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On Boundary Value Problems for Second-order Fuzzy Linear Differential Equations with Constant Coefficients

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ABSTRACT

In this paper we investigate the solutions of boundary value problems for second-order fuzzy linear differential equations with constant coefficients. There are four different solutions for the problems by using a generalized differentiability. Solutions and several comparison results are presented. Some examples are provided for which the solutions are found.

Keywords:

Fuzzy boundary value problems; Second order fuzzy differential equations; Generalized differentiability.

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1. INTRODUCTION

There are several approaches to studying fuzzy differential equations [1,4,5,8,12,14].The first approach was the use of Hukuhara derivative for fuzzy-number-valued functions. This approach has a drawback: the solution becomes fuzzier as time goes by [2,7]. Hence, the solution behaves quite differently from crisp solution. To solve the drawback, Bede and Gal [2] introduced a generalized definition of fuzzy derivative for fuzzy-number-valued function. He showed that the new generalization allows us to have $f^{'}(x)=cg^{'}(x)$ for all $x\in(a,b)$ when $g:[a,b]\to\mathbb{R}$ is differentiable and $f(x)=cg(x)$, where c is a fuzzy number.

O' Regan et. al. [15] showed that a two-point fuzzy boundary value problem is equivalent to a fuzzy integral equation. Bede [3] presented a counterexample to show that this statement does not hold. Also, Bede proved that a large class of fuzzy two-point boundary value problems cannot have a solution under Hukuhara derivative concept.

In this paper, a investigation is made on the solution of two-point fuzzy boundary value problems by using generalized differentiability.

As the fuzzy boundary value problems are given as the form

(1) $$y^{''}(t)=\lambda y(t),\ y(0)=A\ ,\ y^{'}(\ell)=B$$

(2) $$y^{''}(t)=-\lambda y(t),\ y(0)=A\ ,\ y^{'}(\ell)=B$$

fuzzy solutions are developed, where $t\in T=[0,\ell]$, A and B are symmetric triangle fuzzy numbers. We show that all solutions are symmetric triangle fuzzy functions of t but that some solutions are no longer a valid fuzzy level set. Several examples are presented.

2. PRELIMINARIES

In this section, we give some definitions and introduce the necessary notation which will be used throughout the paper.

Definition 2.1

A fuzzy number is a function $u:\mathbb{R}\to[0,1]$ satisfying the following properties:

1) u is normal,
2) u is convex fuzzy set,
3) u is upper semi-continuous on $\mathbb{R}$,
4) $cl\left\{x\in\mathbb{R}\,\middle|\,u(x)>0\right\}$ is compact where cl denotes the closure of a subset.

Let $\mathbb{R}_F$ denote the space of fuzzy numbers.

Definition 2.2

Let $u\in\mathbb{R}_F$. The α-level set of u, denoted $[u]^{\alpha}$, $0<\alpha\leq 1$, is $[u]^{\alpha}=\left\{x\in\mathbb{R}\,\middle|\,u(x)\geq\alpha\right\}$. If α=0, the support of u is defined $[u]^{0}=cl\left\{x\in\mathbb{R}\,\middle|\,u(x)>0\right\}$. The notation, $[u]^{\alpha}=[\underline{u}_{\alpha},\overline{u}_{\alpha}]$ denotes explicitly the α-level set of u. We refer to $\underline{u}$ and $\overline{u}$ as the lower and upper branches of u, respectively.

The following remark shows when $[\underline{u}_{\alpha},\overline{u}_{\alpha}]$ is a valid α-level set.

Remark 2.1

The sufficient and necessary conditions for $[\underline{u}_{\alpha},\overline{u}_{\alpha}]$ to define the parametric form of a fuzzy number as follows:

1) $\underline{u}_{\alpha}$ is bounded monotonic increasing (nondecreasing) left-continuous function on (0,1] and right-continuous for $\alpha=0$,

2) $\overline{u}_{\alpha}$ is bounded monotonic decreasing (nonincreasing) left-continuous function on (0,1] and right-continuous for $\alpha=0$,

3) $\underline{u}_{\alpha}\leq\overline{u}_{\alpha}$, $0\leq\alpha\leq 1$.

Definition 2.3

If A is a symmetric triangular number with support $[\underline{a},\overline{a}]$, the α-level set of A is $[A]^{\alpha}=\left[\underline{a}+\left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha,\overline{a}-\left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right]$.

Definition 2.4

For $u,v\in\mathbb{R}_F$ and $\lambda\in\mathbb{R}$, the sum $u+v$ and the product λu are defined by $[u+v]^{\alpha}=[u]^{\alpha}+[v]^{\alpha}$, $[\lambda u]^{\alpha}=\lambda[u]^{\alpha}$, $\forall\alpha\in[0,1]$, where $[u]^{\alpha}+[v]^{\alpha}$ means the usual addition of two intervals (subsets) of $\mathbb{R}$ and $\lambda[u]^{\alpha}$ means the usual product between a scalar and a subset of $\mathbb{R}$.

The metric structure is given by the Hausdorff distance

$$D:\mathbb{R}_F\times\mathbb{R}_F\rightarrow\mathbb{R}_+\cup\{0\},$$

by

$$D(u,v)=\sup_{\alpha\in[0,1]}\max\left\{\left|\underline{u}_{\alpha}-\underline{v}_{\alpha}\right|,\left|\overline{u}_{\alpha}-\overline{v}_{\alpha}\right|\right\}.$$

Definition 2.5

Let $u,v\in\mathbb{R}_F$. If there exist $w\in\mathbb{R}_F$ such that $u=v+w$, then w is called the H-difference of u and v and it is denoted $u\underset{H}{-}v$.

Definition 2.6

Let I=(a,b), for $a,b\in\mathbb{R}$, and $F:I\rightarrow\mathbb{R}_F$ be a fuzzy function. We say F is differentiable at $t_0\in I$ if there exists an element $F'(t_0)\in\mathbb{R}_F$ such that the limits

$$\lim_{h\rightarrow0^+}\frac{F(t_0+h)-F(t_0)}{h}\text{ and }\lim_{h\rightarrow0^+}\frac{F(t_0)-F(t_0-h)}{h}$$

exist and equal $F'(t_0)$. Here the limits are taken in the metric space $(\mathbb{R}_F,D)$.

The above definition is a straightforward generalization of the Hukuhara differentiability of a set-valued function. Note that this definition of derivative is restrictive; for instance, in [2], the authors showed that if $f(t)=cg(t)$, where c is a fuzzy number and $g:[a,b]\rightarrow\mathbb{R}^+$ is a function with $g'(t_0)<0$, then f is not differentiable. To overcome this inconvenient, they [2] introduced a more general definition of derivative for fuzzy-number-valued function. In this paper, we consider the following definition [6].

Definition 2.7

Let I=(a,b) and $F:I\rightarrow\mathbb{R}_F$ be a fuzzy function. We say F is (1)- differentiable at $t_0\in I$, if there exists an element $F'(t_0)\in\mathbb{R}_F$ such that for all h>0 sufficiently near to 0, there exist $F(t_0+h)-F(t_0)$, $F(t_0)-F(t_0-h)$ and the limits (in the metric D)

$$\lim_{h\rightarrow0^+}\frac{F(t_0+h)-F(t_0)}{h}=\lim_{h\rightarrow0^+}\frac{F(t_0)-F(t_0-h)}{h}=F'(t_0).$$

F is (2)-differentiable if for all h<0 sufficiently near to 0, there exist $F(t_0+h)-F(t_0)$, $F(t_0)-F(t_0-h)$ and the limits (in the metric D)

$$\lim_{h\rightarrow0^-}\frac{F(t_0+h)-F(t_0)}{h}=\lim_{h\rightarrow0^-}\frac{F(t_0)-F(t_0-h)}{h}=F'(t_0).$$

Theorem 2.1

Let $f : I \to \mathbb{R}_F$ be a function and denote $\left[f(t)\right]^\alpha = [\underline{f}_\alpha(t), \overline{f}_\alpha(t)]$, for each $\alpha \in [0,1]$. Then

1) If f is (1)-differentiable, then $\underline{f}_\alpha$ and $\overline{f}_\alpha$ are differentiable functions and $\left[f'(t)\right]^\alpha = [\underline{f}'_\alpha(t), \overline{f}'_\alpha(t)]$.

2) If f is (2)-differentiable, then $\underline{f}_\alpha$ and $\overline{f}_\alpha$ are differentiable functions and $\left[f'(t)\right]^\alpha = [\overline{f}'_\alpha(t), \underline{f}'_\alpha(t)]$

Proof

See [6].

