ISSN 2347-1921

New Variants of the Schroder Method for finding Zeros of Nonlinear Equations having unknown Multiplicity

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ABSTRACT

There are two aims of this paper, firstly, we define new variants of the Schroder method for finding zeros of nonlinear equations having unknown multiplicity and secondly, we introduce a new formula for approximating multiplicity *m*. Using the new formula, the five particular well-established methods are identical to the classical Schroder method. In terms of computational cost the new iterative method requires three evaluations of functions per iteration. It is proved that the each of the methods has a convergence of order two. Numerical examples are given to demonstrate the performance of the methods with and without multiplicity *m*.

Keywords: Schroder method; Modified Newton method; Root-finding; Nonlinear equations; Multiple roots; Order of convergence.

Mathematics Subject Classifications: AMS (MOS): 65H05, 41A25.

Council for Innovative Research
Peer Review Research Publishing System

Journal: Journal of Advances in Mathematics

Vol 8, No 3

editor@cirjam.org

www.cirjam.com, www.cirworld.com

ISSN 2347-1921

1 INTRODUCTION

In recent years, some modifications of Newton's method for multiple roots have been proposed and, therefore, finding the roots of nonlinear equations is very important in numerical analysis and has many applications in engineering and other applied sciences. In this paper, we consider *m*-free methods to find a multiple root α of multiplicity *m*, i.e., $f^{(j)}(\alpha)=0,\ j=0,1,\dots,m-1$ and $f^{(m)}(\alpha)\neq 0,$ of a nonlinear equation

$$f(x)=0. \tag{1}$$

where $f: I \subset \mathbb{R} \to \mathbb{R}$ is a scalar function on an open interval *I* and it is sufficiently smooth in a neighbourhood of $\alpha.$ In recent years, some modifications of the Newton method for multiple roots have been proposed and analysed [1-7]. However, there are not many methods known to handle the case of unknown multiplicity. If the multiplicity is unknown of the function *f* is difficult to compute or is expensive to obtain, then the classical Schroder method is required. In this study, we define a new formula for approximating the multiplicity *m* and present new variants of the classical Schroder method. Simply replacing the multiplicity component *m* with the new formula $\widehat{m}$ in the well-established third order methods, namely the Osada third order method [4], the Halley third order method [6], the Euler-Chebyshev third order method [6] and the Chun-Neta third order method [1], we have obtained new variants of the classical Schroder method. In fact, we have obtained five identical new methods which are independent of the value of *m*. The drawback is that the third-order methods actually reduce a second-order method. In addition, these new Schroder-type methods have an efficiency index equivalent to the classical Schroder method presented in [5,6].

We begin the classical modified Newton's method

$$x_n = x_{n-1} - m\frac{f(x_{n-1})}{f'(x_{n-1})}, \tag{2}$$

In fact, this is dependent on the value of *m* and it is well known that the modified Newton's method (2) have a convergence order of two [2,6]. Whereas, the Schroder method is given as

$$x_n = \frac{f(x_{n-1})\,f'(x_{n-1})}{f'(x_{n-1})^2 - f(x_{n-1})\,f''(x_{n-1})}, \tag{3}$$

It is well established that this method is independent of value *m* and also has a second order convergence [5,6].

Conjecture (Multiplicity *m*)

Assume that the function $f: I \subset \mathbb{R} \to \mathbb{R}$ for an open interval *I* has a multiple root $\alpha \in I$. Let $f(x)$ have first and second derivatives in the interval *I*, then the multiplicity *m* may be approximated by

$$\widehat{m}_n \approx \left|\frac{f'(x_{n-1})^2}{f'(x_{n-1})^2 - f(x_{n-1})\,f''(x_{n-1})}\right|, \tag{4}$$

Hence, for all *n* sufficiently large, there exists a multiplicity *m*

$$\lim_{n\to\infty} \widehat{m}_n = m. \tag{5}$$

Contents of the paper are summarized as follows: Some basic definitions relevant to the present work are presented in section 2. In section 3 we describe the five equivalent methods of Schroder-type method and show that the new methods are independent of *m*. Finally, in section 4 we demonstrate the performance of each of the methods with the approximation of the multiplicity *m*.

