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An elastic plate with a curvilinear hole and flowing heat in S-plane

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Abstract

In the present paper, we apply complex variable method, Cauchy method, to derive exact expressions for Goursat functions for the boundary value problems of an infinite elastic plate weakened by a curvilinear hole. The hole considered is conformally mapped on the area of the right half-plane. Also, when an initial heat is uniformly flowing in the perpendicular direction of the hole, the thermo potential function and the stress components, in this case, are obtained. Some applications are considered, and the work of many previous authors is established as special cases of this work. Also, when the hole is conformally mapped inside and outside the unit circle is established from this work.

Keywords: Goursat functions; conformal mapping; thermo potential function; right half-plane.

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Introduction and Basic Equations

Problems dealing with isotropic homogeneous perforated infinite plate have been investigated and discussed by many authors [1-6]. Some of them used the transformation mapping $z = c\omega(\zeta)$, $c>0$, $\zeta=\xi+i\eta$ to conform the curvilinear hole in the infinite elastic plate into a unit circle γ, $|\zeta|<1$, such that $\omega'(\zeta)\neq 0$ inside the circle. Then, they obtained the Goursat functions by using Laurent's theorem; see [3, 4, 7, 8]. Others used the conformal mapping $z = c\omega(\zeta)$, $c>0$, $\zeta=\xi+i\eta$ to conform the curvilinear hole outside the unit circle γ, $|\zeta|>1$, such that $\omega'(\zeta)\neq 0,\infty$ outside the unit circle. Then, by using the complex variables method, the Goursat functions can be obtained, see [1, 6, 9-12]. In the same way, some authors used the conformal mapping $z = c\omega(s)$, $\mathrm{Re}(s)>0$, $\omega'(s)$ does not vanish on the right half-plane and $\omega(\infty)$ is bounded, to conform the curvilinear hole on the domain of the right half-plane and obtained the Goursat functions, for the first fundamental problems using complex variable methods, see [13-15].

Consider a thin infinite plate of thickness h with a curvilinear hole C, where the region lies inside the hole, conformally mapped into the domain of the right half plane by the rational mapping function, see Figs. (1-4)

$$z = c\,\omega(s) = c\,\frac{(s+1)^2+m\,(s-1)^2}{s^2-1-n\,(s-1)^2} \qquad (\,c>0,|n|<1, s=\sigma+i\,\tau\,), \qquad (1)$$

