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On pseudo-slant submanifolds of nearly trans-Sasakian manifolds

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ABSTRACT

This paper consist the study of pseudo-slant submanifolds of nearly trans-Sasakian manifolds. Integrability conditions of the distributions on these submanifolds are worked out. Some interesting results regarding such manifolds have also been deduced. An example of a pseudo-slant submanifolds of nearly trans-Sasakian manifold is constracted.

Indexing terms/Keywords

Nearly trans-Sasakian manifolds; contact metric manifolds; pseudo-slant submanifolds.

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1. INTRODUCTION

In 1990, Chen [3] introduced the concept of slant immersions as a generalization of both holomorphic and totally real immersions. Many authors have studied slant immersions in Hermitian manifolds. Lotta [12], introduced the notion of slant immersions in contact manifolds. In paper [1,2], slant submanifolds of $K-$contact and Sasakian manifolds have been characterized by Caberizo et al. Recently, Carriazo [4] defined and studied bi-slant immersions in almost Hermitian manifolds and simultaneously gave the notion of pseudo-slant submanifolds in almost Hermitian manifolds. The contact version of pseudo-slant submanifolds have been studied by V.A.Khan and M.A.Khan [5]. Slant submanifolds of trans-Sasakian manifolds have been study by Gupta et al. [6]. In 1985, Oubina introduced a new class of almost contact metric manifold known as trans-Sasakian manifold [7]. This class contains $\alpha-$Sasakian and $\beta-$Kenmotsu manifolds. Pseudo-slant submanifolds of trans-Sasakian manifolds have been studied by U.C.De and Avijit Sarkar [9]. A nearly trans-Sasakian manifold [10] is a more general concept. In this paper, we study pseudo-slant submanifolds of nearly trans-Sasakian manifolds. The present paper is organized as follows:

Section 1, is introductory. Preliminaries are given in section 2. In section 3, we have defined pseudo-slant submanifolds of nearly trans-Sasakian manifolds. Section 4, deals with integrability conditions of the distributions of such manifolds. Section 5, contains an example of pseudo-slant submanifolds of nearly trans-Sasakian manifolds.

2. PRELIMINARIES

Let $\widetilde{M}$ be a $(2n+1)-$dimensional almost contact metric manifold [11] with almost contact metric structure (ϕ,ξ,η,g), where ϕ is a (1,1) tensor field, ξ is a vector field, η is a 1-form and g is a compatible Riemannian metric on $\widetilde{M}$ such that

$$\phi^2=-I+\eta\otimes\xi, \tag{2.1}$$

$$g(\phi X,\phi Y)=g(X,Y)-\eta(X)\eta(Y), \tag{2.2}$$

$$g(\phi X,Y)=-g(X,\phi Y), g(X,\xi)=\eta(X), \tag{2.3}$$

for any $X,Y\in TM$.

An almost contact metric structure (ϕ,ξ,η,g) on $\widetilde{M}$ is called trans-Sasakian if

$$(\widetilde{\nabla}_X\phi)Y=\alpha\{g(X,Y)\xi-\eta(Y)X\}-\beta\{g(\phi X,Y)\xi-\eta(Y)\phi X\}, \tag{2.4}$$

where α and β are smooth functions and ∇ denotes the Riemannian connection of g on $\widetilde{M}$. Further, an almost contact metric manifold $\widetilde{M}$ (ϕ,ξ,η,g) is called nearly trans-Sasakian manifold if [10]

$$(\widetilde{\nabla}_X\phi)Y+(\widetilde{\nabla}_Y\phi)X=\alpha\{2g(X,Y)\xi-\eta(Y)X-\eta(X)Y\}-\beta\{\eta(Y)\phi X+\eta(X)\phi Y\}, \tag{2.5}$$

for certain function α and β on $\widetilde{M}$. If $\beta=0$, then the structure is called nearly $\alpha-$Sasakian. If $\alpha=0$, then the structure is called nearly $\beta-$Kenmotsu. If both α and β are zero, then the manifold reduces to be a neary cosympletic manifold [2]. If α and β are not simultaneously zero, then nearly trans-Sasakian manifold becomes proper nearly trans-Sasakian manifold. We know that a nearly trans-Sasakian structure satiesfies

$$\widetilde{\nabla}_X\xi=-\alpha\phi X+\beta(X-\eta(X)\xi)-\phi((\widetilde{\nabla}_\xi\phi)X), \tag{2.6}$$

for any $X\in T\widetilde{M}$ and ξ is the structure vector field.