Theorem 2.2

Let $f' : I \to \mathbb{R}_F$ be a function, where f is (1)-differentiable or (2)-differentiable and $\left[f(t)\right]^\alpha = [\underline{f}_\alpha(t), \overline{f}_\alpha(t)]$. Then

1) If f and f' (1)-differentiable , then $\underline{f}'_\alpha$ and $\overline{f}'_\alpha$ are differentiable functions and $\left[f''(t)\right]^\alpha = [\underline{f}''_\alpha(t), \overline{f}''_\alpha(t)]$.

2) If f (1)-differentiable and f' (2)-differentiable, then $\underline{f}'_\alpha$ and $\overline{f}'_\alpha$ are differentiable functions and $\left[f''(t)\right]^\alpha = [\overline{f}''_\alpha(t), \underline{f}''_\alpha(t)]$.

3) If f (2)-differentiable and f' (1)-differentiable, then $\underline{f}'_\alpha$ and $\overline{f}'_\alpha$ are differentiable functions and $\left[f''(t)\right]^\alpha = [\overline{f}''_\alpha(t), \underline{f}''_\alpha(t)]$.

4) If f and f' (2)-differentiable , then $\underline{f}'_\alpha$ and $\overline{f}'_\alpha$ are differentiable functions and $\left[f''(t)\right]^\alpha = [\underline{f}''_\alpha(t), \overline{f}''_\alpha(t)]$.

Proof

See [10].

3. Fuzzy Boundary Value Problems For Second-Order Fuzzy Linear Differential Equations with Constant Coefficients

1) The case of positive constant coefficients

Consider the fuzzy boundary value problem

(3) $$y''(t) = \lambda y(t),\ y(0)=A\ , y'(\ell) = B$$

where $\lambda > 0$ and boundary conditions A and B are symmetric triangular numbers. The α-level set of A and B are $[A]^\alpha = \left[\underline{a} + \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha, \overline{a} - \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right]$ and $[B]^\alpha = \left[\underline{b} + \left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha, \overline{b} - \left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right]$, respectively.

Here, (i,j)-solution i,j=1,2 means that y is (i)-differentiable in and y' is (j)-differentiable.

Theorem 3.1

Let $\left[y(t)\right]^\alpha = [\underline{y}_\alpha(t), \overline{y}_\alpha(t)]$ be a solution of (3), where $\underline{y}_\alpha(t)$ and $\overline{y}_\alpha(t)$ are the lower and upper solutions.

For (1,1)-solution, the lower and upper solutions are

$$\underline{y}_\alpha(t) = a_1(\alpha)e^{\sqrt{\lambda}t} + a_2(\alpha)e^{-\sqrt{\lambda}t}$$
$$\overline{y}_\alpha(t) = a_3(\alpha)e^{\sqrt{\lambda}t} + a_4(\alpha)e^{-\sqrt{\lambda}t}$$

where

$$a_1(\alpha)=\frac{\sqrt{\lambda}e^{-\sqrt{\lambda}\ell}\left(\left(\underline{a}+\left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right)+\left(\underline{b}+\left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)\right)}{\sqrt{\lambda}(e^{\sqrt{\lambda}\ell}+e^{-\sqrt{\lambda}\ell})}$$

$$a_2(\alpha)=\frac{\sqrt{\lambda}e^{\sqrt{\lambda}\ell}\left(\left(\underline{a}+\left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right)-\left(\underline{b}+\left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)\right)}{\sqrt{\lambda}(e^{\sqrt{\lambda}\ell}+e^{-\sqrt{\lambda}\ell})}$$

$$a_3(\alpha)=\frac{\sqrt{\lambda}e^{-\sqrt{\lambda}\ell}\left(\left(\overline{a}-\left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right)+\left(\overline{b}-\left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)\right)}{\sqrt{\lambda}(e^{\sqrt{\lambda}\ell}+e^{-\sqrt{\lambda}\ell})}$$

$$a_4(\alpha)=\frac{\sqrt{\lambda}e^{\sqrt{\lambda}\ell}\left(\left(\overline{a}-\left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right)-\left(\overline{b}-\left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)\right)}{\sqrt{\lambda}(e^{\sqrt{\lambda}\ell}+e^{-\sqrt{\lambda}\ell})}$$

For the (1,2)-solution, the lower and upper solutions are

$$\underline{y}_\alpha(t)=b_1(\alpha)e^{\sqrt{\lambda}t}+b_2(\alpha)e^{-\sqrt{\lambda}t}-b_3(\alpha)\sin(\sqrt{\lambda}t)-b_4(\alpha)\cos(\sqrt{\lambda}t)$$

$$\overline{y}_\alpha(t)=b_1(\alpha)e^{\sqrt{\lambda}t}+b_2(\alpha)e^{-\sqrt{\lambda}t}+b_3(\alpha)\sin(\sqrt{\lambda}t)+b_4(\alpha)\cos(\sqrt{\lambda}t)$$

where

$$b_1(\alpha)=\frac{\sqrt{\lambda}e^{-\sqrt{\lambda}\ell}\left(\overline{a}+\underline{a}\right)+\left(\overline{b}+\underline{b}\right)}{2\sqrt{\lambda}(e^{\sqrt{\lambda}\ell}+e^{-\sqrt{\lambda}\ell})}\ ,\qquad b_2(\alpha)=\frac{\sqrt{\lambda}e^{\sqrt{\lambda}\ell}\left(\overline{a}+\underline{a}\right)-\left(\overline{b}+\underline{b}\right)}{2\sqrt{\lambda}(e^{\sqrt{\lambda}\ell}+e^{-\sqrt{\lambda}\ell})}$$

$$b_3(\alpha)=\frac{(1-\alpha)\left[\sqrt{\lambda}\left(\overline{a}-\underline{a}\right)\sin\left(\sqrt{\lambda}\ell\right)+\left(\overline{b}-\underline{b}\right)\right]}{2\sqrt{\lambda}\cos\left(\sqrt{\lambda}\ell\right)},\quad b_4(\alpha)=\left(\frac{1-\alpha}{2}\right)\left(\overline{a}-\underline{a}\right)$$

For (2,2)-solution, the lower and upper solutions are

$$\underline{y}_\alpha(t)=c_1(\alpha)e^{\sqrt{\lambda}t}+c_2(\alpha)e^{-\sqrt{\lambda}t}$$

$$\overline{y}_\alpha(t)=c_3(\alpha)e^{\sqrt{\lambda}t}+c_4(\alpha)e^{-\sqrt{\lambda}t}$$

where

$$c_1(\alpha)=\frac{\sqrt{\lambda}e^{-\sqrt{\lambda}\ell}\left(\left(\underline{a}+\left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right)+\left(\overline{b}-\left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)\right)}{\sqrt{\lambda}(e^{\sqrt{\lambda}\ell}+e^{-\sqrt{\lambda}\ell})}$$

$$c_2(\alpha)=\frac{\sqrt{\lambda}e^{\sqrt{\lambda}\ell}\left(\left(\underline{a}+\left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right)-\left(\overline{b}-\left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)\right)}{\sqrt{\lambda}(e^{\sqrt{\lambda}\ell}+e^{-\sqrt{\lambda}\ell})}$$

$$c_3(\alpha)=\frac{\sqrt{\lambda}e^{-\sqrt{\lambda}\ell}\left(\bar{a}-\left(\frac{\bar{a}-\underline{a}}{2}\right)\alpha\right)+\left(\underline{b}+\left(\frac{\bar{b}-\underline{b}}{2}\right)\alpha\right)}{\sqrt{\lambda}(e^{\sqrt{\lambda}\ell}+e^{-\sqrt{\lambda}\ell})}$$

$$c_4(\alpha)=\frac{\sqrt{\lambda}e^{\sqrt{\lambda}\ell}\left(\bar{a}-\left(\frac{\bar{a}-\underline{a}}{2}\right)\alpha\right)+\left(\underline{b}+\left(\frac{\bar{b}-\underline{b}}{2}\right)\alpha\right)}{\sqrt{\lambda}(e^{\sqrt{\lambda}\ell}+e^{-\sqrt{\lambda}\ell})}$$