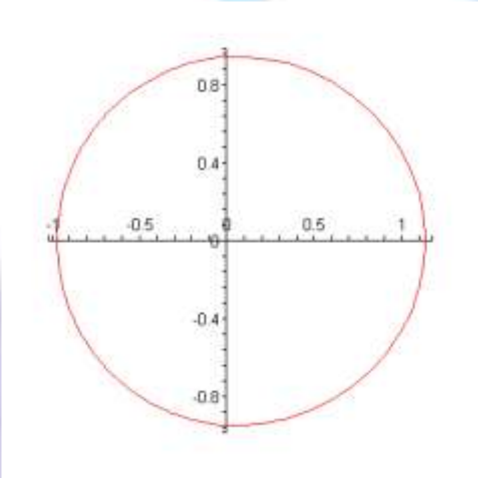

Fig. 1 ***(n= 0.075, m= 0.05)***

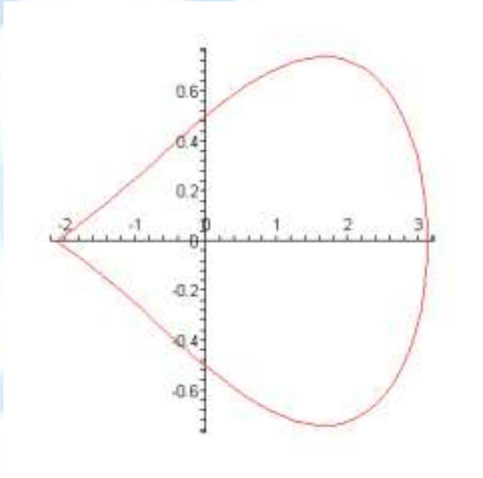

Fig.2 ***(n= 0.2, m= 1.5)***

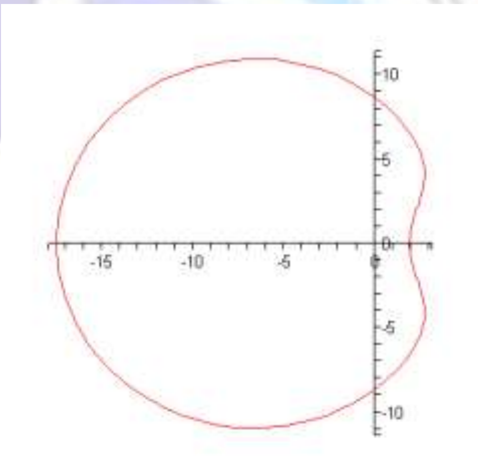

Fig. 3 ***(n= 0.8, m= -4,5)***

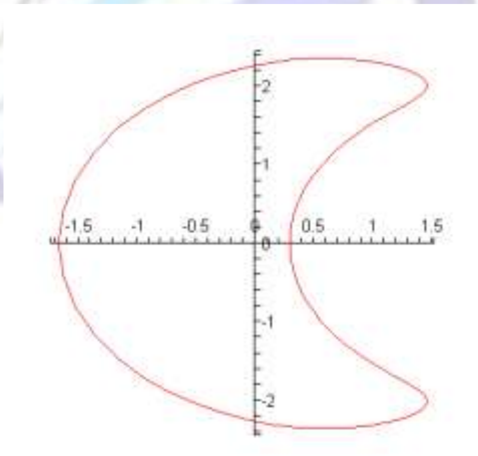

Fig.4 ***(n= 0.7, m= -1.5)***

Here, m, n are real parameters subject to the conditions that $\omega'(s)$ does not vanish on the right half-plane (i.e. $\mathrm{Re}\ s\geq 0$) and $\omega(\infty)$ is bounded.

If a heat, $\Theta = q\, y$, is flowing uniformly in the direction of the negative y-axis, where the increasing temperature Θ is assumed to be constant across the thickness of the plate i.e. $\Theta = \Theta(x, y)$, and q is the constant temperature gradient. Here, we take the x-axis to be the horizontal axis which is perpendicular to the y-axis.

The uniform flow heat is distributed by the pressure p on an insulated curvilinear hole C, and the heat equation satisfies

(i) $\nabla^2\Theta = 0$, $\quad \nabla^2 = \dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2}{\partial \theta^2}$

(2)

(ii) $\dfrac{\partial \Theta}{\partial r} = 0$ on the boundary $r = r_o$ = constant . sin θ

After neglecting the variation of the strain and stress with respect to the thickness of the plate and considering the faces of the plate are free of applied loads, the thermo elastic potential function Φ satisfies the formula, see [4,5]

$$\nabla^2\Phi = (1+\nu)\alpha\, \Theta \ . \tag{3}$$

Here, α is called the coefficient of the thermal expansion and ν is denoted as the Poisson's ratio.

It is known that [4], the first and second boundary value problems, in the plane of thermoelasticity, are equivalent to finding two analytic functions $\phi_1(z)$ and $\psi_1(z)$, $z = x + i\,y$, $i = \sqrt{-1}$. These analytic functions satisfy the boundary conditions

$$K\varphi_1(t) - t\overline{\phi_1^{'}(t)} - \overline{\psi_1(t)} \;=\; f(t), \tag{4}$$

where

$$\phi_1(t) \;=\; -\frac{S_x + iS_y}{2\pi(1+\kappa)}\ln t + c\Gamma t + \phi(t),$$

(5)

$$\psi_1(t) \;=\; \frac{\kappa(S_x - i\,S_y)}{2\pi(1+\chi)}\ln t + c\Gamma^{*} t + \psi(t).$$

Here, for the first boundary value problem, K = -1, $f(t)$ is a given function of stresses, while, for the second boundary value problem, $K = \kappa$, $f(t)$ is a given function of the displacement, the constant κ is called the thermal conductivity of the material and t denoting the affix of a point on the boundary. Also, S_x and S_y are the components of the resultant vectors of all external forces acting on the boundary; Γ and Γ^* are complex constants and represent the stresses at infinity.