Let M be a submanifold immersed in a $(2n+1)-$dimensional contact metric manifold $\widetilde{M}$. And g denote the same induced metric on M. TM is the tangent bundle of the manifold M and $T^{\perp}M$ is the set of vector fields normal to M. Then the Gauss and Weingarten formulae are given by

$$\widetilde{\nabla}_XY=\nabla_XY+h(X,Y), \tag{2.7}$$

$$\widetilde{\nabla}_X N = -A_N X + \nabla^{\perp}{}_X Y, \qquad (2.8)$$

for any $X, Y \in TM$ and $N \in T^{\perp}M$, where $\nabla^{\perp}$is the connection in the normal bundle. The second fundamental form h and A_N are related by

$$g\left(A_N X\right) = g(h(X,Y), N). \qquad (2.9)$$

For any X∈TM and $N \in T^{\perp}M$, we write

$$\phi X = TX + NX, (TX \in TM \text{ and } NX \in T^{\perp}M) \qquad (2.10)$$

$$\phi N = tX + nX, (tX \in TM \text{ and } nX \in T^{\perp}M) \qquad (2.11)$$

The submanifold M is invariant if N is identically zero. On the other hand, M is anti-invariant if T is identically zero. From (2.3) and (2.10), we have

$$g\left(X, TY\right) = -g\left(TX, Y\right), \qquad (2.12)$$

for any $X, Y \in TM$.

If we put $Q = T^2$, we have

$$\left(\nabla_X Q\right)Y = \nabla_X QY - Q\nabla_X Y, \qquad (2.13)$$

$$\left(\nabla_X T\right)Y = \nabla_X TY - T\nabla_X Y, \qquad (2.14)$$

$$\left(\nabla_X N\right)Y = \nabla^{\perp}{}_X NY - N\nabla_X Y, \qquad (2.15)$$

for any $X, Y \in TM$. In view of (2.6),(2.7) and (2.10), it follows that

$$\nabla_X \xi = -\alpha TX + \beta\left(X - \eta\left(X\right)\xi\right) - T\left(\left(\widetilde{\nabla}_{\xi}\phi\right)X\right), \qquad (2.16)$$

$$h\left(X, \xi\right) = -\alpha NX - N\left(\left(\widetilde{\nabla}_{\xi}\phi\right)X\right). \qquad (2.17)$$

3. Pseudo-slant submanifols of nearly trans-Sasakian manifolds

Definitions 3.1: We say that M is a pseudo-slant submanifolds of nearly trans-Sasakian manifold $\widetilde{M}$, if there exist two orthogonal distributions D_1 and D_2 on M such that [8]

(1) TM admits the orthogonal direct decomposition

$$TM = D_1 \oplus D_2 \oplus \langle \xi \rangle,$$

(2) the distribution D_1 is anti-invariant, that is

$$\phi D_1 \subseteq T^{\perp}M,$$

(3) the distribution D_2 is with slant angle $\theta \neq \frac{\pi}{2}$, that is, the angle between D_2 and $\phi\left(D_2\right)$is a constant θ.

From the above definition it is clear that if $\theta = 0$, then the pseudo-slant submanifold is a semi-invariant submanifold. On the other hand, if we denote the dimension of D_i by d_i for $i = 1, 2,$, then we find the following cases:

(a) If $d_2 = 0,$ then M is an anti-invariant submanifold.

(b) If $d_1 = 0,$ and $\theta = 0$, then M is an invariant submanifold.

