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Higher dimensional fixed point results in complete ordered metric spaces

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ABSTRACT

The aim of this paper is to define the concept of compatible maps for n-tupled maps, a new notion propounded by M.Imdad et.al.[13] and prove n-tupled coincidence and n-tupled fixed point theorems in partially ordered metric spaces. Our results generalize, extend and improve the results of [3,7,8,12,13,19.25,26].

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1. INTRODUCTION:

The Banach contraction principle is the most natural and significant result of fixed point theory. It has become one of the most fundamental and powerful tools of nonlinear analysis because of its wide range of applications to nonlinear equations arising in physical and biological processes ensuring the existence and uniqueness of solutions. It is widely considered as the source of metric fixed point theory. Also, its significance lies in its vast applicability in a number of branches of mathematics. Generalization of the above principle has been done by various mathematicians see [1,3,4,7,8,20-26]. Existence of a fixed point for contraction type mappings in partially ordered metric space and applications have been considered by many authors. There already exists an extensive literature on this topic, but keeping in view the relevance of this paper, we merely refer to [3-6,10-13,15,18,19,25-26].

Bhaskar and Lakshmikantham [25] introduced the notions of mixed monotone property and coupled fixed point for the contractive mapping $F: X \times X$, where X is a partially ordered metric space, and proved some coupled fixed point theorems for a mixed monotone operator. As an application of the coupled fixed point theorems, they determined the existence and uniqueness of the solution of a periodic boundary value problems. It is very natural to extend the definition of 2-dimensional fixed point (coupled fixed point), 3-dimensional fixed point (tripled fixed point), 4-dimensional fixed point (quadrupled fixed point) and so on to multidimensional fixed point (n-tuple fixed point) ,(see also [5,9,16, 17,19]). The last remarkable result of this trend was given by M.Imdad et al. [13] by introducing the notion of multidimensional fixed points.(see also[1,12,15,,18]).

The purpose of this paper is to establish some n-tupled coincidence and fixed point results for compatible maps in complete partially ordered metric spaces. Our results generalize and improve the results of [3,7,8,12,13,19,25,26].

2. PRELIMINARIES AND DEFINITIOS:

As usual, this section is devoted to preliminaries which include some basic definitions and results related to coupled fixed point and n-tupled fixed point in partially ordered metric spaces.

Definition 2.1 [26] Let $(X, \preceq)$ be a partially ordered set equipped with a metric d such that

(X, d) is a metric space. Further, equip the product space $X \times X$ with the following partial ordering:

For $(x, y), (u, v) \in X \times X$, define $(u, v) \preceq (x, y) \Leftrightarrow x \succeq u, y \preceq v$.

Definition 2.2 [26] Let $(X, \preceq)$ be a partially ordered set and F: X →X then F enjoys the mixed monotone property if F(x, y) is monotonically non-decreasing in x and monotonically non-increasing in y, that is, for any x, y ∈X,

$$x_1, x_2 \epsilon X, x_1 \preceq x_2 \Rightarrow F(x_1, y) \preceq F(x_2, y) \text{ and } y_1, y_2 \epsilon X, y_1 \preceq y_2 \Rightarrow F(x, y_1) \succeq F(x, y_2).$$

Definition 2.3 [26] Let $(X, \preceq)$ be a partially ordered set and $F: X \times X \to X$, then (x, y) ∈ $X \times X$ is called a coupled fixed point of the mapping F if $F(x, y) = x$ and $F(y, x) = y$.

Definition 2.4 [26] Let $(X, \preceq)$ be a partially ordered set and $F: X \times X \to X$ and $g: X \to X$ then F enjoys the mixed g-monotone property if $F(x, y)$ is monotonically g-non-decreasing in x and monotonically g-non-increasing in y, that is ,for any $x, y \in X$,

$x_1, x_2 \in X, g(x_1) \preceq g(x_2) \Rightarrow F(x_1, y) \preceq F(x_2, y)$, for any $y \in X$,

$y_1, y_2 \in X, g(y_1) \preceq g(y_2) \Rightarrow F(x, y_1) \succeq F(x, y_2)$, for any $x \in X$.

Definition 2.5 [26] Let $(X, \preceq)$ be a partially ordered set and $F: X \times X \to X$ and $g: X \to X$, then $(x, y) \in X \times X$ is called a coupled coincidence point of the maps F and g if $F(x, y) = gx$ and $F(y, x) = gy$.

Definition 2.6 [26] Let $(X, \preceq)$ be a partially ordered set, then $(x, y) \in X \times X$ is called a coupled fixed point of the maps $F: X \times X \to X$ and $g: X \to X$ if $gx = F(x, y) = x$ and $gy = F(y, x) = y$.

Throughout the paper, r stands for a general even natural number

Definition 2.7 [13] Let (X, $\preceq$) be a partially ordered set and $F: \prod_{i=1}^{r} X^i \to X$ then F is said to have the mixed monotone property if F is non-decreasing in its odd position arguments and non-increasing in its even positions arguments , that is, if,

(i) For all $x_1^1, x_2^1 \in X, x_1^1 \preceq x_2^1 \Rightarrow F(x_1^1, x^2, x^3, \dots, x^r) \preceq F(x_2^1, x^2, x^3, \dots, x^r)$,

(ii) For all $x_1^2, x_2^2 \in X, x_1^2 \preceq x_2^2 \Rightarrow F(x^1, x_1^2, x^3, \dots x^r) \succeq F(x^1, x_2^2, x^3, \dots, x^r)$,

(iii) For all $x_1^3, x_2^3 \in X, x_1^3 \preceq x_2^3 \Rightarrow F(x^1, x^2, x_1^3, x^4, \dots, x^r) \preceq F(x^1, x^2, x_2^3, x^4, \dots, x^r)$,

…

For all $x_1^r, x_2^r \in X, x_1^r \preceq x_2^r \Rightarrow F(x^1, x^2, x^3, \dots, x_1^r) \succeq F(x^1, x^2, x^3, \dots, x_2^r)$.

Definition 2.8 [13] Let (X, $\preceq$) be a partially ordered set and $F:\prod_{i=1}^{r} X^i \to X$ and g: X→X be two maps. Then F is said to have the mixed g-monotone property if F is g-non-decreasing in its odd position arguments and g-non-increasing in its even positions arguments , that is, if ,

(i) For all $x_1^1, x_2^1 \in X, gx_1^1 \preceq gx_2^1 \Rightarrow F(x_1^1, x^2, x^3, \dots, x^r) \preceq F(x_2^1, x^2, x^3, \dots, x^r)$,

(ii) For all $x_1^2, x_2^2 \in X, gx_1^2 \preceq gx_2^2 \Rightarrow F(, x^1, x_1^2, x^3, \dots, x^r) \succeq F(x^1, x_2^2, x^3, \dots, x^r)$,

(iii) For all $x_1^3, x_2^3 \in X, gx_1^3 \preceq gx_2^3 \Rightarrow F(x^1, x^2, x_1^3, \dots, x^r) \preceq F(x^1, x^2, x_2^3, \dots, x^r)$,

For all $x_1^r, x_2^r \in X, gx_1^r \preceq gx_2^r \Rightarrow F(x^1, x^2, x^3, \dots, x_1^r) \succeq F(x^1, x^2, \dots, x_2^r)$.