For the (2,1)-solution, the lower and upper solutions are

$$\underline{y}_\alpha(t)=d_1(\alpha)e^{\sqrt{\lambda}t}+d_2(\alpha)e^{-\sqrt{\lambda}t}-d_3(\alpha)\sin(\sqrt{\lambda}t)-d_4(\alpha)\cos(\sqrt{\lambda}t)$$

$$\bar{y}_\alpha(t)=d_1(\alpha)e^{\sqrt{\lambda}t}+d_2(\alpha)e^{-\sqrt{\lambda}t}+d_3(\alpha)\sin(\sqrt{\lambda}t)+d_4(\alpha)\cos(\sqrt{\lambda}t)$$

where

$$d_1(\alpha)=\frac{\sqrt{\lambda}e^{-\sqrt{\lambda}\ell}\left(\bar{a}+\underline{a}\right)+\left(\bar{b}+\underline{b}\right)}{2\sqrt{\lambda}(e^{\sqrt{\lambda}\ell}+e^{-\sqrt{\lambda}\ell})},\qquad d_2(\alpha)=\frac{\sqrt{\lambda}e^{\sqrt{\lambda}\ell}\left(\bar{a}+\underline{a}\right)-\left(\bar{b}+\underline{b}\right)}{2\sqrt{\lambda}(e^{\sqrt{\lambda}\ell}+e^{-\sqrt{\lambda}\ell})}$$

$$d_3(\alpha)=\frac{(1-\alpha)\left[\sqrt{\lambda}\left(\bar{a}-\underline{a}\right)\sin\left(\sqrt{\lambda}\ell\right)+\left(\underline{b}-\bar{b}\right)\right]}{2\sqrt{\lambda}\cos\left(\sqrt{\lambda}\ell\right)},\qquad d_4(\alpha)=\left(\frac{1-\alpha}{2}\right)\left(\bar{a}-\underline{a}\right)$$

Proof

For (1,1)-solution, using Theorem 2.1 and Theorem 2.2, the lower solution and upper solution of (3) , satisfy the following equations

$$\begin{cases}\underline{y}''_\alpha(t)=\lambda\underline{y}_\alpha(t),\ \underline{y}_\alpha(0)=\underline{a}+\left(\frac{\bar{a}-\underline{a}}{2}\right)\alpha,\ \underline{y}'_\alpha(\ell)=\underline{b}+\left(\frac{\bar{b}-\underline{b}}{2}\right)\alpha\\ \bar{y}''_\alpha(t)=\lambda\bar{y}_\alpha(t),\ \bar{y}_\alpha(0)=\bar{a}-\left(\frac{\bar{a}-\underline{a}}{2}\right)\alpha,\ \bar{y}'_\alpha(\ell)=\bar{b}-\left(\frac{\bar{b}-\underline{b}}{2}\right)\alpha\end{cases}$$

respectively. Hence the solutions can be obtained

$$\underline{y}_\alpha(t)=a_1(\alpha)e^{\sqrt{\lambda}t}+a_2(\alpha)e^{-\sqrt{\lambda}t}$$

$$\bar{y}_\alpha(t)=a_3(\alpha)e^{\sqrt{\lambda}t}+a_4(\alpha)e^{-\sqrt{\lambda}t}$$

Using boundary conditions, coefficients $a_1(\alpha)$, $a_2(\alpha)$, $a_3(\alpha)$ and $a_4(\alpha)$ are solved as

$$a_1(\alpha)=\frac{\sqrt{\lambda}e^{-\sqrt{\lambda}\ell}\left(\underline{a}+\left(\frac{\bar{a}-\underline{a}}{2}\right)\alpha\right)+\left(\underline{b}+\left(\frac{\bar{b}-\underline{b}}{2}\right)\alpha\right)}{\sqrt{\lambda}(e^{\sqrt{\lambda}\ell}+e^{-\sqrt{\lambda}\ell})}$$

$$a_2(\alpha)=\frac{\sqrt{\lambda}e^{\sqrt{\lambda}\ell}\left(\underline{a}+\left(\frac{\bar{a}-\underline{a}}{2}\right)\alpha\right)-\left(\underline{b}+\left(\frac{\bar{b}-\underline{b}}{2}\right)\alpha\right)}{\sqrt{\lambda}(e^{\sqrt{\lambda}\ell}+e^{-\sqrt{\lambda}\ell})}$$

$$a_3(\alpha)=\frac{\sqrt{\lambda}e^{-\sqrt{\lambda}\ell}\left(\overline{a}-\left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right)+\left(\overline{b}-\left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)}{\sqrt{\lambda}(e^{\sqrt{\lambda}\ell}+e^{-\sqrt{\lambda}\ell})}$$

$$a_4(\alpha)=\frac{\sqrt{\lambda}e^{\sqrt{\lambda}\ell}\left(\overline{a}-\left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right)-\left(\overline{b}-\left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)}{\sqrt{\lambda}(e^{\sqrt{\lambda}\ell}+e^{-\sqrt{\lambda}\ell})}$$

For (1,2)-solution, using Theorem 2.1 and Theorem 2.2, the fuzzy boundary value problem (3) is transformed into a linear system of real-valued differential equations

$$\begin{cases}\underline{y}''_{\alpha}(t)=-\lambda\overline{y}_{\alpha}(t)\\ \overline{y}''_{\alpha}(t)=-\lambda\underline{y}_{\alpha}(t)\end{cases}$$

with

$$\underline{y}_{\alpha}(0)=\underline{a}+\left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha,\quad \underline{y}'_{\alpha}(\ell)=\underline{b}+\left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha$$

$$\overline{y}_{\alpha}(0)=\overline{a}-\left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha,\quad \overline{y}'_{\alpha}(\ell)=\overline{b}-\left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha$$

Hence the solutions can be obtained

$$\underline{y}_{\alpha}(t)=b_1(\alpha)e^{\sqrt{\lambda}t}+b_2(\alpha)e^{-\sqrt{\lambda}t}-b_3(\alpha)\sin(\sqrt{\lambda}t)-b_4(\alpha)\cos(\sqrt{\lambda}t)$$

$$\overline{y}_{\alpha}(t)=b_1(\alpha)e^{\sqrt{\lambda}t}+b_2(\alpha)e^{-\sqrt{\lambda}t}+b_3(\alpha)\sin(\sqrt{\lambda}t)+b_4(\alpha)\cos(\sqrt{\lambda}t)$$

Using boundary conditions, coefficients $b_1(\alpha)$, $b_2(\alpha)$, $b_3(\alpha)$ and $b_4(\alpha)$ are solved as

$$b_1(\alpha)=\frac{\sqrt{\lambda}e^{-\sqrt{\lambda}\ell}\left(\overline{a}+\underline{a}\right)+\left(\overline{b}+\underline{b}\right)}{2\sqrt{\lambda}(e^{\sqrt{\lambda}\ell}+e^{-\sqrt{\lambda}\ell})},\qquad b_2(\alpha)=\frac{\sqrt{\lambda}e^{\sqrt{\lambda}\ell}\left(\overline{a}+\underline{a}\right)-\left(\overline{b}+\underline{b}\right)}{2\sqrt{\lambda}(e^{\sqrt{\lambda}\ell}+e^{-\sqrt{\lambda}\ell})}$$

$$b_3(\alpha)=\frac{(1-\alpha)\left[\sqrt{\lambda}\left(\overline{a}-\underline{a}\right)\sin\left(\sqrt{\lambda}\ell\right)+\left(\overline{b}-\underline{b}\right)\right]}{2\sqrt{\lambda}\cos\left(\sqrt{\lambda}\ell\right)},\quad b_4(\alpha)=\left(\frac{1-\alpha}{2}\right)\left(\overline{a}-\underline{a}\right)$$

Similarly, for (2,2)-solution and (2,1)-solution, the following sistems are solved

$$\begin{cases}\underline{y}''_{\alpha}(t)=\lambda\underline{y}_{\alpha}(t),\ \underline{y}_{\alpha}(0)=\underline{a}+\left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha,\ \underline{y}'_{\alpha}(\ell)=\overline{b}-\left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\\ \overline{y}''_{\alpha}(t)=\lambda\overline{y}_{\alpha}(t),\ \overline{y}_{\alpha}(0)=\overline{a}-\left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha,\ \overline{y}'_{\alpha}(\ell)=\underline{b}+\left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\end{cases}$$

$$\begin{cases}\underline{y}''_{\alpha}(t)=\lambda\overline{y}_{\alpha}(t),\ \underline{y}_{\alpha}(0)=\underline{a}+\left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha,\ \underline{y}'_{\alpha}(\ell)=\overline{b}-\left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\\ \overline{y}''_{\alpha}(t)=\lambda\underline{y}_{\alpha}(t),\ \overline{y}_{\alpha}(0)=\overline{a}-\left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha,\ \overline{y}'_{\alpha}(\ell)=\underline{b}+\left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\end{cases}$$

respectively.