The first and second boundary value problems in the presence of thermo elastic potential, respectively take the form

$$\phi(t) - t\overline{\phi'(t)} - \overline{\psi(t)} = \frac{\partial\Phi}{\partial x} + i\frac{\partial\Phi}{\partial y} + \frac{1}{2G}\int_0^s\left[X(s) - Y(s)\right]ds\,, \tag{6}$$

$$\kappa\phi(t) - t\overline{\phi'(t)} - \overline{\psi(t)} = u + i\,v - \frac{\partial\Phi}{\partial x} - i\frac{\partial\Phi}{\partial y}\,, \tag{7}$$

where $X(s)$ and $Y(s)$ are called the applied stresses and prescribed on the boundary of the plane, s is the length measured from arbitrary point, and u, v are the displacement components and G is the shear modulus. Also the applied forces stresses $X(s)$ and $Y(s)$ must satisfy the following, see [4]

$$X(s) = \sigma_{xx}\frac{dy}{ds} - \sigma_{xy}\frac{dx}{ds}\,, \qquad Y(s) = \sigma_{yx}\frac{dy}{ds} - \sigma_{yy}\frac{dx}{ds}. \tag{8}$$

Where, σ_{xx}, σ_{yy} and σ_{xy} are called the stress components and given as, see [4, 12].

$$\sigma_{xx} + \sigma_{yy} = 4G\left[\phi'(z) + \overline{\phi'(z)} - \lambda\Theta\right],$$

(9)

$$\sigma_{yy} - \sigma_{xx} + 2i\,\sigma_{xy} = 2G\left[\frac{\partial^2\Phi}{\partial y^2} - \frac{\partial^2\Phi}{\partial x^2} + 2i\,\frac{\partial^2\Phi}{\partial x\,\partial y}\right] + 4G\left[\bar{z}\,\varphi''(z) + \psi''(z)\right],$$

Where, $\lambda = \frac{\alpha}{2}(1+\nu)$ is the coefficient of heat transfer.

When the heat is perpendicular to the plate, the solution of (2) is given in the form, see [16, 17]

$$\Theta(x,y) = q\,\text{Im}\left(z + \frac{r_o^2}{z}\right), \quad r_o = constant.\sin\theta \tag{10}$$

Substituting (10) in (3) and then integrating the result, using polar coordinates, finally we have

$$\Phi(z,\bar{z}) = (1+\nu)\,\alpha\,q\,r_o^2\ \text{Im}\,z\left[\ln(z-\bar{z}) - 1\right]. \tag{11}$$

Using the formulas (10) and (11) in (9), the stress components take the forms, see [16, 17]

$$\sigma_{xx} = G\left[-\left(\frac{\partial^2\Phi}{\partial y^2} - \frac{\partial^2\Phi}{\partial x^2} + 2\lambda\Theta\right) + 2\text{Re}\left(2\phi'(z) - M(z,\bar{z})\right)\right],$$

$$\sigma_{yy} = G\left[\left(\frac{\partial^2\Phi}{\partial y^2} - \frac{\partial^2\Phi}{\partial x^2} - 2\lambda\Theta\right) + 2\text{Re}\left(2\varphi'(z) + M(z,\bar{z})\right)\right],$$

$$\sigma_{xy} = 2G\left[\frac{\partial^2\Phi}{\partial x\,\partial y} + \text{Im}M(z,\bar{z})\right], \tag{12}$$

where,

$$M(z,\bar{z}) = \bar{z}\,\phi''(z) + \psi''(z).$$

After obtaining the Goursat functions and using (10), (11), the components of stress of Eq. (12) are completely determined.

In the remain part of this paper, the complex variables method has been applied to obtain the two analytic complexes of Goursat functions $\phi(z)$ and $\psi(z)$ and the three stress components of Eq. (12) for the first and second boundary value problems in thermoelastic plate. The infinite plate weakened by a curvilinear hole C conformally mapped in the domain of the right half-plane $\text{Re}(s) \geq 0$ using the conformal mapping of Eq. (1), and when a heat $\Theta = q\,y$ is flowing uniformly in the negative direction of y-axis.