Definition 2.9 [12] Let X be a nonempty set. An element $(x^1, x^2, x^3, \dots \dots \dots, x^r) \in \prod_{i=1}^{r} X^i$ is called an r-tupled fixed point of the mapping $F:\prod_{i=1}^{r} X^i \to X$ if

$$x^1 = F(x^1, x^2, x^3, \dots, x^r),$$

$$x^2 = F(x^2, x^3, \dots, x^r, x^1),$$

$$x^3 = F(x^3, \dots, x^r, x^1, x^2),$$

$$\dots$$

$$x^r = F(x^r, x^1, x^2, \dots, x^{r-1}).$$

Example 1 Let (R, d) be a partial ordered metric space under natural setting and let $F:\prod_{i=1}^{r} X^i \to X$ be mapping defined by $F(x^1, x^2, x^3, \dots, x^r) = \sin(x^1 \cdot x^2 \cdot x^3 \cdot \dots \cdot x^r)$, for any $x^1, x^2, x^3, \dots, x^r \in X$,

then (0,0,0,…,0) is an r-tupled fixed point of F.

Definition 2.10 [13] Let X be a nonempty set. An element $(x^1, x^2, x^3, \dots, x^r) \in \prod_{i=1}^{r} X^i$ is called an r-tupled coincidence point of the maps $F:\prod_{i=1}^{r} X^i \to X$ and $g: X \to X$ if

$$gx^1 = F(x^1, x^2, x^3, \dots, x^r),$$

$$gx^2 = F(x^2, x^3, \dots, x^r, x^1),$$

$$gx^3 = F(x^3, \dots, x^r, x^1, x^2),$$

$$\dots$$

$$gx^r = F(x^r, x^2, x^3, \dots, x^{r-1}).$$

Example 2 Let (R, d) be a partial ordered metric space under natural setting and let $F:\prod_{i=1}^{r} X^i \to X$ and $g: X \to X$ be maps defined by

$F(x^1, x^2, x^3, \dots, x^r) = \sin x^1 \cdot cos x^2 \cdot \sin x^3 \cdot \cos x^4 \cdot \dots \cdot \sin x^{r-1} \cdot \cos x^r$,

$g(x) = \sin x$,

for any $x^1, x^2, x^3, \dots, x^r \in X$, then $\{(x^1, x^2, x^3, \dots, x^r), x^i = m\pi, m \in N, 1 \le i \le r\}$ is an r-tupled coincidence point of F and g.

Definition 2.11 [13] Let X be a nonempty set. An element $(x^1, x^2, x^3, \dots \dots \dots, x^r) \in \prod_{i=1}^{r} X^i$ is called an r-tupled fixed point of the maps $F:\prod_{i=1}^{r} X^i \to X$ and $g: X \to X$ if

$$x^1 = gx^1 = F(x^1, x^2, x^3, \dots, x^r),$$

$$x^2 = gx^2 = F(x^2, x^3, \dots, x^r, x^1),$$

$$x^3 = gx^3 = F(x^3, \dots, x^r, x^1, x^2),$$

$$\dots$$

$$x^r = gx^r = F(x^r, x^1, x^2, \dots, x^{r-1}).$$

Now, we define the concept of compatible maps for r-tupled maps.

Definition 2.11 Let (X, ≤) be a partially ordered set, then the maps $F: \prod_{i=1}^{r} X^i \to X$ and $g: X \to X$ are called compatible if

$$\lim_{n\to\infty} g\big(F(x_n^1, x_n^2, \ldots, x_n^r), F(gx_n^1, gx_n^2, \ldots, gx_n^r)\big) = 0,$$

$$\lim_{n\to\infty} g\big(F(x_n^2, x_n^3, \ldots, x_n^r, x_n^1), F(gx_n^2, gx_n^3, \ldots, gx_n^r, gx_n^1)\big) = 0,$$

$$\lim_{n\to\infty} g\big(F(x_n^3, \ldots, x_n^r, x_n^1, x_n^2,), F(gx_n^3, \ldots, gx_n^r, gx_n^1, gx_n^2)\big) = 0,$$

…

$$\lim_{n\to\infty} g\big(F(x_n^r, x_n^1, x_n^2 \ldots, x_n^{r-1}), F(gx_n^r, gx_n^1, gx_n^2, \ldots, gx_n^{r-1})\big) = 0,$$

whenever, $\{x_n^1\}, \{x_n^2\}, \{x_n^3\}, \ldots, \{x_n^r\}$ are sequences in X such that

$$\lim_{n\to\infty} F(x_n^1, x_n^2, x_n^3, \ldots, x_n^r) = \lim_{n\to\infty} g(x_n^1) = x^1,$$

$$\lim_{n\to\infty} F(x_n^2, x_n^3, \ldots, x_n^r, x_n^1) = \lim_{n\to\infty} g(x_n^2) = x^2,$$

$$\lim_{n\to\infty} F(x_n^3, x_n^4, \ldots, x_n^1,, x_n^2) = \lim_{n\to\infty} g(x_n^3) = x^3,$$

…

$$\lim_{n\to\infty} F(x_n^r, x_n^1, x_n^2, \ldots, x_n^{r-1}) = \lim_{n\to\infty} g(x_n^r) = x^r.$$

For some $x^1, x^2, x^3, \ldots, x^r \in X$.

3. MAIN RESULTS:

Imdad et al. [13] proved the following theorem:

Theorem 3.1 Let (X, ≤) be a partially ordered set equipped with a metric d such that (X, d) is a complete metric space. Assume that there is a function $\varphi:[0,\infty)\to[0,\infty)$ with $\varphi(t)<t$ and $\lim_{r\to t^+} \varphi(r) < t$ for each t > 0. Further let $F: \prod_{i=1}^{r} X^i \to X$ and g: X→X be two maps such that F has the mixed g-monotone property satisfying the following conditions:

(i) $F(\prod_{i=1}^{r} X^i) \subseteq g(X)$,

(ii) g is continuous and monotonically increasing ,

(iii) the pair (g, F) is commuting,

(iv) $d\big(F(x^1, x^2, x^3, \ldots, x^r), F(y^1, y^2, y^3, \ldots, y^r)\big) \leq \varphi\left(\frac{1}{r}\sum_{n=1}^{r} d\big(g(x^n), g(y^n)\big)\right)$

for all $x^1, x^2, x^3, \ldots, x^r, y^1, y^2, y^3, \ldots, y^r \in X$, with $gx^1 \preceq gy^1$, $gx^2 \succeq gy^2$, $gx^3 \preceq gy^3, \ldots, gx^r \succeq gy^r$. Also, suppose that either

(a) F is continuous or

(b) X has the following properties:

(i) If a non-decreasing sequence $\{x_n\} \to x$ then $x_n \preceq x$ for all $n \geq 0$.

(ii) If a non-inecreasing sequence $\{y_n\} \to y$ then $y \preceq y_n$ for all $n \geq 0$.