2 BASIC DEFINITIONS

In order to establish the order of convergence of the new iterative methods, we state some of the definitions:

Definition 2 Let $f(x)$ be a real function with a simple root α and let $\{x_n\}$ be a sequence of real numbers that converge towards $\alpha.$ The order of convergence *p* is given by

$$\lim_{n\to\infty}\frac{x_{n+1}-\alpha}{(x_n-\alpha)^p}=\zeta\neq 0 \tag{6}$$

where $p\in\mathbb{R}^+$ and ζ is the asymptotic error constant [2,6].

Definition 3 Let $e_k = x_k - \alpha$ be the error in the *k*th iteration, then the relation

$$e_{k+1}=\zeta e_k^p+\mathrm{O}\left(e_k^{p+1}\right), \tag{7}$$

is the error equation. If the error equation exists then *p* is the order of convergence of the iterative method [2,6].

Definition 4 Let *r* be the number of function evaluations of the new method. The efficiency of the new method is measured by the concept of efficiency index [6] and defined as

$$\sqrt[r]{p} \tag{8}$$

where *p* is the order of the method.

Definition 5 Suppose that x_{n-1}, x_n and x_{n+1} are three successive iterations closer to the root α of (1), then the computational order of convergence [7] may be approximated by

$$\mathrm{COC}\approx\frac{\ln\left|(x_{n+1}-\alpha)(x_n-\alpha)^{-1}\right|}{\ln\left|(x_n-\alpha)(x_{n-1}-\alpha)^{-1}\right|},\qquad n\in\mathbb{N}. \tag{9}$$

3 EQUIVALENCE OF THE METHODS

In this section we demonstrate the similarity between five particular third order methods for finding multiple roots of a nonlinear equation. The new *m*-free methods are based on well-known methods, namely the modified Newton method [2,6], the Osada third order method [4], the Halley third order method [6], the Euler-Chebyshev third order method [6] and the Chun-Neta third order method [1]. We apply the new multiplicity *m* formula to construct the new *m*-free methods.

3.1 THE NEWTON *m*-FREE METHOD

The first of the method we consider is the classical modified Newton method

$$x_n=x_{n-1}-m\frac{f(x_{n-1})}{f'(x_{n-1})}, \tag{10}$$

Replacing the *m* with the approximating formula (4), we obtain

$$x_n=x_{n-1}-\hat{m}\frac{f(x_{n-1})}{f'(x_{n-1})}, \tag{11}$$

and in functional form

$$x_n=x_{n-1}-\left(\frac{f'(x_{n-1})^2}{f'(x_{n-1})^2-f(x_{n-1})f''(x_{n-1})}\right)\left(\frac{f(x_{n-1})}{f'(x_{n-1})}\right), \tag{12}$$

Simplifying (12), we obtain

$$x_n=x_{n-1}-\left(\frac{f'(x_{n-1})f(x_{n-1})}{f'(x_{n-1})^2-f(x_{n-1})f''(x_{n-1})}\right), \tag{13}$$

We observe the expression (13) and find that the formula is identical to the classical Schroder method given by (3).

3.2 THE OSADA *m*-FREE METHOD

The second of the new multiplicity-free iterative method is based on the Osada third order method [4]. The original Osada third order method is given as

$$x_n = x_{n-1} - \left(\frac{m}{2}\right)(m+1)\left(\frac{f(x_{n-1})}{f'(x_{n-1})}\right) + \left(\frac{1}{2}\right)(m-1)^2\left(\frac{f'(x_{n-1})}{f''(x_{n-1})}\right), \tag{14}$$

and we express the new multiplicity-free iterative method as,

$$x_n = x_n - \left(\frac{\widehat{m}}{2}\right)(\widehat{m}+1)\left(\frac{f(x_{n-1})}{f'(x_{n-1})}\right) + \left(\frac{1}{2}\right)(\widehat{m}-1)^2\left(\frac{f'(x_{n-1})}{f''(x_{n-1})}\right), \tag{15}$$

where $\widehat{m}$ is given by (4), x_0 is the initial approximation, provided that the denominators of (15) are not equal to zero. We shall show that the (15) is equivalent to the classical Schroder method (3).

