ISSN 2347-1921

Generalization of Retractable and Coretractable Modules

Nasr Zeyada, O. Al-mallah, and A. Al-Zmammy

Cairo University, Faculty of Science, Department of Mathematics, Egypt,

King Abdulaziz University, Faculty of Science, Department of Mathematics, Saudi Arabia

nzeyada@kau.edu.sa

King Faisal University, Faculty of Science, Department of Mathematics, Saudi Arabia

omarmallah@yahoo.com

King Faisal University, Faculty of Science, Department of Mathematics, Saudi Arabia

al.zmammy@gmail.com

ABSTRACT

In this work, we extend the notion of retractability to s- retractability. An R-module M is called s-retractable if $\hom_R(M,U)\neq 0$ for all nonzero $U\subseteq Z(M)$. Also we extend coretractable modules to semi-coretractable modules. An R-module M is called semi-coretractable if $\hom_R(M/K,M)\neq 0$ for all maximal essential submodule K of M. We investigate theseclasses of modules and extend some of main theorems on retractable and coretractable modules to s-retractable and semi-coretractable modules, respectively.

Keywords:

Retractable Module; s-retractable Module; Coretractable Module; Semi-coretractable Module; Kasch Ring.

2000 Mathematics Subject Classification: 13C11, 16D50, 16L60

INTRODUCTION

Throughout this paper, all rings are associative with identity and all modules are unitary right modules unless stated otherwise. Semisimple rings are in the sense of Wedderburn and Artin. By a right max ring, we mean a ring R such that every nonzero right R-module has a maximal submodule. For a module M_R we write $soc(M_R)$, $Z(M_R)$ and $rad(M_R)$ for the socle, the singular and the Jacobson radical of M_R, respectively. Also $J(R)$ denotes the Jacobson radical of a ring R. The notation $K \leq M$, $K \subseteq^{sm} M$, and $K \subseteq^{ess} M$ denote that K is a submodule, a small submodule and an essential submodule of M, respectively. Following Khuri [7], an R-module M is said to be retractable if $\hom_R(M,N) \neq 0$ for any nonzero submodule N of M. Retractable modules have been investigated by some authors (see, for example, [2, 3, 4, 8, 9, 11]). In section 2 we define the concept of s-retractability of modules. Coretractable modules are the dual notion of retractable modules and it defined by B. Amini, M. Ershad and H. Sharif in the paper [1] as follows: An R-module M is called coretractable if, for any proper submodule K of M, there exists a nonzero homomorphism $f: M \to M$ with $f(K)=0$, that is, $\hom_R(M/K,M) \neq 0$. In section 3, we introduce the notion of semi-coretractability of a module M_R. A module M is called semi-coretractable if $\hom_R(M/K,M) \neq 0$ for all maximal essential submodule K of M_R. We show that a ring R is semi-coretractable as a right R-module if and if only R is right Kasch ring.

Definition 1. An R-module M_R is called s-retractable (sssrad-retractable) if $\hom_R(M,U) \neq 0$ for all nonzero $U \subseteq Z(M)$ ($U \subseteq rad(M)$).

Example 1.

1. Every nonsingular right R-module is s-retractable.
2. Every right R-module with zero radical is rad-retractable.
3. Every retractable right R-module is s-retractable.
4. Since Q is nonsingular Z-module so Q is s-retractable.

We have $\hom_Z(Q,Z)=0$, then Q is not retractable.

Lemma 1. If the R-module M is uniform with $Z(M) \neq 0$ and $\hom_R(M,K) \neq 0$ for all $K \subseteq Z(M)$, then M is retractable.

Proof. Take $0 \neq K \subseteq Z(M)$ then there exist a nonzero homomorphism $f: \mathrm{M} \to \mathrm{K}$. Since M is uniform so $K \subseteq^{ess} M$ and for any submodule L of M, $L \cap K \neq 0$. Hence, there exist a nonzero homomorphism $g: M \to L \cap K \to L$. Therefore, M is retractable.

Corollary 1. If $Z(M) \subseteq^{ess} M$, then M is retractable if and if only M is s-retractable

Proof. Let M be an s-retractable and K be a nonzero submodule of M. Then, $K \cap Z(M) \neq 0$ and there exist a nonzero homomorphism $g: M \to K \cap Z(M) \to K$. Therefore, M is retractable.

Recall that an R-module M is Goldie torsion if $M = Z_2(M)$, where $Z_2(M)$ is the second singular submodule of M.

Corollary 2. Let M be Goldie torsion. Then M is retractable if and if only M is s-retractable.

Proof. Let M be an s-retractable and K be a nonzero submodule of M. Then, $K \cap Z(M) \neq 0$ and there exist a nonzero homomorphism $g: M \to K \cap Z(M) \to K$. Therefore, M is retractable.