If there exist $x_0^1, x_0^2, x_0^3, \ldots, x_0^r \in X$ such that

(iv)
$$gx_0^1 \preceq F(x_0^1, x_0^2, x_0^3, \ldots, x_0^r),$$
$$gx_0^2 \succeq F(x_0^2, x_0^3, \ldots, x_0^r, x_0^1),$$
$$gx_0^3 \preceq F(x_0^3, \ldots, x_0^r, x_0^1, x_0^2),$$
…
$$gx_0^r \succeq F(x_0^r, x_0^1, x_0^2, x_0^3, \ldots, x_0^{r-1}).$$

Then F and g have a r-tupled coincidence point, i. e there exist $x^1, x^2, x^3, \ldots, x^r \in X$ such that

(v)
$$gx^1 = F(x^1, x^2, x^3, \ldots, x^r),$$
$$gx^2 = F(x^2, x^3, \ldots, x^r, x^1),$$

$gx^3 = F(x^3, \ldots, x^r, x^1, x^2)$,

…

$gx^r = F(x^r, x^1, x^2, x^3, \ldots, x^{r-1})$.

Now, we prove our main result as follows:

Theorem 3.2 Let (X, ≤) be a partially ordered set equipped with a metric d such that (X, d) is a complete metric space. Assume that there is a function $\varphi: [0,\infty) \to [0,\infty)$ with $\varphi(t) < t$ and $\lim_{r\to t^+}\varphi(r) < t$ for each t > 0. Further let $F: \prod_{i=1}^{r} X^i \to X$ and g: X → X be two maps such that F has the mixed g-monotone property satisfying the following conditions:

(3.1) $F(\prod_{i=1}^{r} X^i) \subseteq g(X)$,

(3.2) g is continuous and monotonically increasing,

(3.3) the pair (g, F) is compatible,

(3.4) $d(F(x^1, x^2, x^3, \ldots, x^r), F(y^1, y^2, y^3, \ldots, y^r)) \le \varphi(max\{\, d(g(x^n), g(y^n))\})$,

For all $x^1, x^2, x^3, \ldots, x^r, y^1, y^2, y^3, \ldots, y^r \in X, n = 1,2, \ldots, r$ and $gx^1 \le gy^1$, $gx^2 \ge gy^2$, $gx^3 \le gy^3, \ldots, gx^r \ge gy^r$. Also, suppose that either

(c) F is continuous or

(d) X has the following properties:

(j) If a non-decreasing sequence $\{x_n\} \to x$ then $x_n \le x$ for all $n \ge 0$.

(iii) If a non-inecreasing sequence $\{y_n\} \to y$ then $y \le y_n$ for all $n \ge 0$.

If there exist $x_0^1, x_0^2, x_0^3, \ldots \ldots, x_0^r \in X$ such that

(3.5)
$$gx_0^1 \le F(x_0^1, x_0^2, x_0^3, \ldots, x_0^r),$$
$$gx_0^2 \ge F(x_0^2, x_0^3, \ldots, x_0^r, x_0^1),$$
$$gx_0^3 \le F(x_0^3, \ldots, x_0^r, x_0^1, x_0^2),$$
$$\ldots$$
$$gx_0^r \ge F(x_0^r, x_0^1, x_0^2, x_0^3, \ldots, x_0^{r-1}).$$

Then F and g have a r-tupled coincidence point, i. e there exist $x^1, x^2, x^3, \ldots, x^r \in X$ such that

(3.6)
$$gx^1 = F(x^1, x^2, x^3, \ldots, x^r),$$
$$gx^2 = F(x^2, x^3, \ldots, x^r, x^1),$$
$$gx^3 = F(x^3, \ldots, x^r, x^1, x^2),$$
$$\ldots$$
$$gx^r = F(x^r, x^1, x^2, x^3, \ldots, x^{r-1}).$$

Proof. Starting with $x_0^1, x_0^2, x_0^3, \ldots, x_0^r \in X$, we define the sequences $\{x_n^1\}, \{x_n^2\}, \{x_n^3\}, \ldots, \{x_n^r\}$ in X as follows:

(3.7)
$$gx_{n+1}^1 = F(x_n^1, x_n^2, x_n^1, \ldots, x_n^r),$$
$$gx_{n+1}^2 = F(x_n^2, x_n^3, \ldots, x_n^r, x_n^1),$$
$$gx_{n+1}^3 = F(x_n^3, \ldots, x_n^r, x_n^1, x_n^2),$$
$$\ldots$$
$$gx_{n+1}^r = F(x_n^r, x_n^1, x_n^2, x_n^3, \ldots, x_n^{r-1}).$$

Now, we prove that for all $n \ge 0$,

(3.8) $gx_n^1 \le gx_{n+1}^1,\ gx_n^2 \ge gx_{n+1}^2,\ gx_n^3 \le gx_{n+1}^3,\ \ldots, gx_n^r \ge gx_{n+1}^r$.

(3.9)
$$gx_0^1 \le F(x_0^1, x_0^2, x_0^3, \ldots, x_0^r) = gx_1^1,$$
$$gx_0^2 \ge F(x_0^2, x_0^3, \ldots, x_0^r, x_0^1) = gx_1^2,$$
$$gx_0^3 \le F(x_0^3, \ldots, x_0^r, x_0^1, x_0^2) = gx_1^3,$$
$$\ldots$$

$$gx_0^r \geq F(x_0^r, x_0^1, x_0^2, x_0^3, \ldots, x_0^{r-1}) = gx_1^r.$$

So (3.8) holds for n = 0. Suppose (3.8) holds for some n > 0. Consider

$$\begin{aligned} gx_{n+1}^1 &= F(x_n^1, x_n^2, x_n^3, \ldots, x_n^r) \\ &\preceq F(x_{n+1}^1, x_n^2, x_n^3, \ldots, x_n^r) \\ &\preceq F(x_{n+1}^1, x_{n+1}^2, x_n^3, \ldots, x_n^r) \\ &\preceq F(x_{n+1}^1, x_{n+1}^2, x_{n+1}^3, \ldots, x_n^r) \\ &\preceq F(x_{n+1}^1, x_{n+1}^2, \ldots, x_{n+1}^r) = gx_{n+2}^1, \end{aligned}$$

$$\begin{aligned} gx_{n+1}^2 &= F(x_n^2, x_n^3, \ldots, x_{n,}^r x_n^1) \\ &\succeq F(x_{n+1}^2, x_n^3, \ldots, x_n^r, x_n^1) \\ &\succeq F(x_{n+1}^2, x_{n+1}^3, \ldots, x_n^r, x_n^1) \\ &\succeq F(x_{n+1}^2, x_{n+1}^3, \ldots, x_{n+1}^r, x_n^1) \\ &\succeq F(x_{n+1}^1, x_{n+1}^2, \ldots, x_{n+1}^r, x_{n+1}^1) = gx_{n+2}^2, \end{aligned}$$