We begin by replacing the *m* with $\widehat{m}$ into (15), we obtain

$$x_n = x_{n-1} - \left(\frac{1}{2}\right)\left(\frac{f'(x_{n-1})^2}{f'(x_{n-1})^2 - f(x_{n-1})f''(x_{n-1})}\right)\left(\frac{f'(x_{n-1})^2}{f'(x_{n-1})^2 - f(x_{n-1})f''(x_{n-1})}+1\right)\left(\frac{f(x_{n-1})}{f'(x_{n-1})}\right)$$
$$+\left(\frac{1}{2}\right)\left(\frac{f'(x_{n-1})^2}{f'(x_{n-1})^2 - f(x_{n-1})f''(x_{n-1})}-1\right)^2\left(\frac{f'(x_{n-1})}{f''(x_{n-1})}\right) \tag{16}$$

Simplifying (16), we have

$$x_n = x_{n-1} - \left(\frac{1}{2}\right)\left(f'(x_{n-1})^2 - f(x_{n-1})f''(x_{n-1})\right)^{-2}\Big[f'(x_{n-1})^2\big(2f'(x_{n-1})^2$$
$$-f(x_{n-1})f''(x_{n-1})\big)\left(\frac{f(x_{n-1})}{f'(x_{n-1})}\right) - \left(f(x_{n-1})f''(x_{n-1})\right)^2\left(\frac{f'(x_{n-1})}{f''(x_{n-1})}\right)\Big] \tag{17}$$

Further simplification of (17) becomes

$$x_n = x_{n-1} - \left(\frac{1}{2}\right)\left(f'(x_{n-1})^2 - f(x_{n-1})f''(x_{n-1})\right)^{-2}\Big[\big(2f(x_{n-1})f'(x_{n-1})^3$$
$$-f(x_{n-1})^2 f'(x_{n-1})f''(x_{n-1})\big) - f(x_{n-1})^2 f'(x_{n-1})f''(x_{n-1})\Big] \tag{18}$$

Collecting appropriates terms of (18), we obtain

$$x_n = x_{n-1} - \left(\frac{1}{2}\right)\left(f'(x_{n-1})^2 - f(x_{n-1})f''(x_{n-1})\right)^{-2}\Big[\big(2f(x_{n-1})f'(x_{n-1})^3 - 2f(x_{n-1})^2 f'(x_{n-1})f''(x_{n-1})\Big] \tag{19}$$

Simplifying (19), we find

$$x_n = x_{n-1} - \left(\frac{1}{2}\right)\left(f'(x_{n-1})^2 - f(x_{n-1})f''(x_{n-1})\right)^{-2}\Big[2f(x_{n-1})f'(x_{n-1})\big(f'(x_{n-1})^2 - f(x_{n-1})f''(x_{n-1})\big)\Big] \tag{20}$$

and in final form,

$$x_n = x_n - f(x_{n-1}) f'(x_{n-1}) \left(f'(x_{n-1})^2 - f(x_{n-1}) f''(x_{n-1}) \right)^{-1} \quad (21)$$

Here also, we find that the equation (21) and (3) are identical. Hence the Osada third order method is identical to the classical Schroder method.

3.3 THE HALLEY *m*-FREE METHOD

The third of the new *m*-free method is based on the Halley third order method [6]. The original Halley third order method is given as

$$x_n = x_{n-1} - \left(\frac{f(x_{n-1})}{\frac{m+1}{2m} f'(x_{n-1}) - \frac{f(x_{n-1}) f''(x_{n-1})}{2 f'(x_{n-1})}} \right), \quad (22)$$

Replacing the m with $\widehat{m}$, we obtain

$$x_n = x_{n-1} - \left(\frac{f(x_{n-1})}{\frac{\widehat{m}+1}{2\widehat{m}} f'(x_{n-1}) - \frac{f(x_{n-1}) f''(x_{n-1})}{2 f'(x_{n-1})}} \right), \quad (23)$$