Proposition 3.1

i) For (1,1)-solution, the solution $[y(t)]^{\alpha}=[\underline{y}_{\alpha}(t),\overline{y}_{\alpha}(t)]$ of (3) is a valid fuzzy level set for all $t\in[0,\ell]$.

ii) For (1,2)-solution, the solution $[y(t)]^{\alpha}=[\underline{y}_{\alpha}(t),\overline{y}_{\alpha}(t)]$ of (3) is no longer a valid fuzzy level set as $t<\frac{1}{\sqrt{\lambda}}\tan^{-1}\left(-\left(\frac{(\overline{a}-\underline{a})\sqrt{\lambda}\cos(\sqrt{\lambda}\ell)}{(\overline{b}-\underline{b})+(\overline{a}-\underline{a})\sqrt{\lambda}\sin(\sqrt{\lambda}\ell)}\right)\right)$.

iii) For (2,2)-solution, the solution $[y(t)]^{\alpha}=[\underline{y}_{\alpha}(t),\overline{y}_{\alpha}(t)]$ of (3) is a valid fuzzy level set if $\sqrt{\lambda}e^{-\sqrt{\lambda}\ell}(\overline{a}-\underline{a})\geq(\overline{b}-\underline{b})$.

iv) For (2,1)-solution,

a) if $(\overline{a}-\underline{a})\sqrt{\lambda}\sin(\sqrt{\lambda}\ell)>\overline{b}-\underline{b}$, the solution $[y(t)]^{\alpha}=[\underline{y}_{\alpha}(t),\overline{y}_{\alpha}(t)]$ of (3) is no longer a valid fuzzy level set as $t<\frac{1}{\sqrt{\lambda}}\tan^{-1}\left(-\left(\frac{(\overline{a}-\underline{a})\sqrt{\lambda}\cos(\sqrt{\lambda}\ell)}{(\underline{b}-\overline{b})+(\overline{a}-\underline{a})\sqrt{\lambda}\sin(\sqrt{\lambda}\ell)}\right)\right)$.

b) if $(\overline{a}-\underline{a})\sqrt{\lambda}\sin(\sqrt{\lambda}\ell)<\overline{b}-\underline{b}$, the solution $[y(t)]^{\alpha}=[\underline{y}_{\alpha}(t),\overline{y}_{\alpha}(t)]$ of (3) is no longer a valid fuzzy level set as $t>\frac{1}{\sqrt{\lambda}}\tan^{-1}\left(-\left(\frac{(\overline{a}-\underline{a})\sqrt{\lambda}\cos(\sqrt{\lambda}\ell)}{(\underline{b}-\overline{b})+(\overline{a}-\underline{a})\sqrt{\lambda}\sin(\sqrt{\lambda}\ell)}\right)\right)$.

Proof

i) For (1,1)-solution, given $t\in[0,\ell]$,

$$\begin{aligned}\overline{y}_{\alpha}(t)-\underline{y}_{\alpha}(t)&=(a_3(\alpha)-a_1(\alpha))e^{\sqrt{\lambda}t}+(a_4(\alpha)-a_2(\alpha))e^{-\sqrt{\lambda}t}\\&=e^{-\sqrt{\lambda}t}\left((a_3(\alpha)-a_1(\alpha))e^{2\sqrt{\lambda}t}+(a_4(\alpha)-a_2(\alpha))\right)\end{aligned}$$

Let $f(t)=(a_3(\alpha)-a_1(\alpha))e^{2\sqrt{\lambda}t}+(a_4(\alpha)-a_2(\alpha))$. Then $f(0)=(\overline{a}-\underline{a})(1-\alpha)>0$ and

$$\begin{aligned}f'(t)&=2\sqrt{\lambda}(a_3(\alpha)-a_1(\alpha))e^{2\sqrt{\lambda}t}\\&=2(1-\alpha)\left(\frac{\sqrt{\lambda}e^{-\sqrt{\lambda}\ell}(\overline{a}-\underline{a})+(\overline{b}-\underline{b})}{e^{\sqrt{\lambda}\ell}+e^{-\sqrt{\lambda}\ell}}\right)e^{2\sqrt{\lambda}t}>0.\end{aligned}$$

Hence, for (1,1)-solution, the solution $[y(t)]^{\alpha}=[\underline{y}_{\alpha}(t),\overline{y}_{\alpha}(t)]$ of (3) is a valid fuzzy level set for all $t\in[0,\ell]$.

ii) For (1,2)-solution, $\overline{y}_{\alpha}(t)-\underline{y}_{\alpha}(t)=2b_3\sin(\sqrt{\lambda}t)+2b_4\cos(\sqrt{\lambda}t)$

$$\overline{y}_{\alpha}(t)-\underline{y}_{\alpha}(t)\geq 0\Leftrightarrow b_3(\alpha)\sin(\sqrt{\lambda}t)\geq -b_4(\alpha)\cos(\sqrt{\lambda}t).$$

As $0<\sqrt{\lambda}t<\sqrt{\lambda}\ell<\frac{\pi}{2}$, we have $\frac{\sqrt{\lambda}\left(\bar{a}-\underline{a}\right)\sin\sqrt{\lambda}\ell+\left(\bar{b}-\underline{b}\right)}{\sqrt{\lambda}\cos\sqrt{\lambda}\ell}>0$ and $\cos\sqrt{\lambda}t>0$. Hence $\frac{\sin\sqrt{\lambda}t}{\cos\sqrt{\lambda}t}\geq-\left(\frac{\left(\bar{a}-\underline{a}\right)\sqrt{\lambda}\cos\sqrt{\lambda}\ell}{\sqrt{\lambda}\left(\bar{a}-\underline{a}\right)\sin\sqrt{\lambda}\ell+\left(\bar{b}-\underline{b}\right)}\right)$; that is $t\geq\frac{1}{\sqrt{\lambda}}\tan^{-1}\left(-\left(\frac{\left(\bar{a}-\underline{a}\right)\sqrt{\lambda}\cos\sqrt{\lambda}\ell}{\sqrt{\lambda}\left(\bar{a}-\underline{a}\right)\sin\sqrt{\lambda}\ell+\left(\bar{b}-\underline{b}\right)}\right)\right)$.

This implies that the solution $[y(t)]^{\alpha}=[\underline{y}_{\alpha}(t),\bar{y}_{\alpha}(t)]$ of (3) is not valid fuzzy level set as $t<\frac{1}{\sqrt{\lambda}}\tan^{-1}\left(-\left(\frac{\left(\bar{a}-\underline{a}\right)\sqrt{\lambda}\cos\left(\sqrt{\lambda}\ell\right)}{\left(\bar{b}-\underline{b}\right)+\left(\bar{a}-\underline{a}\right)\sqrt{\lambda}\sin\left(\sqrt{\lambda}\ell\right)}\right)\right)$.

For **iii)** (2,2)-solution and **iv)** (2,1)-solution, proof is similar.

Proposition 3.2

For any $t\in[0,\ell]$, the solution $[y(t)]^{\alpha}=[\underline{y}_{\alpha}(t),\bar{y}_{\alpha}(t)]$ of (3) is a symmetric triangle fuzzy number.

Proof

For (1,1)-solution, we have

$$\underline{y}_1(t)=\frac{\sqrt{\lambda}e^{-\sqrt{\lambda}\ell}\left(\frac{\bar{a}+\underline{a}}{2}\right)+\left(\frac{\bar{b}+\underline{b}}{2}\right)}{\sqrt{\lambda}(e^{\sqrt{\lambda}\ell}+e^{-\sqrt{\lambda}\ell})}e^{\sqrt{\lambda}t}+\frac{\sqrt{\lambda}e^{\sqrt{\lambda}\ell}\left(\frac{\bar{a}+\underline{a}}{2}\right)-\left(\frac{\bar{b}+\underline{b}}{2}\right)}{\sqrt{\lambda}(e^{\sqrt{\lambda}\ell}+e^{-\sqrt{\lambda}\ell})}e^{-\sqrt{\lambda}t}=\bar{y}_1(t)$$

and

$$\underline{y}_1(t)-\underline{y}_{\alpha}(t)=(1-\alpha)\left(\frac{\sqrt{\lambda}e^{-\sqrt{\lambda}\ell}\left(\frac{\bar{a}-\underline{a}}{2}\right)+\left(\frac{\bar{b}-\underline{b}}{2}\right)}{\sqrt{\lambda}(e^{\sqrt{\lambda}\ell}+e^{-\sqrt{\lambda}\ell})}e^{\sqrt{\lambda}t}+\frac{\sqrt{\lambda}e^{\sqrt{\lambda}\ell}\left(\frac{\bar{a}-\underline{a}}{2}\right)-\left(\frac{\bar{b}-\underline{b}}{2}\right)}{\sqrt{\lambda}(e^{\sqrt{\lambda}\ell}+e^{-\sqrt{\lambda}\ell})}e^{-\sqrt{\lambda}t}\right)=\bar{y}_{\alpha}(t)-\bar{y}_1(t)$$

For (1,2)-solution we have

$$\underline{y}_1(t)=b_1(\alpha)e^{\sqrt{\lambda}t}+b_2(\alpha)e^{-\sqrt{\lambda}t}=\bar{y}_1(t)$$

and

$$\underline{y}_1(t)-\underline{y}_{\alpha}(t)=b_3(\alpha)\sin(\sqrt{\lambda}t)+b_4(\alpha)\cos(\sqrt{\lambda}t)=\bar{y}_{\alpha}(t)-\bar{y}_1(t).$$

For (2,2)-solution and (2,1)-solution, proof is similar.