$$\begin{aligned} gx_{n+1}^3 &= F(x_n^3, \ldots, x_{n,}^r x_n^1, x_n^2) \\ &\preceq F(x_{n+1}^3, \ldots, x_n^r, x_n^1, x_n^2) \\ &\preceq F(x_{n+1}^3, x_{n+1}^4, \ldots, x_n^r, x_n^1, x_n^2) \\ &\preceq F(x_{n+1}^3, x_{n+1}^4, \ldots, x_{n+1}^r, x_n^1, x_n^2) \\ &\preceq F(x_{n+1}^3, x_{n+1}^4, x_n^5, \ldots, x_{n+1}^r, x_{n+1}^1, x_n^2) \\ &\preceq F(x_{n+1}^3, x_{n+1}^4, \ldots, x_{n+1}^r, x_{n+1}^1, x_{n+1}^2) = gx_{n+2}^3, \\ &\ldots \end{aligned}$$

$$\begin{aligned} gx_{n+1}^r &= F(x_n^r, x_n^1, x_n^2, \ldots, x_n^{r-1}) \\ &\succeq F(x_{n+1}^r, x_n^1, x_n^2, \ldots, x_n^{r-1}) \\ &\succeq F(x_{n+1}^r, x_{n+1}^1, x_n^2, x_n^3 \ldots, x_n^{r-1}) \\ &\succeq F(x_{n+1}^r, x_{n+1}^1, x_{n+1}^2 \ldots, x_{n,}^{r-1}) \\ &\succeq F(x_{n+1}^r, x_{n+1}^1, x_n^2, \ldots, x_{n+1}^{r-1}) = gx_{n+2}^r, \end{aligned}$$

Thus by induction (3.8) holds for all $n \geq 0$. Using (3.7) and (3.8)

(3.10) $$\mathrm{d}\left(\mathrm{g}(\mathrm{x}_\mathrm{m}^1), \mathrm{g}(\mathrm{x}_{\mathrm{m}+1}^1)\right) = \mathrm{d}\left(\mathrm{F}(\mathrm{x}_{\mathrm{m}-1}^1, \mathrm{x}_{\mathrm{m}-1}^2, \ldots, \mathrm{x}_{\mathrm{m}-1}^\mathrm{r}), \mathrm{F}(\mathrm{x}_\mathrm{m}^1, \mathrm{x}_\mathrm{m}^2, \ldots, \mathrm{x}_\mathrm{m}^\mathrm{r})\right)$$
$$\leq \varphi\left(\max\{\mathrm{d}(\mathrm{g}(\mathrm{x}_{\mathrm{m}-1}^\mathrm{n}), \mathrm{g}(\mathrm{x}_\mathrm{m}^\mathrm{n}))\}\right).$$

Similarly, we can inductively write

(3.11) $$d\left(g(x_m^2), g(x_{m+1}^2)\right) \leq \varphi\left(\max\{d(g(x_{m-1}^n), g(x_m^n))\}\right),$$

$$\ldots$$

$$d\left(g(x_m^r), g(x_{m+1}^r)\right) \leq \varphi\left(\max\{d(g(x_{m-1}^n), g(x_m^n))\}\right).$$

Therefore, by putting

(3.12) $$\gamma_m = max\left\{d\left(g(x_m^1), g(x_{m+1}^1)\right), d\left(g(x_m^2), g(x_{m+1}^2)\right), \ldots, d\left(g(x_m^r), g(x_{m+1}^r)\right)\right\}.$$

We have,

(3.13) $$\gamma_m = max\left\{d\left(g(x_m^1), g(x_{m+1}^1)\right), d\left(g(x_m^2), g(x_{m+1}^2)\right), \ldots, d\left(g(x_m^r), g(x_{m+1}^r)\right)\right\}$$
$$\leq \varphi\left(\max\{d(g(x_{m-1}^n), g(x_m^n))\}\right) = \varphi(\gamma_{m-1}).$$

Since $\varphi(t) < t$ for all t > 0, therefore, $\gamma_m \leq \gamma_{m-1}$ for all m so that $\{\gamma_m\}$ is a non-increasing sequence. Since it is bounded below, there is some $\gamma \geq 0$ such that

(3.14) $$\lim_{n\to\infty} \gamma_m = +\gamma.$$

We shall show that $\gamma = 0$. Suppose, if possible $\gamma > 0$. Taking limit as $m \to \infty$ of both sides of (3.13) and keeping in mind our supposition that $\lim_{r\to t^+} \varphi(r) < t$ for all t > 0, we have

(3.15) $$\gamma = \lim_{n\to\infty} \gamma_m \le \varphi(\gamma_{m-1}) = \varphi(\gamma) < \gamma,$$

this contradiction gives $\gamma = 0$ and hence

(3.16) $$\lim_{n\to\infty}\left[max\left\{d\left(g(x_m^1), g(x_{m+1}^1)\right), d\left(g(x_m^2), g(x_{m+1}^2)\right), \dots, d\left(g(x_m^r), g(x_{m+1}^r)\right)\right\}\right] = 0.$$

Next we show that all the sequences $\{g(x_m^1)\}, \{g(x_m^2)\}, \{g(x_m^3)\}, \dots \dots \dots, and\ \{g(x_m^r)\}$ are Cauchy sequences. If possible, suppose that at least one of $\{g(x_m^1)\}, \{g(x_m^2)\}, \dots, and$

$\{g(x_m^r)\}$ is not a Cauchy sequence. Then there exist $\epsilon > 0$ and sequences of positive integers $\{l(k)\}$ and $\{m(k)\}$ such that for all positive integers k,

$m(k) > l(k) > k,$

(3.17) $$\max\{d(gx_{l(k)}^1, gx_{m(k)}^1), d(gx_{l(k)}^2, gx_{m(k)}^2), \dots, d(gx_{l(k)}^r, gx_{m(k)}^r)\} \ge \varepsilon \text{ and}$$

$$\max\{d(gx_{l(k)}^1, gx_{m(k)-1}^1), d(gx_{l(k)}^2, gx_{m(k)-1}^2), \dots, d(gx_{l(k)}^r, gx_{m(k)-1}^r)\} < \varepsilon.$$

Now,

(3.18) $$d(gx_{l(k)}^1, gx_{m(k)}^1) = d\left(F(x_{l(k)-1}^1, x_{l(k)-1}^2, \dots, x_{l(k)-1}^r), F(x_{m(k)-1}^1, x_{m(k)-1}^2, \dots, x_{m(k)-1}^r)\right)$$

$$\le \varphi(max\{d(gx_{l(k)-1}^n, gx_{m(k)-1}^n)\}),\ n = 1,2, \dots, r.$$

Similiarly, $d(gx_{l(k)}^2, gx_{m(k)}^2) \le \varphi(max\{d(gx_{l(k)-1}^n, gx_{m(k)-1}^n)\}),\ n = 1,2, \dots, r$

…

$$d(gx_{l(k)}^r, gx_{m(k)}^r) \le \varphi(max\{d(gx_{l(k)-1}^n, gx_{m(k)-1}^n)\}),\ n = 1,2, \dots, r$$