In alternative forms, the equation (23) becomes

$$x_n = x_{n-1} - \left(\frac{2\widehat{m} f(x_{n-1}) f'(x_{n-1})}{(\widehat{m}+1) f'(x_{n-1})^2 - \widehat{m} f(x_{n-1}) f''(x_{n-1})} \right), \quad (24)$$

$$x_n = x_{n-1} - \left(2\widehat{m} f(x_{n-1}) f'(x_{n-1}) \right) \left[(\widehat{m}+1) f'(x_{n-1})^2 - \widehat{m} f(x_{n-1}) f''(x_{n-1}) \right]^{-1} \quad (25)$$

Substituting (4) into (25), we attain

$$x_n = x_{n-1} - \left(2 \left(\frac{f'(x_{n-1})^2}{f'(x_{n-1})^2 - f(x_{n-1}) f''(x_{n-1})} \right) f(x_{n-1}) f'(x_{n-1}) \right) \left[\left(\left(\frac{f'(x_{n-1})^2}{f'(x_{n-1})^2 - f(x_{n-1}) f''(x_{n-1})} \right) + 1 \right) \times f'(x_{n-1})^2 - \left(\frac{f'(x_{n-1})^2}{f'(x_{n-1})^2 - f(x_{n-1}) f''(x_{n-1})} \right) f(x_{n-1}) f''(x_{n-1}) \right]^{-1} \quad (26)$$

Collecting appropriates terms of (26), we obtain

$$x_n = x_{n-1} - \left(2 f(x_{n-1}) f'(x_{n-1})^3 \right) \left[\left(2 f'(x_{n-1})^2 - f(x_{n-1}) f''(x_{n-1}) \right) f'(x_{n-1})^2 - f(x_{n-1}) f'(x_{n-1})^2 f''(x_{n-1}) \right]^{-1} \quad (27)$$

Simplifying (27), we find

$$x_n = x_{n-1} - \left(2 f(x_{n-1}) f'(x_{n-1}) \right) \left[\left(2 f'(x_{n-1})^2 - f(x_{n-1}) f''(x_{n-1}) \right) - f(x_{n-1}) f''(x_{n-1}) \right]^{-1} \quad (28)$$

and in final form,

$$x_n = x_{n-1} - \left(f(x_{n-1}) f'(x_{n-1}) \right) \left[f'(x_{n-1})^2 - f(x_{n-1}) f''(x_{n-1}) \right]^{-1} \quad (29)$$

x_0 is the initial approximation, provided that the denominators of (19) are not equal to zero. Here also, we find that the equation (29) and (3) are identical. Hence the Halley third order method is identical to the classical Schroder method.

ISSN 2347-1921

3.4 THE EULER-CHEBYSHEV *m*-FREE METHOD

The fourth of the new *m*-free method is based on the Euler-Chebyshev third order method [6]. The original Euler-Chebyshev third order method is given as

$$x_n = x_{n-1} - \left(\frac{m}{2}\right)(3-m)\left(\frac{f(x_{n-1})}{f'(x_{n-1})}\right) - \left(\frac{m^2}{2}\right)\left(\frac{f(x_{n-1})^2 f''(x_{n-1})}{f'(x_{n-1})^3}\right), \quad (30)$$

The new *m*-free iterative method is given as,

$$x_n = x_{n-1} - \left(\frac{\widehat{m}}{2}\right)(3-\widehat{m})\left(\frac{f(x_{n-1})}{f'(x_{n-1})}\right) - \left(\frac{\widehat{m}^2}{2}\right)\left(\frac{f(x_{n-1})^2 f''(x_{n-1})}{f'(x_{n-1})^3}\right), \quad (31)$$

where *m* is given by (4), x_0 is the initial approximation, provided that the denominators of (31) are not equal to zero. We shall show that the (31) is equivalent to the classical Schroder method.

We begin by replacing the *m* with $\widehat{m}$

$$x_n = x_{n-1} - \left(\frac{1}{2}\right)\left(\frac{f'(x_{n-1})^2}{f'(x_{n-1})^2 - f(x_{n-1})f''(x_{n-1})}\right)\left(3-\left(\frac{f'(x_{n-1})^2}{f'(x_{n-1})^2 - f(x_{n-1})f''(x_{n-1})}\right)\right)\left(\frac{f(x_{n-1})}{f'(x_{n-1})}\right)$$
$$-\left(\frac{1}{2}\right)\left(\frac{f'(x_{n-1})^2}{f'(x_{n-1})^2 - f(x_{n-1})f''(x_{n-1})}\right)^2\left(\frac{f(x_{n-1})^2 f''(x_{n-1})}{f'(x_{n-1})^3}\right) \quad (32)$$