Many special cases will be derived from the results. Also, when the conformal mapping is conformally outside the unit circle $|\zeta| > 1$ or inside the unit circle $|\zeta| < 1$ will be established from this work. Many applications for the first and second boundary value problems can be discussed.

Goursat Complex Functions

To obtain the Goursat functions, we write the following expression

$$\frac{\overline{z(i\tau)}}{z'(i\tau)} = \frac{\overline{\omega(i\tau)}}{\omega'(i\tau)} = \overline{\alpha(i\tau)} + \beta(i\tau), \tag{13}$$

where,

$$\alpha(i\tau)=\frac{h}{a+i\tau}\ ,\qquad a=\frac{1+n}{1-n}\ ,\quad |n|<1$$

$$h=4n^2a^2\left(m+n^2\right)\left(1-2n^2-mn^2\right)^{-1}\ ,$$

and $\beta(s)$ is a regular function within the right half-plane except at infinity.

Using (5) in (4), the first and second boundary value problems in terms of Goursat functions take the forms

$$K\,\phi(i\tau)-\alpha(i\tau)\overline{\phi'(i\tau)}-\overline{\psi_*(i\tau)}\ =f_*(i\tau)\ , \tag{14}$$

where,

$$\psi_*(i\tau)=\psi(i\tau)+\beta(i\tau)\,\phi'(i\tau)\ ,$$

$$f_*(i\tau)=F(i\tau)-Kc\Gamma\,\omega(i\tau)+c\,\overline{\Gamma}^*\,\overline{\omega(i\tau)}+\omega(i\tau)\,\overline{N(i\tau)}\ ,$$

$$\overline{N(i\tau)}\ =c\overline{\Gamma}-\frac{S_x-iS_y}{2\pi(1+\kappa)}\frac{1}{\overline{\omega(i\tau)}},\qquad F(i\tau)=f(\omega(i\tau))\ . \tag{15}$$

The function $F(i\tau)$ with its derivatives must satisfy the Hölder condition and we assume $\phi(\infty)=\psi(\infty)=0$.

Multiplying both sides of (14) by $\dfrac{1}{2\pi(s-i\tau)}$, then integrating with respect to τ from $-\infty$ to ∞, to get

$$K\,\phi(s)-\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\alpha(i\tau)\overline{\phi'(i\tau)}\,d\tau}{(s-i\tau)}\ =\ \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{f_*(i\tau)\,d\tau}{(s-i\tau)}\ . \tag{16}$$

Using $f_*(i\tau)$ of Eq. (15) and assuming

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\alpha(i\tau)\overline{\phi'(i\tau)}}{(s-i\tau)}d\tau\ =\ \frac{h\,b}{s+a}\ . \tag{17}$$

Hence, the formula (16) becomes

$$K\,\phi(s)=A(s)-\frac{2c\,\overline{\Gamma}^*}{s+1}+\frac{M}{s+a}+\frac{h\,b}{s+a}\ . \tag{18}$$

Where,

$$M=\frac{2c\left(m+n^2\right)}{(1-n)^2}\left(K\,\Gamma-\overline{\Gamma}\right)+\frac{n\,(1+n)\left(m+n^2\right)\left(S_x-i\,S_y\right)}{\pi(1+\kappa)\left(1+mn^2\right)(1-n)}\ , \tag{19}$$

$$A(s)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{F(i\tau)\,d\tau}{(s-i\tau)}\ .$$

Using (18) in (17), the complex constant b takes the form

$$b=\left(16a^4K^2-h^2\right)^{-1}\Big[-4a^2K\,\overline{M}+h\,M-2c\,(1+n)^2\left(h\,\overline{\Gamma}^*-4a^2K\,\Gamma^*\right)$$

$$-4a^2\left(h\,A'(-a)-4a^2K\,\overline{A'(-a)}\right)\Big]\ . \tag{20}$$

Also, the function $\psi(i\tau)$ can be determined from equation (14) in the form

$$\psi(s)=\overline{A(s)}-c\Gamma^{*}\left(\omega(s)+\frac{2}{\bar{s}+1}\right)+\frac{\overline{M}+h\bar{b}}{\bar{s}+a}+\left(cK\,\bar{\Gamma}-N(s)\right)\overline{\omega(s)}-\frac{\overline{\omega(s)}}{\omega'(s)}\,\phi'(s)\ . \tag{21}$$