(c) If $d_1 = 0$ and $\theta \neq 0,$, then M is a proper slant submanifold, with the slant angle $\theta \neq 0.$

A pseudo-slant submanifold is proper if $d_1 d_2 \neq 0$ and $\theta \neq 0.$

4. Inttgrability of Distributions

Theorem 4.1: Let M be a pseudo-slant submanifold of a nearly trans-Sasakian manifold $\widetilde{M}$. Then

$$\begin{aligned} &A_{\phi Y} X - A_{\phi X} Y = \nabla_X (TY) + h(X, TY) \\ &- A_{NY} X + \nabla^{\perp}_X (NY) - T(\nabla_X Y) \\ &- N(\nabla_X Y) - T\big(h(X,Y) - N(h(X,Y))\big), \end{aligned} \tag{4.1}$$

for all X,Y∈ D_1.

Proof: In view of (2.9),

$$g(A_{\phi Y} X, Z) = g(h(X,Z), \phi Y) = -g(\phi h(X,Z), Y). \tag{4.2}$$

By virtue of (2.7), (4.2) reduce to

$$g(A_{\phi Y} X, Z) = -g(\phi \widetilde{\nabla}_Z X, Y) + g(\phi \nabla_Z X, Y).$$

Since $\phi \nabla_Z X \in T^{\perp} M$

$$= -g(\phi \widetilde{\nabla}_Z X, Y),$$

$$= g((\widetilde{\nabla}_Z \phi) X, Y) - g(\widetilde{\nabla}_Z \phi X, Y). \tag{4.3}$$

Now, for $X \in D_1$, $\phi X \in T^{\perp} M$. Hence, from (2.8) we have

$$\widetilde{\nabla}_Z \phi X = -A_{\phi X} Z + \widetilde{\nabla}_Z \phi X. \tag{4.4}$$

Combining (4.3) and (4.4), we obtain

$$g(A_{\phi Z} X, Z) = g((\widetilde{\nabla}_Z \phi) X, Y) + g(A_{\phi X} Z, Y). \tag{4.5}$$

Since $h(X,Y) = h(Y,X)$, it follows from (2.9) that

$$g(A_{\phi X} Z, Y) = g(A_{\phi X} Y, Z).$$

Hence from (4.5) we obtain, with the help of (2.5)

$$g\left(A_{\phi Y}X,Z\right)-g\left(A_{\phi X}Y,Z\right)=2\alpha\eta\left(Y\right)g\left(Z,X\right)$$
$$-\alpha\eta\left(X\right)g\left(Z,Y\right)-\alpha\eta\left(Z\right)g\left(X,Y\right)-\beta\eta\left(X\right)g\left(\phi Z,Y\right) \quad (4.6)$$
$$-\eta\left(Z\right)g\left(\phi X,Y\right)+g\left(\left(\tilde{\nabla}_X\phi\right)Y,Z\right)$$

$$g\left(A_{\phi Y}X,Z\right)-g\left(A_{\phi X}Y,Z\right)=2\alpha\eta\left(Y\right)g\left(Z,X\right)$$
$$-\alpha\eta\left(X\right)g\left(Z,Y\right)-\alpha\eta\left(Z\right)g\left(X,Y\right)-\beta\eta\left(X\right)g\left(\phi Z,Y\right) \quad (4.7)$$
$$-\eta\left(Z\right)g\left(\phi X,Y\right)+g(\tilde{\nabla}_X\left(TY\right)+\tilde{\nabla}_X\left(NY\right)-\phi\left(\nabla_X Y\right)-\phi\left(h\left(X,Y\right)\right),Z),$$

$$g\left(A_{\phi Y}X,Z\right)-g\left(A_{\phi X}Y,Z\right)=2\alpha\eta\left(Y\right)g\left(Z,X\right)-\alpha\eta\left(X\right)g\left(Z,Y\right)-\alpha\eta\left(Z\right)g\left(X,Y\right)$$
$$-\beta\eta\left(X\right)g\left(\phi Z,Y\right)-\eta\left(Z\right)g\left(\phi X,Y\right)+g(\nabla_X\left(TY\right)+h\left(X,TY\right)-A_{NY}X+\nabla^{\perp}{}_X\left(NY\right) \quad (4.8)$$
$$-T\left(\nabla_X Y\right)-N\left(\nabla_X Y\right)-T\left(h\left(X,Y\right)-N\left(h\left(X,Y\right)\right)\right),Z),$$

Since $X,Y,Z\in D_1$, an orthonormal distribution to the distribution $\langle\xi\rangle$ it follows that $\eta\left(X\right)=\eta\left(Y\right)=0.$.Therefore, the above equation reduces to

$$A_{\phi Y}X-A_{\phi X}Y=\nabla_X\left(TY\right)+h\left(X,TY\right)$$
$$-A_{NY}X+\nabla^{\perp}{}_X\left(NY\right)-T\left(\nabla_X Y\right) \quad (4.9)$$
$$-N\left(\nabla_X Y\right)-N\left(h\left(X,Y\right)\right),$$

we get the theorem.