Hence solutions are symmetric fuzzy function of t.

Example 3.1

Consider the fuzzy boundary value problem

$$\begin{cases} y''(t)=y(t),\ \ t\in\left(0,\frac{\pi}{4}\right) \\ y(0)=\left[1+\frac{1}{2}\alpha,2-\frac{1}{2}\alpha\right],\ \ y'(\frac{\pi}{4})=[3+\frac{1}{2}\alpha,4-\frac{1}{2}\alpha] \end{cases}$$

For (1,1)-solution , the fuzzy solution is obtained as

$$\underline{y}_\alpha(t) = a_1(\alpha)e^t + a_2(\alpha)e^{-t}$$
$$\overline{y}_\alpha(t) = a_3(\alpha)e^t + a_4(\alpha)e^{-t}$$

where

$$a_1(\alpha) = \frac{e^{-\frac{\pi}{4}}\left(1+\frac{1}{2}\alpha\right)+\left(3+\frac{1}{2}\alpha\right)}{e^{\frac{\pi}{4}}+e^{-\frac{\pi}{4}}}, \qquad a_2(\alpha) = \frac{e^{\frac{\pi}{4}}\left(1+\frac{1}{2}\alpha\right)-\left(3+\frac{1}{2}\alpha\right)}{e^{\frac{\pi}{4}}+e^{-\frac{\pi}{4}}}$$

$$a_3(\alpha) = \frac{e^{-\frac{\pi}{4}}\left(2-\frac{1}{2}\alpha\right)+\left(4-\frac{1}{2}\alpha\right)}{e^{\frac{\pi}{4}}+e^{-\frac{\pi}{4}}}, \qquad a_4(\alpha) = \frac{e^{\frac{\pi}{4}}\left(2-\frac{1}{2}\alpha\right)-\left(4-\frac{1}{2}\alpha\right)}{e^{\frac{\pi}{4}}+e^{-\frac{\pi}{4}}}$$

For (1,2)-solution, fuzzy solution is obtained as

$$\underline{y}_\alpha(t) = b_1(\alpha)e^t + b_2(\alpha)e^{-t} - b_3(\alpha)\sin(t) - b_4(\alpha)\cos(t)$$
$$\overline{y}_\alpha(t) = b_1(\alpha)e^t + b_2(\alpha)e^{-t} + b_3(\alpha)\sin(t) + b_4(\alpha)\cos(t)$$

where

$$b_1(\alpha) = \frac{3e^{-\frac{\pi}{4}}+7}{2(e^{\frac{\pi}{4}}+e^{-\frac{\pi}{4}})}, \; b_2(\alpha) = \frac{3e^{\frac{\pi}{4}}-7}{2(e^{\frac{\pi}{4}}+e^{-\frac{\pi}{4}})}, \; b_3(\alpha) = \left(\sqrt{2}+1\right)\left(\frac{1-\alpha}{2}\right), \; b_4(\alpha) = \frac{1-\alpha}{2}$$

For (2,2)-solution, fuzzy solution is obtained as

$$\underline{y}_\alpha(t) = c_1(\alpha)e^t + c_2(\alpha)e^{-t}$$
$$\overline{y}_\alpha(t) = c_3(\alpha)e^t + c_4(\alpha)e^{-t}$$

where

$$c_1(\alpha) = \frac{e^{-\frac{\pi}{4}}\left(1+\frac{1}{2}\alpha\right)+\left(4-\frac{1}{2}\alpha\right)}{e^{\frac{\pi}{4}}+e^{-\frac{\pi}{4}}}, \qquad c_2(\alpha) = \frac{e^{\frac{\pi}{4}}\left(1+\frac{1}{2}\alpha\right)-\left(4-\frac{1}{2}\alpha\right)}{e^{\frac{\pi}{4}}+e^{-\frac{\pi}{4}}}$$

$$c_3(\alpha) = \frac{e^{-\frac{\pi}{4}}\left(2-\frac{1}{2}\alpha\right)+\left(3+\frac{1}{2}\alpha\right)}{e^{\frac{\pi}{4}}+e^{-\frac{\pi}{4}}}, \qquad c_4(\alpha) = \frac{e^{\frac{\pi}{4}}\left(2-\frac{1}{2}\alpha\right)-\left(3+\frac{1}{2}\alpha\right)}{e^{\frac{\pi}{4}}+e^{-\frac{\pi}{4}}}$$

For (2,1)-solution, fuzzy solution is obtained as

$$\underline{y}_\alpha(t) = d_1(\alpha)e^t + d_2(\alpha)e^{-t} - d_3(\alpha)\sin(t) - d_4(\alpha)\cos(t)$$
$$\overline{y}_\alpha(t) = d_1(\alpha)e^t + d_2(\alpha)e^{-t} + d_3(\alpha)\sin(t) + d_4(\alpha)\cos(t)$$

where

$$d_1(\alpha) = \frac{3e^{-\frac{\pi}{4}}+7}{2(e^{\frac{\pi}{4}}+e^{-\frac{\pi}{4}})}, \; d_2(\alpha) = \frac{3e^{\frac{\pi}{4}}-7}{2(e^{\frac{\pi}{4}}+e^{-\frac{\pi}{4}})}, \; d_3(\alpha) = \left(1-\sqrt{2}\right)\left(\frac{1-\alpha}{2}\right), \; d_4(\alpha) = \frac{1-\alpha}{2}$$

Using Proposition 3.1, (1,1)-solution is a valid fuzzy level set for all $t \in \left[0, \frac{\pi}{4}\right]$, (1,2)-solution is not a valid fuzzy level set when $t < \tan^{-1}\left(1-\sqrt{2}\right)$, (2,2)-solution is not a valid fuzzy level set since $e^{-\frac{\pi}{4}} < 1$, (2,1)-solution is not a valid fuzzy level set when $t > \tan^{-1}\left(1+\sqrt{2}\right)$.

All solutions are symmetric triangle fuzzy function of t.

2) The case of negative constant coefficients

Consider the fuzzy boundary value problem

(4) $$y''(t) = -\lambda y(t),\ y(0)=A,\ y'(\ell) = B$$

where $\lambda > 0$ and boundary conditions A and B are symmetric triangular numbers. The α-level set of A and B are $[A]^\alpha = \left[\underline{a} + \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha, \overline{a} - \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right]$ and $[B]^\alpha = \left[\underline{b} + \left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha, \overline{b} - \left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right]$, respectively.

Here, (i,j)-solution i,j=1,2 means that y is (i)-differentiable in and y' is (j)-differentiable.

Theorem 3.2

Let $\left[y(t)\right]^\alpha = [\underline{y}_\alpha(t), \overline{y}_\alpha(t)]$ be a solution of (4), where $\underline{y}_\alpha(t)$ and $\overline{y}_\alpha(t)$ are the lower and upper solutions.