Applications for Goursat functions

1- Curvilinear hole for an infinite plate subjected to a uniform tensile stress:

For $K = -1$, Γ = P/4, $\Gamma^{*}=-\frac{1}{2}Pe^{-2i\vartheta}$ and $S_x = S_y = f = 0$, we have an infinite plate stretched at infinity by the application of a uniform tensile stress of intensity P, making an angle θ with the x-axis. The plate is weakened by a curvilinear hole C which is free from stress. The Goursat functions take the form

$$\phi(s)=\frac{2c\,\overline{\Gamma^{*}}}{s+1}-\frac{M}{s+a}-\frac{h\,b}{s+a}\ , \tag{22}$$

$$\psi(s)=-c\,\Gamma^{*}\left(\omega(s)+\frac{2}{1+\bar{s}}\right)+\frac{\overline{M}+h\,\bar{b}}{\bar{s}+a}-\frac{\overline{\omega(s)}}{\omega'(s)}\varphi'(s)-\left(c\,\bar{\Gamma}+\overline{N(s)}\right)\overline{\omega(s)}\ . \tag{23}$$

Where, the complex constants M and b have been determined by Eqs. (19) and (20) and their values were calculated by using Maple 9.5.

For $c = 2$, $m = 1.6$, $n = -0.05$, $P = 1/4$, $\vartheta = \pi/4$, $G = 1/2$, $q = 2$, $v = 1$, $\alpha = 1/2$, $\lambda = 1/2$ and $r_o = 1/\sqrt{2}$ the relation between the stress components σ_{xx}, σ_{yy}, σ_{xy} and σ and τ are considered in Figs. (5-9).

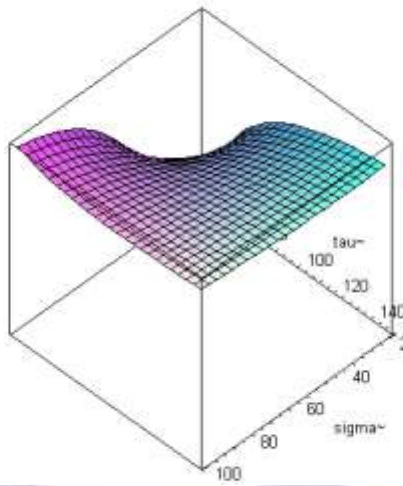

Fig. 5

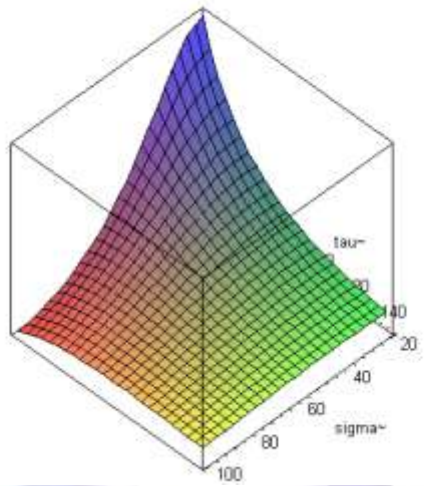

Fig. 6

Maximum value of σ_{xx} is
[-17.30052, [σ = 100., τ = 150.]]
Minimum value of σ_{xx} is
[-17.31421, [σ = 20.78959, τ = 50.]]

Maximum value of σ_{yy} is
[-0.04200, [σ = 20.07141, τ = 50.]]
Minimum value of σ_{yy} is
[-0.05677, [σ = 100., τ = 150.]]