Remark 4.2: As particular cases the above result holds for nearly $\alpha-$Sasakian manifold, nearly $\beta-$Kenmotsu and nearly cosymplectic manifolds.

Since $h\left(X,Y\right)=h\left(Y,X\right)$, in view of (2.7), we see that

$$\nabla_X Y-\nabla_Y X=\tilde{\nabla}_X Y-\tilde{\nabla}_Y X. \quad (4.10)$$

Let $X\in D_1, Y\in D_2$, then

$$\left(\tilde{\nabla}_X g\right)\left(Y,Z\right)=\tilde{\nabla}_X g\left(Y,Z\right)-g\left(\tilde{\nabla}_X Y,Z\right)-g\left(Y,\tilde{\nabla}_X Z\right),$$

or,

$$0=0-g\left(\tilde{\nabla}_X Y,Z\right)-g\left(Y,\tilde{\nabla}_X Z\right).$$

$$g\left(\tilde{\nabla}_X Y,Z\right)=-g\left(Y,\tilde{\nabla}_X Z\right). \quad (4.11)$$

Theorem 4.3: Let M be a pseudo-slant submanifold of a nearly trans-Sasakian manifold $\tilde{M}$. Then for any $X,Y\in D_1\oplus D_2$.

$$g\left(\left[X,Y\right],\xi\right)=2ag\left(X,TY\right). \quad (4.12)$$

Proof: We have

$$g([X,Y],\xi)=g(\nabla_X Y,\xi)-g(\nabla_Y X,\xi). \quad (4.13)$$

In view of (4.11), we have from above

$$g([X,Y],\xi)=-g(\nabla_X\xi,Y)+g(\nabla_Y\xi,X). \quad (4.14)$$

By (2.16),(4.14) yields

$$g([X,Y],\xi)=2ag(X,TY).$$

The above equation gives the following:

Corollary 4.4: In a proper nearly trans-Sasakian manifold and nearly $\alpha-$Sasakian manifold the distribution $D_1\oplus D_2$ is not integrable.

Suppose $\alpha=0,$ that is, the manifold is nearly β-Kenmotsu. Then $g([X,Y],\xi)=0.$. This implies that $[X,Y]\in D_1\oplus D_2$ for $X,Y\in D_1\oplus D_2$. In other words, we have the following:

Corollary 4.5: In a nearly $\beta-$Kenmotsu manifold the distribution $D_1\oplus D_2$ is integrable.

Again, in a similar manner we have

Corollary 4.6: In a nearly cosympletic manifold the distribution $D_1\oplus D_2$ is integrable.

Theorem 4.7: Let M be a pseudo-slant submanifold of a nearly trans-Sasakian manifold $\widetilde{M}$. Then the anti-invariant distribution D_1 is not integrable.

Proof: For any $X\in TM$, let

$$X=P_1X+P_2X+\eta(X)\xi, \quad (4.15)$$

where $P_i, i=1,2$ are projection maps on the distibution D_i. From (4.15) it follows that

$$\phi X=NP_1X+TP_2X+NP_2X,$$

$$TX=TP_2X,\ NX=NP_1X+NP_2X.$$

Now for any $X,Y\in D_1$ and $Z\in D_2,$

$$g([X,Y],TZ)=g([X,Y],TP_2Z)=-g(\phi[X,Y],P_2Z). \quad (4.16)$$

Now

$$\phi[X,Y]=\phi\nabla_X Y-\phi\nabla_Y X, \quad (4.17)$$

$$=\phi\widetilde{\nabla}_X Y-\phi\widetilde{\nabla}_Y X,$$

$$=\widetilde{\nabla}_X\phi Y-(\widetilde{\nabla}_X\phi)Y-\widetilde{\nabla}_Y\phi X+(\widetilde{\nabla}_Y\phi)X.$$