Thus,

(3.19) $$\varepsilon \le \max\{d(gx_{l(k)}^1, gx_{m(k)}^1), d(gx_{l(k)}^2, gx_{m(k)}^2), \dots, d(gx_{l(k)}^r, gx_{m(k)}^r)\}$$

$$\le \varphi(max\{d(gx_{l(k)-1}^n, gx_{m(k)-1}^n)\}),\ n = 1,2, \dots, r$$

Again, the triangular inequality and (3.17) gives

(3.20) $$d(gx_{l(k)-1}^1, gx_{m(k)-1}^1) \le d(gx_{l(k)-1}^1, gx_{l(k)}^1) + d(gx_{l(k)}^1, gx_{m(k)-1}^1)$$

$$\le d(gx_{l(k)-1}^1, gx_{l(k)}^1) + \varepsilon,\ n = 1,2, \dots, r \quad \text{and}$$

$$d(gx_{l(k)-1}^2, gx_{m(k)-1}^2) \le d(gx_{l(k)-1}^n, gx_{l(k)}^n) + \varepsilon,\ n = 1,2, \dots, r$$

…

$$d(gx_{l(k)-1}^r, gx_{m(k)-1}^r) \le d(gx_{l(k)-1}^n, gx_{l(k)}^n) + \varepsilon,\ n = 1,2, \dots, r.$$

i. e, we have

(3.21) $$max\{d(gx_{l(k)-1}^1, gx_{m(k)-1}^1), d(gx_{l(k)-1}^2, gx_{m(k)-1}^2), \dots, d(gx_{l(k)-1}^r, gx_{m(k)-1}^r)\}$$

$$\le max\{\le d(gx_{l(k)-1}^n, gx_{l(k)}^n)\} + \varepsilon,\ n = 1,2, \dots, r$$

Also,

(3.22) $$d(gx_{m(k)}^1, gx_{l(k)}^1) \le d(gx_{m(k)}^1, gx_{m(k)-1}^1) + d(gx_{m(k)-1}^1, gx_{l(k)-1}^1)$$

$$+ d(gx_{l(k)-1}^1, gx_{l(k)}^1)$$

$$d(gx_{m(k)}^2, gx_{l(k)}^2) \le d(gx_{m(k)}^2, gx_{m(k)-1}^2) + d(gx_{m(k)-1}^2, gx_{l(k)-1}^2) + d(gx_{l(k)-1}^2, gx_{l(k)}^2)$$

…

$$d(gx_{m(k)}^r, gx_{l(k)}^r) \le d(gx_{m(k)}^r, gx_{m(k)-1}^r) + d(gx_{m(k)-1}^r, gx_{l(k)-1}^r) + d(gx_{l(k)-1}^r, gx_{l(k)}^r)$$

Using (3.17),(3.19) and (3.22), we have

(3.23) $$\varepsilon \le max\{d(gx_{l(k)}^1, gx_{m(k)}^1), d(gx_{l(k)}^2, gx_{m(k)}^2), \dots, d(gx_{l(k)}^r, gx_{m(k)}^r)\}$$

$$\le max\{d(gx_{m(k)}^1, gx_{m(k)-1}^1), d(gx_{m(k)}^2, gx_{m(k)-1}^2), \dots, d(gx_{m(k)}^r, gx_{m(k)-1}^r)\}$$

$$+ max\{d(gx^1_{m(k)-1}, gx^1_{l(k)-1}), d(gx^2_{m(k)-1}, gx^2_{l(k)-1}), \dots, d(gx^r_{m(k)-1}, gx^r_{l(k)-1})\}$$
$$+ max\{d(gx^1_{l(k)-1}, gx^1_{l(k)}), d(gx^2_{l(k)-1}, gx^2_{l(k)}), \dots, d(gx^r_{l(k)-1}, gx^r_{l(k)})\}.$$

Letting $k \to \infty$ in above equation, we get

(3.24) $lim_{k\to\infty}(max\{d(gx^1_{l(k)-1}, gx^1_{m(k)-1}), d(gx^2_{l(k)-1}, gx^2_{m(k)-1}), \dots, d(gx^r_{l(k)-1}, gx^r_{m(k)-1})\}) = \varepsilon$

Finally, letting $k \to \infty$ in (3.17) and using (3.19) and (3.23) , we get

(3.25) $$\varepsilon \leq \max\{d(gx^1_{l(k)}, gx^1_{m(k)}), d(gx^2_{l(k)}, gx^2_{m(k)}), \dots, d(gx^r_{l(k)}, gx^r_{m(k)})\} \leq \varphi(\varepsilon) < \varepsilon,$$

which is a contradiction. Therefore, $\{g(x^1_m)\}, \{g(x^2_m)\}, \{g(x^3_m)\}, \dots\dots, \{g(x^r_m)\}$ are Cauchy sequences. Since the metric space (X, d) is complete, so there exist $x^1, x^2, \dots\dots, x^r \in X$ such that

(3.26) $$\lim_{m\to\infty} g(x^1_m) = x^1, \lim_{m\to\infty} g(x^2_m) = x^2, \dots\dots\dots\dots\dots, \lim_{m\to\infty} g(x^r_m) = x^r.$$

As g is continuous, so from (3.26), we have

(3.27) $$\lim_{m\to\infty} g(g(x^1_m)) = g(x^1), \lim_{m\to\infty} g(g(x^2_m)) = g(x^2), \dots, \lim_{m\to\infty} g(g(x^r_m)) = g(x^r).$$

By the compatibility of g and F, we have

(3.28) $$\lim_{n\to\infty} d\left(g(F(x^1_m, x^2_m, \dots, x^r_m)), F(g(x^1_m), g(x^2_m), \dots, g(x^r_m))\right) = 0,$$
$$\lim_{n\to\infty} d\left(g(F(x^2_m, \dots, x^r_m, x^1_m)), F(g(x^2_m), \dots, g(x^r_m), g(x^1_m))\right) = 0,$$
$$\dots$$
$$\lim_{n\to\infty} d\left(g\left((F(x^r_m, x^1_m, \dots, x^{r-1}_m))\right), F(g(x^r_m), g(x^1_m), \dots, g(x^{r-1}_m))\right) = 0.$$

Now, we show that F and g have an r-tupled coincidence point. To accomplish this, suppose (a) holds, i. e F is continuous, then using (3.28) and (3.8), we see that

$$d(g(x^1), F(x^1, x^2, \dots, x^r))$$
$$= \lim_{n\to\infty} d\left(g\left(g(x^1_{m+1})\right), F(g(x^1_m), g(x^2_m), \dots, g(x^r_m))\right)$$
$$= \lim_{n\to\infty} d\left(g(F(x^1_m, x^2_m, \dots, x^r_m)), F(g(x^1_m), g(x^2_m), \dots, g(x^r_m))\right) = 0,$$

which gives $g(x^1) = F(x^1, x^2, \dots, x^r)$. Similarly, we can prove $g(x^2) = F(x^2, \dots, x^r, x^1), \dots,$

$$g(x^r) = F(x^r, x^1, \dots, x^{r-1})$$

Hence $(x^1, x^2, \dots, x^r) \in \prod_{i=1}^r X^i$ is an r-tupled coincidence point of the maps F and g.