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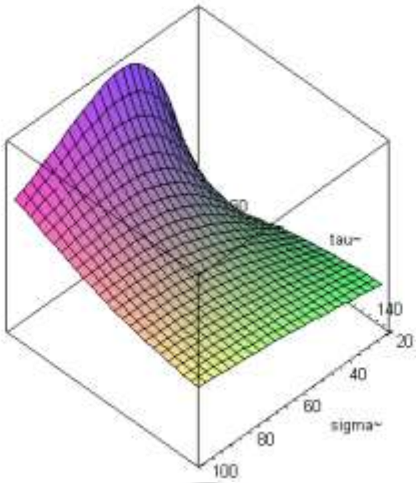


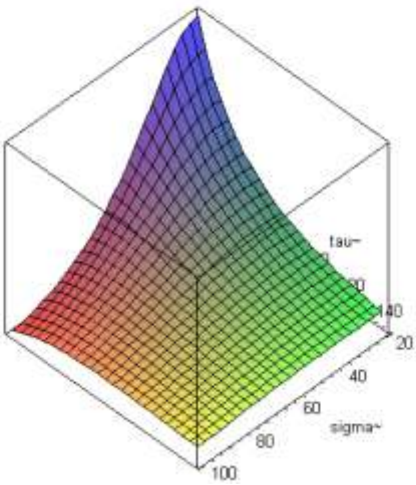
Fig. 7

Maximum value of σ_{xy} is [0.00711, [σ = 53.0625, τ = 50.]]

Minimum value of σ_{xy} is [-0.00222, [σ = 20., τ = 51.7025]]

σ_{xx} / σ_{yy}

σ_{yy} / σ_{xx}



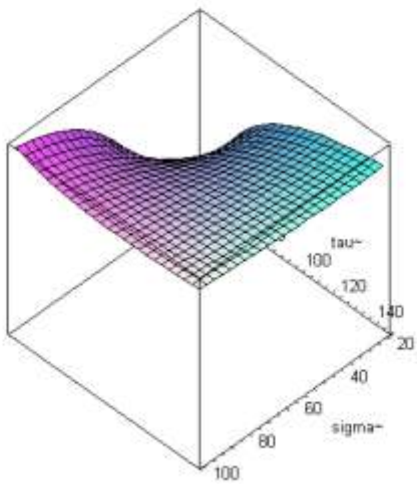


Fig. 8

Fig. 9

2- When the external force acts on the center of the curvilinear :

For $\Gamma = \Gamma^* = f = 0$, $K = \kappa$, we have the second fundamental problem when the force acts on the curvilinear kernel. It will be assumed that the stresses vanish at infinity .The Goursat functions become

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$$\phi(s) = \frac{1}{\kappa}\left(\frac{M}{s+a} + \frac{h\,b}{s+a}\right), \tag{24}$$

$$\psi(s) = \frac{\overline{M} + h\,\overline{b}}{\overline{s} + a} - \frac{\overline{\omega(s)}}{\omega'(s)}\phi'(s) - \overline{\omega(s)}N(s), \tag{25}$$

where

$$N(s) = -\frac{S_x + iS_y}{2\pi(1+\kappa)}\frac{1}{\omega(s)}, \tag{26}$$

and complex constants *M* and *b* have been determined by Eqs. (19) and (20) and their values were calculated by using Maple 9.5 .

For $c = 2$, $m = 1.6$, $n = -0.05$, $\kappa = 2$, $S_x = S_y = 10$, $G = 1/2$, $q = 2$, $v = 1$ $\alpha = 1/2$, $\lambda = 1/2$ and $r_o = 1/\sqrt{2}$ the relation between the stress components σ_{xx}, σ_{yy}, σ_{xy} and σ and τ are considered in Figs. (10-14).

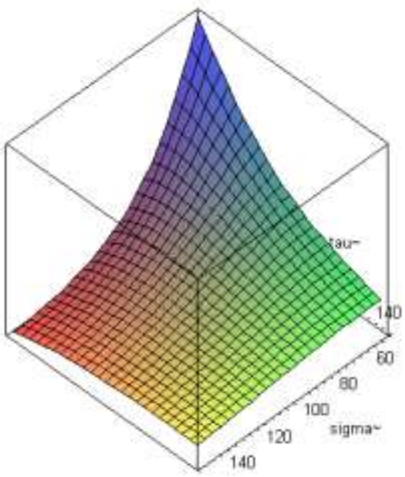

Fig. 10

Maximum value of σ_{xx} is

[-17.29877, [σ = 50., τ = 50.]]

Minimum value of σ_{xx} is

[-17.29938, [σ = 150., τ = 50.]]