In view of (2.5) and(2.8) and keeping in mind $g(U,V)=0$ for $U\in D_1$ and $V\in D_2$, we obtain from (4.16)

$$g\left(\left[X,Y\right],TP_2Z\right)=-g(-A_{\phi Y}X+A_{\phi X}Y$$
$$+\alpha\eta\left(Y\right)X+\alpha\eta\left(X\right)Y$$
$$+\beta\eta\left(Y\right)\phi X+\beta\eta\left(X\right)\phi Y$$
$$+2\left(\widetilde{\nabla}_X\phi\right)X,P_2Z). \tag{4.18}$$

For $X,Y\in D_1$, we get $\eta\left(X\right)=\eta\left(Y\right)=0$. Hence from the above equation, we have

$$g\left(\left[X,Y\right],TZ\right)=-g\left(-A_{\phi Y}X+A_{\phi X}Y+2\left(\widetilde{\nabla}_X\phi\right)X,P_2Z\right),$$

and $-A_{\phi Y}X+A_{\phi X}Y=-\left(\widetilde{\nabla}_X\phi\right)X,$

hence $g\left(\left[X,Y\right],TZ\right)=-g\left(\left(\widetilde{\nabla}_Y\phi\right)X,P_2Z\right),$

$$g\left(\left[X,Y\right],TZ\right)=-g\left(\nabla_Y\left(TX\right)\right)+h\left(Y,TX\right)-A_{NX}Y$$
$$+\nabla^{\perp}_Y\left(NX\right)-T\left(\nabla_YX\right)-N\left(\nabla_YX\right)$$
$$-T(h\left(Y,X\right)-N\left(h\left(Y,X\right),P_2Z\right),$$

$$g\left(\left[X,Y\right],TZ\right)=-g\left(\nabla_Y\left(TX\right),P_2Z\right)+g\left(A_{NX}Y,P_2Z\right)$$
$$+g\left(T\left(\nabla_YX\right),P_2Z\right)+g\left(T(h\left(Y,X\right),P_2Z\right).$$

Therefore the distribution D_1 is not integrable.

Corollary 4.8: On a pseudo-slant submanifold M of a nearly trans-Sasakian manifold $\widetilde{M}$, the distribution $D_2\oplus\langle\xi\rangle$, is not integrable.

Remark 4.9: The above result also holds for nearly cosympletic, nearly α-Sasakian and nearly β-Kenmotsu manifolds. For a Sasakian manifold the above result was proved by V.A.Khan and M.A.Khan [5].

Theorem 4.10: Let M be a pseudo-slant submanifold of a nearly trans-Sasakian manifold $\widetilde{M}$. Then the slant distribution D_2 is not integrable.

Proof: Since $g\left(\left[X,Y\right],\xi\right)=2\alpha g\left(X,TY\right)$, by the definition of pseudo-slant submanifold the proof follows.

From the above theorem

Corollary 4.11: In an nearly α-Sasakian manifold the slant distribution D_2 is not integrable.

Theorem 4.12: Let M be a submanifold of an almost contact metric manifold $\widetilde{M}$, such that $\xi\in TM$. Then M is a pseudo-slant submanifold if and only if there exists a constant $\lambda\in(0,1]$, such that

(a) $D=\{X\in TM,T^2X=-\lambda X\}$ is a distribution on M.

(b) For any $X\in TM$, orthogonal to $D,TX=0.$

Furthermore, in this case $\lambda=\cos^2\theta$, where θ denotes the slant angle of D.

Proof: Follows from [5].

Theorem 4.13: Let M be pseudo-slant submanifold of a nearly trans-Sasakian manifold $\widetilde{M}$. Then $\nabla Q=0,$, if and only if M is an anti-invariant submanifold.