Collecting appropriates terms, we obtain

$$x_n = x_{n-1} - \left(\frac{1}{2}\right)\left(f'(x_{n-1})^2 - f(x_{n-1})f''(x_{n-1})\right)^{-2}\Big[f'(x_{n-1})^2\left(f'(x_{n-1})^2\right)$$
$$-3f(x_{n-1})f''(x_{n-1})\Big)\left(\frac{f(x_{n-1})}{f'(x_{n-1})}\right) + \left(f'(x_{n-1})^4\right)\left(\frac{f(x_{n-1})^2 f''(x_{n-1})}{f'(x_{n-1})^3}\right)\Big] \quad (33)$$

Simplifying (33),

$$x_n = x_{n-1} - \left(\frac{1}{2}\right)\left(f'(x_{n-1})^2 - f(x_{n-1})f''(x_{n-1})\right)^{-2}\Big[\Big(2f(x_{n-1})f'(x_{n-1})^3$$
$$-3f(x_{n-1})^2 f'(x_{n-1})f''(x_{n-1})\Big) + f(x_{n-1})^2 f'(x_{n-1})f''(x_{n-1})\Big] \quad (34)$$

Further simplification of (34), we obtain

$$x_n = x_{n-1} - \left(\frac{1}{2}\right)\left(f'(x_{n-1})^2 - f(x_{n-1})f''(x_{n-1})\right)^{-2}\Big[\Big(2f(x_{n-1})f'(x_{n-1})^3 - 2f(x_{n-1})^2 f'(x_{n-1})f''(x_{n-1})\Big] \quad (35)$$

and in final form

$$x_n = x_{n-1} - f(x_{n-1})f'(x_{n-1})\left(f'(x_{n-1})^2 - f(x_{n-1})f''(x_{n-1})\right)^{-1} \quad (36)$$

Here also, we find that the equation (36) and (3) are identical. Hence the Euler-Chebyshev third order method is identical to the classical Schroder method.

3.5 THE CHUN-NETA *m*-FREE METHOD

The fifth of the new *m*-free method is based on the Chun-Neta third order method [1]. The original Chun-Neta third order method is given as

$$x_n = x_{n-1} - \left(\frac{2m^2 f(x_{n-1})^2 f''(x_{n-1})}{m(3-m) f(x_{n-1}) f'(x_{n-1}) f''(x_{n-1}) + (m-1)^2 f'(x_{n-1})^3} \right), \tag{37}$$

Replacing the m with $\widehat{m}$, we obtain

$$x_n = x_{n-1} - \left(\frac{2\widehat{m}^2 f(x_{n-1})^2 f''(x_{n-1})}{\widehat{m}(3-\widehat{m}) f(x_{n-1}) f'(x_{n-1}) f''(x_{n-1}) + (\widehat{m}-1)^2 f'(x_{n-1})^3} \right), \tag{38}$$

In alternative forms, the equation (38) becomes

$$x_n = x_{n-1} - \left(2\widehat{m}^2 f(x_{n-1})^2 f''(x_{n-1})\right)\left[\widehat{m}(3-\widehat{m}) f(x_{n-1}) f'(x_{n-1}) f''(x_{n-1}) + (\widehat{m}-1)^2 f'(x_{n-1})^3\right]^{-1} \tag{39}$$

Substituting the $\widehat{m}$ into (39), we obtain

$$\begin{aligned} x_n = x_{n-1} - &\left(2\left(\frac{f'(x_{n-1})^2}{f'(x_{n-1})^2 - f(x_{n-1}) f''(x_{n-1})} \right)^2 f(x_{n-1})^2 f''(x_{n-1}) \right) \\ &\times \left[\left(\frac{f'(x_{n-1})^2}{f'(x_{n-1})^2 - f(x_{n-1}) f''(x_{n-1})} \right) \left(3 - \left(\frac{f'(x_{n-1})^2}{f'(x_{n-1})^2 - f(x_{n-1}) f''(x_{n-1})} \right) \right) f(x_{n-1}) f'(x_{n-1}) f''(x_{n-1}) \right. \\ &\quad \left. + \left(\left(\frac{f'(x_{n-1})^2}{f'(x_{n-1})^2 - f(x_{n-1}) f''(x_{n-1})} \right) - 1 \right)^2 f'(x_{n-1})^3 \right]^{-1} \end{aligned} \tag{40}$$

