005) 1–7 article 41.
- [25] Srivastava, H. M. Gupta, K. C. and Goyal S. P., *The H-Functions of One and Two Variables with Applications*. South Asian Publishers, New Delhi and Madras, 1982.
- [26] Srivastava, H. M. Karlsson P. W., *Multiple Gaussian Hypergeometric Series*. Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1985.
- [27] Srivastava, H. M. and Manocha, H. L., *A Treatise on Generating Functions*. Halsted Press ,Ellis Horwood, Limited, Chichester, 1984.
- [28] Wang, Z. Gao, C. and Liao, M., On certain generalized class of non-Bazilević functions, *Acta Math. Acad. Paed. Nyireg. New Ser.* 21 (2) (2005) 147–154.