If (b) holds. Since $g(x^i_m)$ is non-decreasing or non-increasing as i is odd or even and $g(x^i_m) \to x^i$ as $m \to \infty$, we have $g(x^i_m) \leq x^i$, when i is odd while $g(x^i_m) \geq x^i$, when i is even. Since g is monotonically increasing, therefore

(3.29) $g\left(g(x^i_m)\right) \leq g(x^i)$ when i is odd,

$g\left(g(x^i_m)\right) \geq g(x^i)$ when i is even.

Now, using triangle inequality together with (3.8), we get

(3.30) $$d(g(x^1), F(x^1, x^2, \dots, x^r))$$
$$\leq d\left(g(x^1), g(x^1_{m+1})\right) + d\left(g(x^1_{m+1}), F(x^1, x^2, \dots, x^r)\right)$$
$$\leq d\left(g(x^1), g(x^1_{m+1})\right) + d\left(g(F(x^1_m, x^2_m, \dots, x^r_m)), F(g(x^1_m), g(x^2_m), \dots, g(x^r_m))\right)$$
$$\to 0 \; as \; n \to \infty.$$

Therefore, $g(x^1) = F(x^1, x^2, \dots, x^r)$. Similarly we can prove $g(x^2) = F(x^2, \dots, x^r, x^1), \dots,$

$g(x^r) = F(x^r, x^1, \dots, x^{r-1})$.Thus the theorem follows.

Now, we furnish our theorem by an example.

Example 3. Let $X = [0,1]$ be complete metric space under usual metric and natural ordering $\leq$ of real numbers. Define the maps $F: \prod_{i=1}^r X^i \to X$ and g: X→X as follows

$g(x) = rx$ and $F(x^1, x^2, x^3, \dots, x^r) = \begin{cases} \frac{x^1-2x^2+3x^3-\dots+(r-1)x^{r-1}-rx^r}{r^2+1}, x^{i+1} \le x^i, i = 1,3,\dots,r-1 \\ 0 \qquad otherwise, \end{cases}$

For all $x^1, x^2, x^3, \dots, x^r \in X$. Then F enjoy the mixed g-monotone property. Also F is g-compatible in X. Now choose $\{x_0^1, x_0^2, \dots, x_0^r\} = \left\{0, \frac{1}{5}, 0, \frac{1}{5}, \dots, \frac{1}{5}\right\}$

Set $\varphi(t) = \frac{(r+1)}{r^2+1}\, t$. Then we see that

$g(x_0^1) = g(0) = 0 = F(x_0^1, x_0^2, \dots, x_0^r),$

$g(x_1^2) = F(x_0^2, \dots, x_0^r, x_0^1) \le g\left(\frac{1}{5}\right) = g(x_0^2),$

$g(x_0^3) = g(0) = 0 = F(x_0^3, x_0^4, \dots, x_0^1),$

…

$g(x_1^r) = F(x_0^r, \dots, x_0^2, x_0^1) \le g\left(\frac{1}{5}\right) = g(x_0^r).$

Now, we check contactive condition(3.4) of theorem 3.2. We take

$x^1, x^2, x^3, \dots, x^r, y^1, y^2, \dots, y^{r-1}, y^r \in X$ such that

$gy^1 \le gx^1, gx^2 \le gy^2, \dots gx^r \le gy^r.$

Let $K = max(d(gx^1, gy^1), d(gx^2, gy^2), \dots, d(gx^r, gy^r))$

$\quad = max\{|gx^1 - gy^1|, |gx^2 - gy^2|, \dots, |gx^r - gy^r|\}$

$\quad = r.max\{|x^1 - y^1|, |x^2 - y^2|, \dots, |x^r - y^r|\}$

The following four cases arise

Case 1: Let $x^1, x^2, x^3, \dots, x^r, y^1, y^2, \dots, y^{r-1}, y^r \in X$ such that $x^{i+1} \le x^i, y^{i+1} \le y^i$,

$i = 1,3, \dots, r-1$. Then

$$d(F(x^1, x^2, \dots, x^{r-1}, x^r), F(y^1, y^2, \dots, y^{r-1}, y^r)) = d\left(\frac{x^1-2x^2+3x^3-\dots+(r-1)x^{r-1}-rx^r}{r^2+1}, \frac{y^1-2y^2+3y^3-\dots+(r-1)y^{r-1}-ry^r}{r^2+1}\right)$$

$$= \frac{1}{r^2+1}\left|\begin{matrix}(x^1 - 2x^2 + 3x^3 - \dots + (r-1)x^{r-1} - rx^r) \\ -(y^1 - 2y^2 + 3y^3 - \dots + (r-1)y^{r-1} - ry^r)\end{matrix}\right|$$

$$= \frac{1}{r^2+1}\{(\lfloor x^1 - y^1\rfloor + 2\lfloor x^2 - y^2\rfloor + \dots + r\lfloor x^r - y^r\rfloor)\}$$

$$= \frac{K}{r^2+1}\left(\textstyle\sum_{n=1}^{r} n\right) = \frac{r(r+1)}{2(r^2+1)}K.$$

Case 2: Let $x^1, x^2, x^3, \dots, x^r, y^1, y^2, \dots, y^{r-1}, y^r \in X$ such that $x^{i+1} \le x^i, i = 1,3, \dots,$

$r-1$ *and* $y^i \le y^{i+1}$ for atleast one i. Then ($for\ y^1 \le y^2$)

$$d(F(x^1, x^2, \dots, x^{r-1}, x^r), F(y^1, y^2, \dots, y^{r-1}, y^r))$$

$$= d\left(\frac{x^1-2x^2+3x^3-\dots+(r-1)x^{r-1}-rx^r}{r^2+1}, 0\right)$$

$$= \frac{1}{r^2+1}|(x^1 - 2x^2 + 3x^3 - \dots + (r-1)x^{r-1} - rx^r) + (2y^2 - y^1)|.$$

$$= \frac{1}{r^2+1}\{(\lfloor x^1 - y^1\rfloor + 2\lfloor x^2 - y^2\rfloor + \dots + r\lfloor x^r - y^r\rfloor)\} = \frac{r(r+1)}{2(r^2+1)}K$$

Case 3: Let $x^1, x^2, x^3, \dots, x^r, y^1, y^2, \dots, y^{r-1}, y^r \in X$ such that $y^{i+1} \le y^i, i = 1,3, \dots,$

$r-1$ *and* $x^i \le x^{i+1}$ for atleast one i. This case is similar to Case 2.

Case 3: Let $x^1, x^2, x^3, \dots, x^r, y^1, y^2, \dots, y^{r-1}, y^r \in X$ such that $x^i \le x^{i+1}, y^i \le y^{i+1}$,

$i = 1,3, \dots, r-1$. Then

$$d(F(x^1, x^2, \dots, x^{r-1}, x^r), F(y^1, y^2, \dots, y^{r-1}, y^r)) = d(0,0) \le \frac{r(r+1)}{2(r^2+1)}K.$$

Thus, in all the cases,

$$d(F(x^1, x^2, \dots, x^{r-1}, x^r), F(y^1, y^2, \dots, y^{r-1}, y^r)) \le \frac{r(r+1)}{2(r^2+1)}K \le \frac{(r+1)}{(r^2+1)}\, r.K \le \varphi(max\{d(gx^n, gy^n)\}).$$

Hence all the conditions of our theorem 3.2 are satisfied and (0,0,…,0) is an r-tuple coincidence point of F and g.