For (1,1)-solution, the lower and upper solutions are

$$\underline{y}_\alpha(t) = -a_1(\alpha)e^{\sqrt{\lambda}t} - a_2(\alpha)e^{-\sqrt{\lambda}t} + a_3(\alpha)\sin(\sqrt{\lambda}t) + a_4(\alpha)\cos(\sqrt{\lambda}t)$$

$$\overline{y}_\alpha(t) = a_1(\alpha)e^{\sqrt{\lambda}t} + a_2(\alpha)e^{-\sqrt{\lambda}t} + a_3(\alpha)\sin(\sqrt{\lambda}t) + a_4(\alpha)\cos(\sqrt{\lambda}t)$$

where

$$a_1(\alpha) = \left(\frac{1-\alpha}{2}\right)\left(\left(\overline{a}-\underline{a}\right) - \left(\frac{\sqrt{\lambda}e^{\sqrt{\lambda}\ell}\left(\overline{a}-\underline{a}\right) - \left(\overline{b}-\underline{b}\right)}{\sqrt{\lambda}\left(e^{\sqrt{\lambda}\ell} + e^{-\sqrt{\lambda}\ell}\right)}\right)\right)$$

$$a_2(\alpha) = \left(\frac{1-\alpha}{2}\right)\left(\frac{\sqrt{\lambda}e^{\sqrt{\lambda}\ell}\left(\overline{a}-\underline{a}\right) - \left(\overline{b}-\underline{b}\right)}{\sqrt{\lambda}\left(e^{\sqrt{\lambda}\ell} + e^{-\sqrt{\lambda}\ell}\right)}\right)$$

$$a_3(\alpha) = \frac{\left(\overline{b}+\underline{b}\right) + \left(\overline{a}+\underline{a}\right)\sqrt{\lambda}\sin\left(\sqrt{\lambda}\ell\right)}{2\sqrt{\lambda}\cos\left(\sqrt{\lambda}\ell\right)}$$

$$a_4(\alpha) = \frac{\overline{a}+\underline{a}}{2}$$

For (1,2)-solution, the lower and upper solutions are

$$\underline{y}_\alpha(t) = b_1(\alpha)\cos(\sqrt{\lambda}t) + b_2(\alpha)\sin(\sqrt{\lambda}t)$$

$$\overline{y}_\alpha(t) = b_3(\alpha)\cos(\sqrt{\lambda}t) + b_4(\alpha)\sin(\sqrt{\lambda}t)$$

where

$$b_1(\alpha)=\underline{a}+\left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha,\quad b_2(\alpha)=\frac{\left(\underline{b}+\left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)+\left(\underline{a}+\left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right)\sqrt{\lambda}\sin\left(\sqrt{\lambda}\ell\right)}{\sqrt{\lambda}\cos\left(\sqrt{\lambda}\ell\right)}$$

$$b_3(\alpha)=\overline{a}-\left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha,\quad b_4(\alpha)=\frac{\left(\overline{b}-\left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)+\left(\overline{a}-\left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right)\sqrt{\lambda}\sin\left(\sqrt{\lambda}\ell\right)}{\sqrt{\lambda}\cos\left(\sqrt{\lambda}\ell\right)}$$

For (2,2)-solution, the lower and upper solutions are

$$\underline{y}_\alpha(t)=-c_1(\alpha)e^{\sqrt{\lambda}t}-c_2(\alpha)e^{-\sqrt{\lambda}t}+c_3(\alpha)\sin(\sqrt{\lambda}t)+c_4(\alpha)\cos(\sqrt{\lambda}t)$$

$$\overline{y}_\alpha(t)=c_1(\alpha)e^{\sqrt{\lambda}t}+c_2(\alpha)e^{-\sqrt{\lambda}t}+c_3(\alpha)\sin(\sqrt{\lambda}t)+c_4(\alpha)\cos(\sqrt{\lambda}t)$$

where

$$c_1(\alpha)=\left(\frac{1-\alpha}{2}\right)\left(\left(\overline{a}-\underline{a}\right)-\left(\frac{\sqrt{\lambda}e^{\sqrt{\lambda}\ell}\left(\overline{a}-\underline{a}\right)+\left(\overline{b}-\underline{b}\right)}{\sqrt{\lambda}\left(e^{\sqrt{\lambda}\ell}+e^{-\sqrt{\lambda}\ell}\right)}\right)\right)$$

$$c_2(\alpha)=\left(\frac{1-\alpha}{2}\right)\left(\frac{\sqrt{\lambda}e^{\sqrt{\lambda}\ell}\left(\overline{a}-\underline{a}\right)+\left(\overline{b}-\underline{b}\right)}{\sqrt{\lambda}\left(e^{\sqrt{\lambda}\ell}+e^{-\sqrt{\lambda}\ell}\right)}\right)$$

$$c_3(\alpha)=\frac{\left(\overline{b}+\underline{b}\right)+\left(\overline{a}+\underline{a}\right)\sqrt{\lambda}\sin\left(\sqrt{\lambda}\ell\right)}{2\sqrt{\lambda}\cos\left(\sqrt{\lambda}\ell\right)}$$

$$c_4(\alpha)=\frac{\overline{a}+\underline{a}}{2}$$

For (2,1)-solution, the lower and upper solutions are

$$\underline{y}_\alpha(t)=d_1(\alpha)\cos(\sqrt{\lambda}t)+d_2(\alpha)\sin(\sqrt{\lambda}t)$$

$$\overline{y}_\alpha(t)=d_3(\alpha)\cos(\sqrt{\lambda}t)+d_4(\alpha)\sin(\sqrt{\lambda}t)$$

where

$$d_1(\alpha)=\underline{a}+\left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\;,d_2(\alpha)=\frac{\left(\overline{b}-\left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)+\left(\underline{a}+\left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right)\sqrt{\lambda}\sin\left(\sqrt{\lambda}\ell\right)}{\sqrt{\lambda}\cos\left(\sqrt{\lambda}\ell\right)}$$

$$d_3(\alpha)=\overline{a}-\left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\;,\; d_4(\alpha)=\frac{\left(\underline{b}+\left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)+\left(\overline{a}-\left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right)\sqrt{\lambda}\sin\left(\sqrt{\lambda}\ell\right)}{\sqrt{\lambda}\cos\left(\sqrt{\lambda}\ell\right)}$$

Proof

For (1,1)-solution, using Theorem 2.1, Theorem 2.2 and $-\lambda\left[\underline{x}_\alpha(t),\overline{x}_\alpha(t)\right]=\left[-\lambda\overline{x}_\alpha(t),-\lambda\underline{x}_\alpha(t)\right],\ \lambda>0$, the fuzzy boundary value problem (4) is transformed into a linear system of real-valued differential equations

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$$\begin{cases} \underline{y}''_{\alpha}(t) = -\lambda \overline{y}_{\alpha}(t) \\ \overline{y}''_{\alpha}(t) = -\lambda \underline{y}_{\alpha}(t) \end{cases}$$

with

$$\underline{y}_{\alpha}(0) = \underline{a} + \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha, \quad \underline{y}'_{\alpha}(\ell) = \underline{b} + \left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha$$

$$\overline{y}_{\alpha}(0) = \overline{a} - \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha, \quad \overline{y}'_{\alpha}(\ell) = \overline{b} - \left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha$$

Hence the solutions can be obtained

$$\underline{y}_{\alpha}(t) = -a_1(\alpha)e^{\sqrt{\lambda}t} - a_2(\alpha)e^{-\sqrt{\lambda}t} + a_3(\alpha)\sin(\sqrt{\lambda}t) + a_4(\alpha)\cos(\sqrt{\lambda}t)$$

$$\overline{y}_{\alpha}(t) = a_1(\alpha)e^{\sqrt{\lambda}t} + a_2(\alpha)e^{-\sqrt{\lambda}t} + a_3(\alpha)\sin(\sqrt{\lambda}t) + a_4(\alpha)\cos(\sqrt{\lambda}t)$$