Expanding (40), we attain

$$\begin{aligned} x_n = x_{n-1} - &\left(2 f'(x_{n-1})^4 f(x_{n-1})^2 f''(x_{n-1})\right)\left[\left(f'(x_{n-1})^2\right)\left(2 f'(x_{n-1})^2 - 3 f(x_{n-1}) f''(x_{n-1})\right)\right. \\ &\left. \times f(x_{n-1}) f'(x_{n-1}) f''(x_{n-1}) + \left(f(x_{n-1}) f''(x_{n-1})\right)^2 f'(x_{n-1})^3\right]^{-1} \end{aligned} \tag{41}$$

Collecting appropriates terms of (41), we obtain

$$\begin{aligned} x_n = x_{n-1} - &\left(2 f'(x_{n-1})^4 f(x_{n-1})^2 f''(x_{n-1})\right)\left[\left(2 f'(x_{n-1})^4 - 3 f(x_{n-1}) f'(x_{n-1})^2 f''(x_{n-1})\right)\right. \\ &\left. \times f(x_{n-1}) f'(x_{n-1}) f''(x_{n-1}) + \left(f(x_{n-1}) f''(x_{n-1})\right)^2 f'(x_{n-1})^3\right]^{-1} \end{aligned} \tag{42}$$

Simplifying (42),

$$\begin{aligned} x_n = x_{n-1} - &\left(2 f'(x_{n-1})^4 f(x_{n-1})^2 f''(x_{n-1})\right) \times \left[2 f(x_{n-1}) f'(x_{n-1})^5 f''(x_{n-1})\right. \\ &\left. - 3 f(x_{n-1})^2 f'(x_{n-1})^3 f''(x_{n-1})^2 + f(x_{n-1})^2 f'(x_{n-1})^3 f''(x_{n-1})^2\right]^{-1} \end{aligned} \tag{43}$$

Further simplification of (43), we obtain

$$\begin{aligned} x_n = x_{n-1} - &\left(2 f'(x_{n-1})^4 f(x_{n-1})^2 f''(x_{n-1})\right) \times \left[2 f(x_{n-1}) f'(x_{n-1})^5 f''(x_{n-1})\right. \\ &\left. - 2 f(x_{n-1})^2 f'(x_{n-1})^3 f''(x_{n-1})^2\right]^{-1} \end{aligned} \tag{44}$$

and in final form

ISSN 2347-1921

$$x_n = x_{n-1} - \left(f(x_{n-1}) f'(x_{n-1}) \right) \left(f'(x_{n-1})^2 - f(x_{n-1}) f''(x_{n-1}) \right)^{-1} \quad (45)$$

Here also, we find that the equation (45) and (3) are identical. Hence the Chun-Neta third order method is identical to the classical Schroder method.

4 APPLICATION OF THE ITERATIVE METHODS

Since the five new *m*-free methods described are equivalent to the classical Schroder method, and the results are identical, hence we shall only display the results of the classical Schroder method. The classical Schroder method is given by (3) and is employed to solve nonlinear equations and compare with the modified Newton method. To demonstrate the performance of these methods, we use ten particular nonlinear equations. In addition, we display the approximation of the multiplicity *m*. We shall determine the consistency and stability of results by examining the convergence of the new iterative methods. The findings are generalised by illustrating the effectiveness of these methods for determining the multiple roots of a nonlinear equation. Consequently, we give estimates of the approximate solutions produced by these methods and list the errors obtained by each of the methods. The numerical computations listed in the tables were performed on an algebraic system called Maple. In fact, the errors displayed are of absolute value and insignificant approximations by the various methods have been omitted in the following tables.