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The following corollary is a generalization of corollary 1[10] and theorem 2.1[9]

Corollary 3.1 Let (X, $\preceq$) be a partially ordered set equipped with a metric d such that (X, d) is a complete metric space. Further let $F:\prod_{i=1}^{r} X^i \to X$ and g: X→X be two maps satisfying all the conditions of theorem 3.2 ,with replaced (3.4) by

(3.31) $\left(d\left(F(x^1,x^2,\dots,x^r),F(y^1,y^2,\dots,y^r)\right) \le k\left(max\{d\left(g(x^n),g(y^n)\right)\}\right), n = 1,2,\dots,r\right.$

$and\ k \in [0,1)$.Then F and g have a r-tupled coincidence point.

Proof: If we put $\varphi(t) = k.t$ with $k \in [0,1)$ in theorem 3.2, we get the corollary.

Uniqueness of r-tupled fixed point

For all $(x^1,x^2,\dots,x^r),(y^1,y^2,\dots,y^r) \in X^r$,

$$(x^1,x^2,\dots,x^r) \preceq (y^1,y^2,\dots,y^r)$$

$$\Leftrightarrow x^1 \le y^1, x^2 \ge y^2, \dots, x^r \ge y^r.$$

We say that $(x^1,x^2,\dots,x^r) = (y^1,y^2,\dots,y^r) \quad \Leftrightarrow x^1 = y^1, x^2 = y^2, \dots, x^r = y^r.$

Theorem 3.3 In addition to the hypothesis of theorem 3.2, suppose that for every

$$(x^1,x^2,\dots,x^r),(y^1,y^2,\dots,y^r) \in X^r$$

There exist $(z^1,z^2,\dots,z^r) \in X^r$ such that

$F\left((z^1,z^2,\dots,z^r),(z^2,z^3,\dots,z^r,z^1),\dots,(z^r,z^1,\dots,z^{r-1})\right)$ is comparable to

$F\left((x^1,x^2,\dots,x^r),F(x^2,x^3,\dots,x^r,x^1),\dots,F(x^r,x^1,\dots,x^{r-1})\right)$ and

$F\left((y^1,y^2,\dots,y^r),F(y^2,y^3,\dots,y^r,y^1),\dots,F(y^r,y^1,\dots,y^{r-1})\right)$.

Then F and g have a unique r- coincidence point, which is a fixed point of $g: X \to X$ and $F:\prod_{i=1}^{r} X^i \to X$. That is there exists a unique $(x^1,x^2,\dots,x^r) \in X^r$ such that

$x^i = g(x^i) = F(x^i,x^1,\dots,x^{i-1})$ for all $i \in \{1,2,\dots,r\}$.

Proof. By theorem 3.2, the set of r-coincidence points is non-empty. Now, suppose that $(x^1,x^2,\dots,x^r)$ and $(y^1,y^2,\dots,y^r)$ are two coincidence points of F and g, that is

$g(x^i) = F(x^i,x^1,\dots,x^{i-1})$ for all $i \in \{1,2,\dots,r\}$and

$g(y^i) = F(y^i,y^1,\dots,y^{i-1})$ for all $i \in \{1,2,\dots,r\}$.

We will show that

(3.32) $g(x^i) = g(y^i)$ for all $i \in \{1,2,\dots,r\}$.

By assumption, there exists $(z^1,z^2,\dots,z^r) \in X^r$ such that

$F\left((z^1,z^2,\dots,z^r),F(z^2,z^3,\dots,z^r,z^1),\dots,F(z^r,z^1,\dots,z^{r-1})\right)$ is comparable to

$F\left((x^1,x^2,\dots,x^r),F(x^2,x^3,\dots,x^r,x^1),\dots,F(x^r,x^1,\dots,x^{r-1})\right)$ and

$F\left((y^1,y^2,\dots,y^r),F(y^2,y^3,\dots,y^r,y^1),\dots,F(y^r,y^1,\dots,y^{r-1})\right)$.

Let $z_0^i = z^i$ for all $i \in \{1,2,\dots,r\}$. Since $F(X^r) \subseteq g(X)$, we can choose $z_1^i \in X$ such that

$g(z_1^i) = F(z_0^i,z_0^1,\dots,z_0^{i-1})$ for all $i \in \{1,2,\dots,r\}$. By a similar reason, we can inductively define sequences $\{g(z_n^i)\}, n \in N$ for all $i \in \{1,2,\dots,r\}$ such that

$g(z_{n+1}^i) = F(z_n^i,z_n^1,\dots,z_n^{i-1})$ for all $i \in \{1,2,\dots,r\}$.

In addition, let $x_0^i = x^i$ and $y_0^i = y^i$ for all $i \in \{1,2,\dots,r\}$ and in the same way, define the sequences $\{g(x_n^i)\}$ and $\{g(y_n^i)\}$, $n \in N$ for all $i \in \{1,2,\dots,r\}$. Since

$F(x^1,x^2,\dots,x^r),F(x^2,x^3,\dots,x^r,x^1),\dots,F(x^r,x^1,\dots,x^{r-1}) = (gx_1^1,gx_1^2,\dots,gx_1^r)$ and
$F(z^1,z^2,\dots,z^r),F(z^2,z^3,\dots,z^r,z^1),\dots,F(z^r,z^1,\dots,z^{r-1}) = (gz_1^1,gz_1^2,\dots,gz_1^r)$

are comparable, then

$g(x_1^i) \le g(z_1^i)$ for all $i \in \{1,2,\dots,r\}$ if i is odd,

$g(x_1^i) \ge g(z_1^i)$ for all $i \in \{1,2,\dots,r\}$ if i is even.

We have

$g(x^{2i-1}) = g(x_1^{2i-1}) \leq g(z_2^{2i-1}) \leq \cdots \leq g(z_r^{2i-1})$,

$g(x^{2i}) = g(x_1^{2i}) \geq g(z_2^{2i}) \geq \cdots \geq g(z_r^{2i})$.

Then $(g(x^1), g(x^2), \dots, g(x^r))$ and $(g(z_n^1), g(z_n^2), \dots, g(z_n^r))$ are comparable for all $n \in N$. It follows from condition (3.4) of theorem 3.2

$$d\left(g(x^i), g(z_{n+1}^i)\right) = d\left(F(x^i, x^1, \dots, x^{i-1}), F(z_n^i, z_n^1, \dots, z_n^{i-1})\right)$$

$\leq \varphi(\{\max d(gx^i, gz_n^i)\})$, for all $i \in \{1,2,\dots,r\}$.