Using boundary conditions, coefficients $a_1(\alpha)$, $a_2(\alpha)$, $a_3(\alpha)$ and $a_4(\alpha)$ are solved as

$$a_1(\alpha) = \left(\frac{1-\alpha}{2}\right)\left(\left(\overline{a}-\underline{a}\right) - \left(\frac{\sqrt{\lambda}e^{\sqrt{\lambda}\ell}\left(\overline{a}-\underline{a}\right) - \left(\overline{b}-\underline{b}\right)}{\sqrt{\lambda}\left(e^{\sqrt{\lambda}\ell} + e^{-\sqrt{\lambda}\ell}\right)}\right)\right)$$

$$a_2(\alpha) = \left(\frac{1-\alpha}{2}\right)\left(\frac{\sqrt{\lambda}e^{\sqrt{\lambda}\ell}\left(\overline{a}-\underline{a}\right) - \left(\overline{b}-\underline{b}\right)}{\sqrt{\lambda}\left(e^{\sqrt{\lambda}\ell} + e^{-\sqrt{\lambda}\ell}\right)}\right)$$

$$a_3(\alpha) = \frac{\left(\overline{b}+\underline{b}\right) + \left(\overline{a}+\underline{a}\right)\sqrt{\lambda}\sin\left(\sqrt{\lambda}\ell\right)}{2\sqrt{\lambda}\cos\left(\sqrt{\lambda}\ell\right)}$$

$$a_4(\alpha) = \frac{\overline{a}+\underline{a}}{2}$$

For (1,2)-solution, using Theorem 2.1, Theorem 2.2 and $-\lambda\left[\underline{x}_{\alpha}(t), \overline{x}_{\alpha}(t)\right] = \left[-\lambda\overline{x}_{\alpha}(t), -\lambda\underline{x}_{\alpha}(t)\right], \lambda > 0$ the lower solution and upper solution of (4) , satisfy the following equations

$$\begin{cases} \underline{y}''_{\alpha}(t) = -\lambda\underline{y}_{\alpha}(t), \ \underline{y}_{\alpha}(0) = \underline{a} + \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha, \ \underline{y}'_{\alpha}(\ell) = \underline{b} + \left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha \\ \overline{y}''_{\alpha}(t) = -\lambda\overline{y}_{\alpha}(t), \ \overline{y}_{\alpha}(0) = \overline{a} - \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha, \ \overline{y}'_{\alpha}(\ell) = \overline{b} - \left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha \end{cases}$$

respectively. Hence the solutions can be obtained

$$\underline{y}_{\alpha}(t) = b_1(\alpha)\cos(\sqrt{\lambda}t) + b_2(\alpha)\sin(\sqrt{\lambda}t)$$

$$\overline{y}_{\alpha}(t) = b_3(\alpha)\cos(\sqrt{\lambda}t) + b_4(\alpha)\sin(\sqrt{\lambda}t)$$

Using boundary conditions, coefficients $b_1(\alpha)$, $b_2(\alpha)$, $b_3(\alpha)$ and $b_4(\alpha)$ are solved as

$$b_1(\alpha)=\underline{a}+\left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha,\quad b_2(\alpha)=\frac{\left(\underline{b}+\left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)+\left(\underline{a}+\left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right)\sqrt{\lambda}\sin\left(\sqrt{\lambda}\ell\right)}{\sqrt{\lambda}\cos\left(\sqrt{\lambda}\ell\right)}$$

$$b_3(\alpha)=\overline{a}-\left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha,\quad b_4(\alpha)=\frac{\left(\overline{b}-\left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\right)+\left(\overline{a}-\left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right)\sqrt{\lambda}\sin\left(\sqrt{\lambda}\ell\right)}{\sqrt{\lambda}\cos\left(\sqrt{\lambda}\ell\right)}$$

Similarly, for (2,2)-solution and (2,1)-solution the following sistems are solved

$$\begin{cases}\underline{y}''_{\alpha}(t)=-\lambda\overline{y}_{\alpha}(t),\ \underline{y}_{\alpha}(0)=\underline{a}+\left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha,\ \underline{y}'_{\alpha}(\ell)=\overline{b}-\left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\\ \overline{y}''_{\alpha}(t)=-\lambda\underline{y}_{\alpha}(t),\ \overline{y}_{\alpha}(0)=\overline{a}-\left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha,\ \overline{y}'_{\alpha}(\ell)=\underline{b}+\left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\end{cases}$$

$$\begin{cases}\underline{y}''_{\alpha}(t)=\lambda\underline{y}_{\alpha}(t),\ \underline{y}_{\alpha}(0)=\underline{a}+\left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha,\ \underline{y}'_{\alpha}(\ell)=\overline{b}-\left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\\ \overline{y}''_{\alpha}(t)=\lambda\overline{y}_{\alpha}(t),\ \overline{y}_{\alpha}(0)=\overline{a}-\left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha,\ \overline{y}'_{\alpha}(\ell)=\underline{b}+\left(\frac{\overline{b}-\underline{b}}{2}\right)\alpha\end{cases}$$

respectively.

Proposition 3.3

i) For (1,1)-solution, the solution $[y(t)]^{\alpha}=[\underline{y}_{\alpha}(t),\overline{y}_{\alpha}(t)]$ of (4) is a valid fuzzy level set for all $t\in[0,\ell]$ if $a_1(\alpha)>0$.

ii) For (1,2)-solution, the solution $[y(t)]^{\alpha}=[\underline{y}_{\alpha}(t),\overline{y}_{\alpha}(t)]$ of (4) is no longer a valid fuzzy level set as

$$t<\frac{1}{\sqrt{\lambda}}\tan^{-1}\left(\frac{(1-\alpha)\left(\overline{a}-\underline{a}\right)}{-c}\right),$$

where

$$c=\frac{\left(\overline{b}-\underline{b}\right)(1-\alpha)+\left(\overline{a}-\underline{a}\right)(1-\alpha)\sqrt{\lambda}\sin\left(\sqrt{\lambda}\ell\right)}{\sqrt{\lambda}\cos\left(\sqrt{\lambda}\ell\right)}.$$

iii) For (2,2)-solution, the solution $[y(t)]^{\alpha}=[\underline{y}_{\alpha}(t),\overline{y}_{\alpha}(t)]$ of (4) is a valid fuzzy level set for all $t\in[0,\ell]$ if $c_1(\alpha)>0$.

iv) For (2,1)-solution,

a) if $\left(\overline{a}-\underline{a}\right)\sqrt{\lambda}\sin\left(\sqrt{\lambda}\ell\right)>\left(\overline{b}-\underline{b}\right)$, the solution $[y(t)]^{\alpha}=[\underline{y}_{\alpha}(t),\overline{y}_{\alpha}(t)]$ of (4) is no longer a valid fuzzy level set as $t<\frac{1}{\sqrt{\lambda}}\tan^{-1}\left(\frac{(1-\alpha)\left(\overline{a}-\underline{a}\right)}{-c}\right)$.

b) if $\left(\overline{a}-\underline{a}\right)\sqrt{\lambda}\sin\left(\sqrt{\lambda}\ell\right)<\left(\overline{b}-\underline{b}\right)$, the solution $\left[y(t)\right]^{\alpha}=[\underline{y}_{\alpha}(t),\overline{y}_{\alpha}(t)]$ of (4) is no longer a valid fuzzy level set as $t>\frac{1}{\sqrt{\lambda}}\tan^{-1}\left(\frac{(1-\alpha)\left(\overline{a}-\underline{a}\right)}{-c}\right)$.

where

$$c=\frac{\left(\overline{b}-\underline{b}\right)(\alpha-1)+\left(\overline{a}-\underline{a}\right)(1-\alpha)\sqrt{\lambda}\sin\left(\sqrt{\lambda}\ell\right)}{\sqrt{\lambda}\cos\left(\sqrt{\lambda}\ell\right)}.$$

Proof

i) For (1,1)-solution, the difference of $\overline{x}_{\alpha}$ and $\underline{x}_{\alpha}$ is

$$\begin{aligned}\overline{y}_{\alpha}(t)-\underline{y}_{\alpha}(t)&=2(a_1(\alpha)e^{\sqrt{\lambda}t}+a_2(\alpha)e^{-\sqrt{\lambda}t})\\&=2e^{-\sqrt{\lambda}t}(a_1(\alpha)e^{2\sqrt{\lambda}t}+a_2(\alpha)).\end{aligned}$$

Let $f(t)=a_1(\alpha)e^{2\sqrt{\lambda}t}+a_2(\alpha)$. Then $f(0)=(\overline{a}-\underline{a})\left(\frac{1-\alpha}{2}\right)>0$ and $f'(t)=2\sqrt{\lambda}a_1(\alpha)e^{2\sqrt{\lambda}t}>0$ as $a_1(\alpha)>0$. Therefore, $\overline{y}_{\alpha}(t)-\underline{y}_{\alpha}(t)>0$ as $a_1(\alpha)>0$.

ii) For (1,2)-solution,

$$\overline{y}_{\alpha}(t)-\underline{y}_{\alpha}(t)=\left(\overline{a}-\underline{a}\right)(1-\alpha)\cos\left(\sqrt{\lambda}t\right)+c\sin\left(\sqrt{\lambda}t\right),$$

where

$$c=\frac{\left(\overline{b}-\underline{b}\right)(1-\alpha)+\left(\overline{a}-\underline{a}\right)(1-\alpha)\sqrt{\lambda}\sin\left(\sqrt{\lambda}\ell\right)}{\sqrt{\lambda}\cos\left(\sqrt{\lambda}\ell\right)}$$ and $\ell\neq\frac{(2n+1)\pi}{2\sqrt{\lambda}}$, for all integer n.

$$\overline{y}_{\alpha}(t)-\underline{y}_{\alpha}(t)\geq 0\Leftrightarrow\left(\overline{a}-\underline{a}\right)(1-\alpha)\cos\left(\sqrt{\lambda}t\right)\geq -c\sin\left(\sqrt{\lambda}t\right).$$

As $0<\sqrt{\lambda}t<\sqrt{\lambda}\ell<\frac{\pi}{2}$, we have $\left(\frac{\left(\overline{a}-\underline{a}\right)(1-\alpha)}{-c}\right)\leq\frac{\sin\sqrt{\lambda}t}{\cos\sqrt{\lambda}t}$; that is $\frac{1}{\sqrt{\lambda}}\tan^{-1}\left(\frac{(1-\alpha)\left(\overline{a}-\underline{a}\right)}{-c}\right)\leq t$. This implies that the solution $\left[y(t)\right]^{\alpha}=[\underline{y}_{\alpha}(t),\overline{y}_{\alpha}(t)]$ of (4) is not a valid fuzzy level set as $t<\frac{1}{\sqrt{\lambda}}\tan^{-1}\left(\frac{(1-\alpha)\left(\overline{a}-\underline{a}\right)}{-c}\right)$.