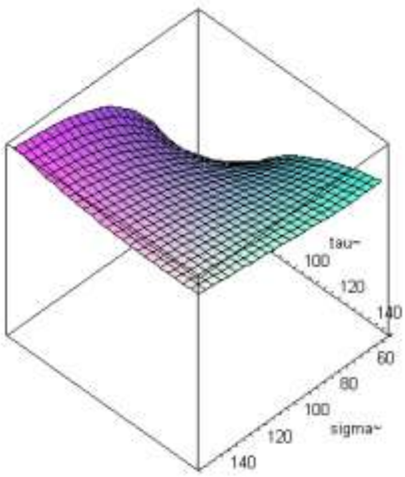

Fig. 11

Maximum value of σ_{yy} is

[-0.058000, [σ = 150., τ = 50.]]

Minimum value of σ_{yy} is

[-0.05864, [σ = 50., τ = 50.]]

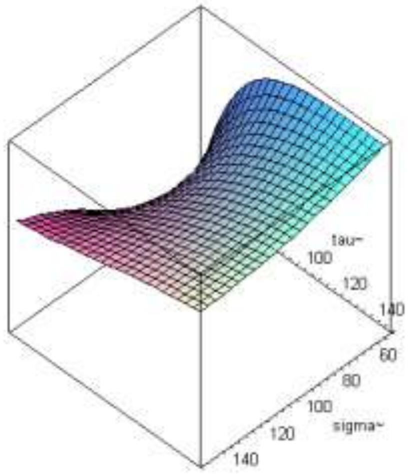

Fig. 12

Maximum value of σ_{xy} is [0.000059, [σ = 50., τ = 112.18327]]

Minimum value of σ_{xy} is [-0.00026, [σ = 66.38680, τ = 50.]]

$\dfrac{\sigma_{xx}}{\sigma_{yy}}$

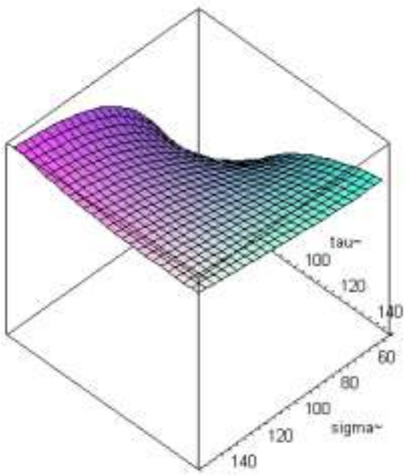

Fig. 13

$\dfrac{\sigma_{yy}}{\sigma_{xx}}$

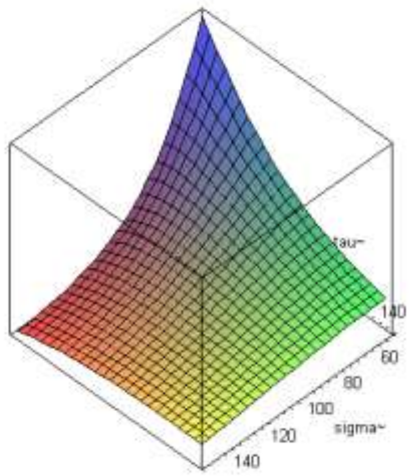

Fig. 14

Conclusion

1- In the theory of two dimensional linear elasticity, one of the most useful techniques for the solution of boundary value problem for a region weakened by a curvilinear hole is transform the region into a simpler shape to get solutions without difficulties.

2- The transformation mapping $z = c\omega(s), c>0, s=\sigma+i\tau$, transforms the domain of the infinite plate with a curvilinear hole into the domain of the right half-plane. While the mapping $z = c\omega(\zeta), c>0, \zeta=\rho e^{i\theta}$, transforms the domain of the infinite plate with a curvilinear hole onto the domain outside (when $|\zeta|>1$) or inside (when $|\zeta|<1$) a unit circle.

3- The transformation $s=\dfrac{\zeta+1}{\zeta-1}$, transforms the domain of the right half-plane onto the domain outside a unit circle. The inverse case can be obtained by using the transformation mapping $\zeta=\dfrac{s+1}{s-1}$.

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