Therefore,

(3.33) $[\max d(g(x^1), g(z_n^1)), d(g(x^2), g(z_n^2)), \dots, d(g(x^r), g(z_n^r))]$

$\leq \varphi(\{\max d(g(x^1), g(z_n^1)), d(g(x^2), g(z_n^2)), \dots, d(g(x^r), g(z_n^r))\})$

…

$\leq \varphi^n(max\{d(g(x^1), g(z_n^1)), d(g(x^2), g(z_n^2)), \dots, d(g(x^r), g(z_n^r))\})$

For all $n \geq 1$. Note that$\varphi(0) = 0, \varphi(t) < t, lim_{r\to t^+}\varphi(r) < t$ for $t > 0$ imply that $lim_{n\to\infty}\varphi^n(t) = 0$ for all $t > 0$. Hence from (3.33) we have

(3.34) $lim_{n\to\infty} d\left(g(x^i), g(z_{n+1}^i)\right) = 0$ for all $i \in \{1,2,\dots,r\}$.

Similarly, one can prove that

(3.35) $lim_{n\to\infty} d\left(g(y^i), g(z_{n+1}^i)\right) = 0$ for all $i \in \{1,2,\dots,r\}$.

Using (3.34), (3.35) and triangle inequality we get

$d\left(g(x^i), g(y^i)\right) \leq d\left(g(x^i), g(z_{n+1}^i)\right) + d\left(g(z_{n+1}^i), g(y^i)\right) \to 0,$

As $n \to \infty$ for all $i \in \{1,2,\dots,r\}$. Hence, $g(x^i) = g(y^i)$.

Since $g(x^i) = F(x^i, x^1, \dots, x^{i-1})$ for all $i \in \{1,2,\dots,r\}$, hence, we have

(3.36) $g\left(g(x^i)\right) = g\left(F(x^i, x^1, \dots, x^{i-1})\right) = F(gx^i, gx^1, \dots, gx^{i-1})$

Denote $gx^i = u^i$ for all $i \in \{1,2,\dots,r\}$. From (3.36), we have

(3.37) $g(u^i) = g(gx^i) = F(u^i, u^1, \dots, u^{i-1})$ for all $i \in \{1,2,\dots,r\}$.

Hence $(u^i, u^1, \dots, u^{i-1})$ is a r-coincidence point of F and g.

It follows $y^i = u^i$ and so

$g(y^i) = g(u^i)$ for all $i \in \{1,2,\dots,r\}$.

This means that

$g(u^i) = u^i$ for all $i \in \{1,2,\dots,r\}$.

Now, from (3.37), we have

$u^i = g(u^i) = F(u^i, u^1, \dots, u^{i-1})$ for all $i \in \{1,2,\dots,r\}$.

Hence, $(u^1, u^2, \dots, u^r)$ is a r-fixed point of F and a fixed point of g.

To prove the uniqueness of the fixed point, assume that $(v^1, v^2, \dots, v^r)$ is another r-fixed point. Then, we have

$u^i = g(u^i) = v^i = g(v^i)$ for all $i \in \{1,2,\dots,r\}$

Thus, $(u^1, u^2, \dots, u^r) = (v^1, v^2, \dots, v^r)$. This completes the proof.

In the following theorem, we replace the continuity of g, the compatibility of F and g and the completeness of X by assuming that g(X) is a complete subspace of X.

Theorem 3.4 Let (X, $\preceq$) be a partially ordered set equipped with a metric d such that (X, d) is a complete metric space. Assume that there is a function $\varphi : [0,\infty) \to [0,\infty)$ with $\varphi(t) < t$ and $\lim_{r\to t^+}\varphi(r) < t$ for each t > 0. Further let $F: \prod_{i=1}^r X^i \to X$ and g: X → X be two maps such that F has the mixed g-monotone property and satisfying (3.1), (3.4) and the following conditions:

(3.38) g(X) is a complete subspace of X, Also, suppose that either X has the following properties:

(k) If a non-decreasing sequence $\{x_n\} \to x$ then $x_n \preceq x$ for all $n \geq 0$.

(iv) If a non-inecreasing sequence $\{y_n\} \to y$ then $y \preceq y_n$ for all $n \geq 0$.

If there exist $x_0^1, x_0^2, x_0^3, \dots \dots, x_0^r \in X$ such that (3.5) holds. Then F and g have a r-tupled coincidence point.

Proof: We construct the sequences $\{x_n^1\}, \{x_n^2\}, \{x_n^3\}, \dots, \{x_n^r\}$ as in theorem 3.2. As in the proof of theorem 3.2, the sequences $\{g(x_m^1)\}, \{g(x_m^2)\}, \{g(x_m^3)\}, \dots \dots \dots, and \{g(x_m^r)\}$ are Cauchy sequences. Since g(X) is complete, there exist $x^1, x^2, \dots \dots, x^r \in X$ such that

(3.39) $\lim_{m\to\infty} g(x_m^1) = gx^1, \lim_{m\to\infty} g(x_m^2) = gx^2, \dots \dots \dots \dots \dots, \lim_{m\to\infty} g(x_m^r) = gx^r.$

Since $g(x_m^i)$ is non-decreasing or non-increasing as i is odd or even and $g(x_m^i) \to x^i$ as $m \to \infty$, we have $g(x_m^i) \preceq x^i$, when i is odd while $g(x_m^i) \succeq x^i$, when i is even. Since g is monotonically increasing, therefore

(3.40) $g\left(g(x_m^i)\right) \preceq g(x^i)$ when i is odd,

$g\left(g(x_m^i)\right) \succeq g(x^i)$ when i is even.

(3.41) $d(\, g(x^1), F(x^1, x^2, \dots, x^r))$

$$\leq d\left(g(x^1), g(x_{m+1}^1)\right) + d\left(g(x_{m+1}^1), F(x^1, x^2, \dots, x^r)\right)$$

$$\leq d\left(g(x^1), g(x_{m+1}^1)\right) + d\left(\left(F(x_m^1, x_m^2, \dots, x_m^r)\right), F(x^1, x^2, \dots, x^r)\right)$$

$$d\left(g(x^1), g(x_{m+1}^1)\right) + \varphi(\, max\{d\left(g(x_m^n), g(x^n)\right)\})$$

$$\to 0 \; as \; n \to \infty.$$

Therefore, $g(x^1) = F(x^1, x^2, \dots, x^r)$. Similarly we can prove $g(x^2) = F(x^2, \dots, x^r, x^1)$, ...,

$g(x^r) = F(x^r, x^1, \dots, x^{r-1})$. Thus the theorem follows.

4. CONCLUSION:

Our work sets analogues, unifies, generalizes, extends and improves several well known results existing in literature, in particular the recent results of [1-4,7-10,12,13,15,19,21,25,26] etc. in the frame work of ordered metric spaces as the notion of compatible maps is more general than commuting and weakly commuting maps. Our theorems 3.2 and 3.3 have been proved by assuming much weaker condition than in analogous results and our corollary 3.1 is a generalization of corollary 1[10] and theorem 2.1 [9]. Also, our theorem 3.4 does not need completeness of space and continuity of maps involved therein. The results concerning commuting and weakly commuting maps being extendable in the spirit of our theorems, can be extended verbatim by simply using wider class of compatibly in place of commuting and weakly commuting maps.

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