Proof: Consider the distribution $D_2 \oplus \langle \xi \rangle$, then from theorem (4.12), we can write

$$T^2 X = -\lambda\left(X - \eta(X)\xi\right). \quad (4.19)$$

Denote by θ the slant angle of M . Then, replacing X by $\nabla_X Y$, we get from (4.19)

$$Q(\nabla_X Y) = -\cos^2\theta(\nabla_X Y) + \cos^2\theta\eta(\nabla_X Y)\xi, \quad (4.20)$$

for any $X, Y \in D_2 \oplus \langle \xi \rangle$.

Equation (4.19), also gives

$$\begin{aligned}\nabla_X QY = &-\cos^2\theta(\nabla_X Y) + \cos^2\theta\eta(\nabla_X Y)\xi \\ &+\cos^2\theta g(Y, \nabla_X \xi)\xi + \cos^2\theta\eta(Y)\nabla_X \xi,\end{aligned} \quad (4.21)$$

beacause

$$X\eta(Y) = \eta(\nabla_X Y) + g(Y, \nabla_X \xi).$$

Now, since M is a submanifold of a nearly trans-Sasakian manifold $\widetilde{M}$

$$\begin{aligned}\nabla_X \xi = &-\alpha TX + \beta\left(X - \eta(X)\xi\right) - T\left(\widetilde{\nabla}_\xi \phi\right)X, \\ &+\cos^2\theta g(Y, \nabla_X \xi) + \cos^2\theta\eta(Y)\nabla_X \xi\end{aligned} \quad (4.22)$$

for any $X \in TM$. Putting the value of $\nabla_X \xi$ in (4.21), we obtain

$$\begin{aligned}\nabla_X QY = &-\cos^2\theta(\nabla_X Y) + \cos^2\theta\eta(\nabla_X Y)\xi \\ &-\alpha\cos^2\theta g(Y, TX)\xi + \beta\cos^2\theta g(X, Y)\xi \\ &-\beta\eta(X)\cos^2\theta\eta(Y)\xi - g\left(Y, T\left(\nabla_\xi \phi\right)X\right)\xi \\ &-\alpha\eta(Y)\cos^2\theta TX + \beta\cos^2\theta\eta(Y)X \\ &-\beta\cos^2\theta\eta(Y)X - \beta\cos^2\theta\eta(Y)\eta(X)\xi \\ &-\cos^2\theta\eta(Y)T\left(\nabla_\xi \phi\right)X.\end{aligned} \quad (4.23)$$

Combining (4.20) and (4.23), we find

$$\begin{aligned}(\nabla_X Q)Y = &-\alpha\cos^2\left(g(Y, TX)\xi + \eta(Y)TX\right) \\ &+\beta\cos^2(g(X, Y)\xi - 2\eta(X)\eta(Y)\xi + \eta(Y)X)) \\ &-g\left(T\left(\nabla_\xi \phi\right)X, Y\right)\xi\cos^2\theta - \cos^2\theta T\left(\nabla_\xi \phi\right)X,\end{aligned}$$

for any $X, Y \in D_2 \oplus \langle \xi \rangle$. Here, we note that

$$g(X, Y)\xi - 2\eta(X)\eta(Y)\xi + \eta(Y)X \neq 0$$

Hence $\nabla Q = 0$, if and only if $\theta = \dfrac{\pi}{2}$ holds in $D_2 \oplus \langle \xi \rangle$. Again D_1 is anti-invariant by definition. Thus, the theorem follows.

As a consequence of theorem (4.13) we obtain the following:

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Corollary 4.14: In a pseudo-slant submanifold of a nearly α-Sasakian manifold $\nabla Q = 0,$ if and only if the submanifold is anti-invariant.

Corollary 4.15: In a pseudo-slant submanifold of a nearly $\beta-$kenmotsu manifold $\nabla Q = 0,$, if and only if the submanifold is anti-invariant.

But for a cosymplectic manifold, we have

Corollary 4.16: In a pseudo-slant submanifold of a nearly cosympletic manifold ∇Q is alwayas zero, whether the submanifold is anti-invariant or not.

Theorem 4.17: Let M be a submanifold of an almost contact metric manifold $\widetilde{M}$ with a slant angle θ. Then at each point $x \in M, Q/_D$ has only one eigenvalue $\lambda = \cos^2\theta$, for the slant distribution D of M.