The new *m*-free methods require three function evaluations and has the order of convergence two. To determine the efficiency index of the new method, we shall use the definition 3. Hence, the efficiency index of the *m*-free methods given is $\sqrt[3]{2} \approx 1.26,$ which is identical to the classical Schroder methods. The test functions and their exact root α are displayed in table 1. The difference between the root α and the approximation x_n, for test functions with initial guess x_0 , are displayed in Table 1. In fact, x_n is calculated by using the same total number of function evaluations (TNFE) for all methods. Furthermore, the computational order of convergence and approximation of multiplicity *m* are displayed in Table 2.

Table 1 Test functions and their roots.

Functions	***m***	***Roots***	***Initial guess***
$f_1(x) = \left(x^3 + 4x^2 - 10\right)^m$	$m = 8$	$\alpha = 1.365230...$	$x_0 = 1.5$
$f_2(x) = \left(xe^{x^2} - \sin(x)^2 + 3\cos(x) + 5\right)^m$	$m = 5$	$\alpha = -1.207647...$	$x_0 = -1.3$
$f_3(x) = \left((x-1)^3 - 1\right)^m$	$m = 6$	$\alpha = 2$	$x_0 = 1.8$
$f_4(x) = \left(\exp(x) + x - 20\right)^m$	$m = 9$	$\alpha = 2.842438...$	$x_0 = 2.6$
$f_5(x) = \left(\cos(x) + x\right)^m$	$m = 15$	$\alpha = -0.739085...$	$x_0 = -0.9$
$f_6(x) = \left(\sin(x)^3 - x^2 + 1\right)^m$	$m = 66$	$\alpha = 1.404491..$	$x_0 = 1.6$
$f_7(x) = \left(e^{-x^2} - e^{x^2} - x^8 + 10\right)^m$	$m = 3$	$\alpha = 1.239417...$	$x_0 = 1.3$
$f_8(x) = \left(6x^5 + 5x^4 - 4x^3 + 3x^2 - 2x + 1\right)^m$	$m = 11$	$\alpha = -1.57248...$	$x_0 = -1.7$
$f_9(x) = \left(\tan(x) - e^x - 1\right)^m$	$m = 39$	$\alpha = 1.371045...$	$x_0 = 1.4$
$f_{10}(x) = \left(x^6 - 1\right)^m$	$m = 2$	$\alpha = -1..$	$x_0 = -0.9$

Table 2 Comparison of iterative methods

f_i	Newton $\lvert\alpha - x_3\rvert$	Schroder $\lvert\alpha - x_3\rvert$	Newton COC	Schroder COC	Multiplicity m	Multiplicity $\widehat{m}$
f_1	0.924e-1	0.572e-9	1.0069	2.0007	8	7.9997
f_2	0.784e-7	0.210e-6	1.9989	1.9904	5	4.9944
f_3	0.102	0.419e-5	0.9555	2.0089	6	5.9755
f_4	0.808e-7	0.424e-7	1.9981	1.9978	9	8.9975
f_5	0.569e-11	0.332e-11	1.9992	2.0010	15	15.000
f_6	0.188	0.134e-6	0.9985	2.0060	66	65.957
f_7	0.687e-7	0.143e-6	1.9961	1.9968	3	2.9964
f_8	0.101	0.847e-5	1.0127	2.0097	11	10.907
f_9	0.882e-7	0.318e-6	1.9984	1.9839	10	9.9730
f_{10}	0.738e-2	0.489e-5	0.9294	1.9969	2	1.9860

5 CONCLUSION

In this paper, we have introduced a new formula for approximating the multiplicity *m* of nonlinear equations with multiple roots. Also, we have demonstrated the similarity between the classical Schroder method and the modified Newton methods, the Osada third order method, the Halley third order method, the Euler-Chebyshev third order method and the Chun-Neta third order method. By simply using the new formula (4) we have obtained new variants of the Schroder method. The prime motive for presenting these new *m*-free methods was to design new *m*-free methods. We have examined the effectiveness of the new *m*-free methods by showing the accuracy of the multiple roots of several nonlinear equations. Furthermore, in all of the test examples, empirically we have found that the best results of the new *m*-free methods are obtained when multiplicity is unknown. The major advantage of the new *m*-free methods is that we do not need to know the multiplicity of the nonlinear equation and the disadvantage is that we have to calculate a second order derivative. Finally, the subject of further research is to produce a higher order convergence of the Schroder-type method.

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