For **iii)** (2,2)-solution and **iv)** (2,1)-solution, proof is similar.

Proposition 3.4

For any $t\in[0,\ell]$, the solution $\left[y(t)\right]^{\alpha}=[\underline{y}_{\alpha}(t),\overline{y}_{\alpha}(t)]$ of (4) is a symmetric triangle fuzzy number.

Proof

For (1,1)-solution, we have

$$\underline{y}_1(t)=a_3(\alpha)\sin\left(\sqrt{\lambda}t\right)+a_4(\alpha)\cos\left(\sqrt{\lambda}t\right)=\overline{y}_1(t)$$

and

$$\underline{y}_1(t)-\underline{y}_\alpha(t)=a_1(\alpha)e^{\sqrt{\lambda}t}+a_2(\alpha)e^{-\sqrt{\lambda}t}=\overline{y}_\alpha(t)-\overline{y}_1(t).$$

For (1,2)-solution, we have

$$\underline{y}_1(t)=\left(\frac{\overline{a}+\underline{a}}{2}\right)\cos\left(\sqrt{\lambda}t\right)\frac{\left(\frac{\overline{b}+\underline{b}}{2}\right)+\left(\frac{\overline{a}+\underline{a}}{2}\right)\sqrt{\lambda}\sin\left(\sqrt{\lambda}\ell\right)}{\sqrt{\lambda}\cos\left(\sqrt{\lambda}\ell\right)}\sin\left(\sqrt{\lambda}t\right)=\overline{y}_1(t)$$

and

$$\underline{y}_1(t)-\underline{y}_\alpha(t)=\left(\frac{\overline{a}-\underline{a}}{2}\right)(1-\alpha)\cos\left(\sqrt{\lambda}t\right)\frac{\left(\frac{\overline{b}-\underline{b}}{2}\right)(1-\alpha)+\left(\frac{\overline{a}-\underline{a}}{2}\right)(1-\alpha)\sqrt{\lambda}\sin\left(\sqrt{\lambda}\ell\right)}{\sqrt{\lambda}\cos\left(\sqrt{\lambda}\ell\right)}\sin\left(\sqrt{\lambda}t\right)=\overline{y}_\alpha(t)-\overline{y}_1(t).$$

For (2,2)-solution and (2,1)-solution, proof is similar.

Hence solutions are symmetric fuzzy function of t.

Example 3.2

Consider the fuzzy boundary value problem

$$\begin{cases} y''(t)=-y(t),\ \ t\in\left(0,\frac{\pi}{4}\right) \\ y(0)=\left[1+\frac{1}{2}\alpha,2-\frac{1}{2}\alpha\right],\ \ y'(\pi)=[3+\frac{1}{2}\alpha,4-\frac{1}{2}\alpha] \end{cases}$$

For (1,1)-solution, the lower and upper solutions are

$$\underline{y}_\alpha(t)=\left(\frac{\alpha-1}{2}\right)\left(1-\left(\frac{e^{\frac{\pi}{4}}-1}{e^{\frac{\pi}{4}}+e^{-\frac{\pi}{4}}}\right)\right)e^{t}+\left(\frac{\alpha-1}{2}\right)\left(\frac{e^{\frac{\pi}{4}}-1}{e^{\frac{\pi}{4}}+e^{-\frac{\pi}{4}}}\right)e^{-t}+\frac{3+7\sqrt{2}}{2}\sin t+\frac{3}{2}\cos t$$

$$\overline{y}_\alpha(t)=\left(\frac{1-\alpha}{2}\right)\left(1-\left(\frac{e^{\frac{\pi}{4}}-1}{e^{\frac{\pi}{4}}+e^{-\frac{\pi}{4}}}\right)\right)e^{t}+\left(\frac{1-\alpha}{2}\right)\left(\frac{e^{\frac{\pi}{4}}-1}{e^{\frac{\pi}{4}}+e^{-\frac{\pi}{4}}}\right)e^{-t}+\frac{3+7\sqrt{2}}{2}\sin t+\frac{3}{2}\cos t$$

For (1,2)-solution, the lower and upper solutions are

$$\underline{y}_\alpha(t)=\left(1+\frac{1}{2}\alpha\right)\cos t+\left(\left(1+3\sqrt{2}\right)+\left(\frac{1+\sqrt{2}}{2}\right)\alpha\right)\sin t$$

$$\overline{y}_\alpha(t)=\left(2-\frac{1}{2}\alpha\right)\cos t+\left(\left(2+4\sqrt{2}\right)-\left(\frac{1+\sqrt{2}}{2}\right)\alpha\right)\sin t$$

For (2,2)-solution, the lower and upper solutions are

$$\underline{y}_{\alpha}(t)=\left(\frac{\alpha-1}{2}\right)\left(1-\left(\frac{e^{\frac{\pi}{4}}+1}{e^{\frac{\pi}{4}}+e^{-\frac{\pi}{4}}}\right)\right)e^{t}+\left(\frac{\alpha-1}{2}\right)\left(\frac{e^{\frac{\pi}{4}}+1}{e^{\frac{\pi}{4}}+e^{-\frac{\pi}{4}}}\right)e^{-t}+\frac{3+7\sqrt{2}}{2}\sin t+\frac{3}{2}\cos t$$

$$\overline{y}_{\alpha}(t)=\left(\frac{1-\alpha}{2}\right)\left(1-\left(\frac{e^{\frac{\pi}{4}}+1}{e^{\frac{\pi}{4}}+e^{-\frac{\pi}{4}}}\right)\right)e^{t}+\left(\frac{1-\alpha}{2}\right)\left(\frac{e^{\frac{\pi}{4}}+1}{e^{\frac{\pi}{4}}+e^{-\frac{\pi}{4}}}\right)e^{-t}+\frac{3+7\sqrt{2}}{2}\sin t+\frac{3}{2}\cos t$$

For (2,1)-solution, the lower and upper solutions are

$$\underline{y}_{\alpha}(t)=\left(1+\frac{1}{2}\alpha\right)\cos t+\left(\left(1+4\sqrt{2}\right)+\left(\frac{1-\sqrt{2}}{2}\right)\alpha\right)\sin t$$

$$\overline{y}_{\alpha}(t)=\left(2-\frac{1}{2}\alpha\right)\cos t+\left(\left(2+3\sqrt{2}\right)-\left(\frac{1-\sqrt{2}}{2}\right)\alpha\right)\sin t$$

Using Proposition 3.3, (1,1)-solution is a valid fuzzy level set since $\left(\frac{1-\alpha}{2}\right)\left(1-\left(\frac{e^{\frac{\pi}{4}}-1}{e^{\frac{\pi}{4}}+e^{-\frac{\pi}{4}}}\right)\right)>0$, (1,2)-solution is not a valid fuzzy level set when $t<\tan^{-1}\left(1-\sqrt{2}\right)$, (2,2)-solution is not a valid fuzzy level set since $\left(\frac{1-\alpha}{2}\right)\left(1-\left(\frac{e^{\frac{\pi}{4}}+1}{e^{\frac{\pi}{4}}+e^{-\frac{\pi}{4}}}\right)\right)<0$, (2,1)-solution is not a valid fuzzy level set when $t>\tan^{-1}\left(1+\sqrt{2}\right)$.

All solutions are symmetric triangle fuzzy function of t.

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