Proof: Follows from [12].

Theorem4.18: In a a pseudo-slant submanifold of a nearly trans-Sasakian manifold

$$\begin{aligned}(\nabla_X T)Y &= A_{NY}X + A_{NX}Y + th(X,Y) - \alpha\big(g(X,Y)\xi - \eta(Y)X\big)\\ &+\beta\big(g(TX,Y)\xi - \eta(Y)TX\big)\\ &-\nabla_Y(TX) + T(\nabla_Y X) + T\big(h(Y,X)\big).\\ &-T(h(Y,X)) - N\big(h(Y,X),P_2 Z\big),\end{aligned} \tag{4.24}$$

Proof: For any $X, Y \in TM$ we have

$$\widetilde{\nabla}_X\phi Y = \widetilde{\nabla}_X\phi Y - \phi\big(\widetilde{\nabla}_X Y\big).$$

By (2.7) and(2.10), we have from above

$$\widetilde{\nabla}_X TY + \widetilde{\nabla}_X NY = \big(\widetilde{\nabla}_X\phi\big)Y + \phi\big(\nabla_X Y + h(X,Y)\big).$$

Again, by (2.10) and (2.11)

$$\widetilde{\nabla}_X TY + \widetilde{\nabla}_X NY = \big(\widetilde{\nabla}_X\phi\big)Y + T\nabla_X Y + N\nabla_X Y + th(X,Y) + nh(X,Y).$$

Using (2.7) and (2.8) from above, we get

$$\begin{aligned}\nabla_X TY + h(X,TY) - A_{NY}X + \nabla^{\perp}{}_X NY &= \alpha\big(2g(X,Y)\xi - \eta(Y)X - \eta(X)Y\big)\\ &-\beta\big(\eta(Y)\phi X + \eta(X)\phi Y\big) + T(\nabla_X Y)\\ &+N(\nabla_X Y) + th(X,Y) + nh(X,Y)\\ &-\nabla_Y(TX) - h(Y,TX) + A_{NX}Y\\ &-\nabla^{\perp}{}_Y(NX) + T(\nabla_Y X) + N(\nabla_Y X)\\ &+T\big(h(Y,X)\big) + N\big(h(Y,X)\big).\end{aligned} \tag{4.25}$$

Comparing tangential and normal parts, we have

$$\begin{aligned}\nabla_X TY - A_{NY}X &= \alpha\big(2g(X,Y)\xi - \eta(Y)X - \eta(X)Y\big)\\ &-\beta\big(\eta(Y)\phi X + \eta(X)\phi Y\big)\\ &+T(\nabla_X Y) + th(X,Y) + \nabla_Y(TX)\\ &+A_{NX}Y + T(\nabla_Y X) + T\big(h(Y,X)\big)\end{aligned} \tag{4.26}$$

That is,

$$\begin{aligned}(\nabla_X T)Y &= A_{NY}X + A_{NX}Y + th(X,Y)\\ &+\alpha\left(2g(X,Y)\xi - \eta(Y)X - \eta(X)Y\right)\\ &-\beta\left(\eta(Y)\phi X + \eta(X)\phi Y\right)\\ &-\nabla_Y(TX) + T(\nabla_Y X) + T\left(h(X,Y)\right).\end{aligned} \tag{4.27}$$

As a consequence of the above theorem we obtain the following:

Corollary 4.19: In a pseudo-slant submanifold of a nearly $\alpha-$Sasakian manifold

$$\begin{aligned}(\nabla_X T)Y &= A_{NY}X + A_{NX}Y + th(X,Y) + \alpha\left(2g(X,Y)\xi - \eta(Y)X - \eta(X)Y\right)\\ &-\nabla_Y(TX) + T(\nabla_Y X) + T\left(h(X,Y)\right).\end{aligned} \tag{4.28}$$

Corollary 4.20: In a pseudo-slant submanifold of a nearly $\beta-$Kenmotsu manifold

$$\begin{aligned}(\nabla_X T)Y &= A_{NY}X + A_{NX}Y + th(X,Y) - \beta\left(\eta(Y)\phi X + \eta(X)\phi Y\right)\\ &-\nabla_Y(TX) + T(\nabla_Y X) + T\left(h(X,Y)\right).\end{aligned} \tag{4.29}$$

Corollary 4.21: In a pseudo-slant submanifold of a nearly cosympletic manifold

$$(\nabla_X T)Y = A_{NY}X + A_{NX}Y + th(X,Y) - \nabla_Y(TX) + T(\nabla_Y X) + T\left(h(X,Y)\right). \tag{4.30}$$

Example

From [7] we know that R^{2n+1} admits a nearly trans-Sasakian structure. Now consider an example of a three-dimensional submanifold of a nearly trans-Sasakian manifold.

Let (X,Y,Z) be Cartesian coordinates of R³ and put

$$\xi = \frac{\partial}{\partial z},$$

$$\eta = dz - ydx,$$

$$\phi = \begin{pmatrix} 0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix},$$

$$g = \begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{pmatrix}.$$

Then $\delta\varphi(\xi) = -1, \delta\eta = -1$ and (ϕ,ξ,η,g) is a nearly trans-Sasakian structure on R^3

of type $\left(-\frac{1}{2},\frac{1}{2}\right)$ [2]. The vector fields

$$e_1 = \frac{\partial}{\partial z}, e_2 = \frac{\partial}{\partial y}, e_3 = \frac{\partial}{\partial x},$$

form an orthonormal frame of TM. We see that $\phi e_1 = 0, \phi e_2 = 0, \phi e_3 = e_2.$

Let $D_1 = \langle e_2 \rangle, D_2 = \langle e_3 \rangle, \langle \xi \rangle = \langle e_1 \rangle$. Suppose $X \in D_1$ and $Y \in TM$. Then we can write $X = ke_2, k$ is a scalar and $Y = re_1 + se_2 + te_3$, where r, s, t are scalars. Now $\cos\angle(\phi X, Y) = \frac{g(\phi X, Y)}{|\phi X||Y|}$. From the component of the metric g see that $g(\phi X, Y) = krg(\phi e_2, e_1) + ks\,g(\phi e_2, e_2) + ktg(\phi e_2, e_3) = 0.$ Hence, the distribution D_1 is anti-invariant.

Again, let us suppose $U \in D_2$ and $V \in TM$.. then we can write $U = ce_3, c$ is a scalar and $V = ke_1 + le_2 + me_3$, where k, l, m are scalars. Now $\cos\angle(\phi U, V) = \frac{g(\phi U, V)}{|\phi U||V|}$. We see that

$$g(\phi U, V) = ckg(\phi e_3, e_1) + c\,\lg(\phi e_3, e_2) + cmg(\phi e_3, e_3) = cl.$$

Therefore $\cos\angle(\phi U, V) = \frac{cl}{|c\phi e_3||ke_1 + le_2 + me_3|}$. which is constant. We see that the distribution D_2 is slant.

In this case, the distribution D_1 is anti-invariant while the distribution D_2 is slant. Hence the submanifold under consideration is pseudo-slant.

Conclusion:

A nearly trans-Sasakian manifolds of type (α, β), generalize both nearly $\alpha-$Sasakian of type $(\alpha, 0)$ and nearly $\beta-$Kenmotsu of type $(0, \beta)$. In this paper we consider the direct orthogonal decomposition of tangent bundle TM as $TM = D_1 \oplus D_2 \oplus \langle \xi \rangle$, where D_1 is anti-invariant distribution, D_2 is the slant distribution with slant angle $\theta \neq \frac{\pi}{2}$, that is, the angle between D_2 and $\varphi(D_2)$ is a constant θ. A pseudo-slant submanifold for which $\theta = 0$, extend the notion of semi-invariant submanifold. We mainly show that the distributions $D_1 \oplus \langle \xi \rangle, D_1$ and D_2 are not integrable. A necessary and sufficient condition for a pseudo-slant submanifold to be anti-invariant is obtained. An example of a pseudo-slant submanifold of a nearly trans-Sasakian manifold is constracted. In manny branches of applied mathematics submanifold theory has an important role. The results obtained in this paper can be used in many problems of dynamical system and critical point theory.

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