Recall that an R-module M is semi-hollow if every finitely generated submodule of M is small.

Corollary 3. Let M be semi-hollow. Then M is retractable if and only if M is rad-retractable.

Corollary 4. If M is semi-hollow and every cyclic submodule of M is injective, then $\hom_R(M,K) \neq 0$ for all

ISSN 2347-1921

$K \subseteq rad(M)$.

Proof. Let $K \subseteq rad(M)$ be a nonzero submodule. If $0 \neq mR \subseteq K$, then mR is injective and it is a summand of M. Thus, there exist a nonzero homomorphism $\pi : M \to mR$. Hence, we have nonzero homomorphism $M \to K.$

Definition 2. A ring R is called right SC-ring if every singular right R-module is semisimple.

Corollary 5. Let R be a right SC-ring. If M is an R-module such that $\hom_R(M,K) \neq 0$ for all $K \subseteq soc(M)$, then M is s-retractable. In particular, R is s-retractable.

S-retractability for group modules

Tamer Kosan and Jan Zemlicka [9], showed that if M is an R-module and G is finite group, then the group module MG is a retractable RG-module if and if only M_R is retractable. In the following we show that MG is s-retractable as R-module if and if only M_R is s-retractable. Throughout this section G is a group and M is a module over a ring R. Let MG denote the set all formal linear combinations of the form $\sum_{g\in G} m_g g$ where $m_g \in M$ and $m_g = 0$ almost for every g. For elements $\sum_{g\in G} m_g g$ and $\sum_{g\in G} n_g g$ in MG, by writing $\sum_{g\in G} m_g g = \sum_{g\in G} n_g g$ we mean that $m_g = n_g$ for all $g \in G$.

We define the sum in MG component wise: $\sum_{g\in G} m_g g + \sum_{g\in G} n_g g = \sum_{g\in G} (m_g g + n_g g)$.

For $\sum_{g\in G} r_g g \in RG$, the scalar product of $\sum_{g\in G} m_g g$ by $\sum_{g\in G} r_g g$ is defined by

$$\left(\sum_{g\in G} m_g g\right)\left(\sum_{g\in G} r_g g\right) = \sum_{g\in G} k_g g \text{ where } k_g = \sum_{hh'=g} m_h r_{h'} .$$

It is routineto check that, with these operations, MG becomes a right module over the group ring RG. Note that M is an R-submodule of MG such that $m = m\cdot 1$, where 1 here denotes the identity element of G. It is well known that the identity element in G is also the identity element of RG.

Lemma 2. [9] If MG is the group module, then $MG \cong_{RG} M \otimes_R RG$.

Lemma 3. [9] Let MG be the group module of G by M over RG. Then for any $x \in MG$ and any $\alpha \in RG$, $\varepsilon_M(x\alpha) = \varepsilon_M(x)\varepsilon(\alpha)$. In particular, ε_M is an R-homomorphism and ε_R is a ring homomorphism.

Proposition 1. Let M be an R-module. MG is s-retractable R-module if and if only M_R is s-retractable.

Proof. If $\sum_{i=1}^{n} m_{g_i} g_i \in Z_R(MG)$, then $r\left(\sum_{i=1}^{n} m_{g_i} g_i\right) \subseteq^{ess} R$. But $x \in r\left(\sum_{i=1}^{n} m_{g_i} g_i\right)$ implies that $\left(\sum_{i=1}^{n} m_{g_i} g_i\right)x = 0 \Rightarrow m_{g_i} x = 0$ for all $i = 1,2,.....,n$ and $x \in r(m_{g_i})$ for all $i = 1,2,.....,n$, thus $r\left(\sum_{i=1}^{n} m_{g_i} g_i\right) \subseteq \bigcap_{i=1}^{n} r(m_{g_i})$ and $r(m_{g_i}) \subseteq^{ess} R$ for all $i = 1,2,.....,n$. Hence, $m_{g_i} \in Z(M)$ and $Z_R(MG) \subseteq Z_R(M)G$. Also the other inclusion is true, therefore $Z_R(MG) = Z_R(M)G$.

Now suppose that MG is s-retractable, so for every nonzero singular submodule K of M there exist a nonzero R-homomorphism $\alpha : MG \to KG$. Then, there exist a nonzero R-homomorphism $\beta = \varepsilon_K \alpha i_M : M \to K$ where i_M is the inclusion $i_M : M \to MG$. For the converse if M is s-retractable, so for every nonzero singular submodule K of M there exist a nonzero R-homomorphism $\alpha : M \to K$. Then, there exist a nonzero R-homomorphism $\beta = i_K \alpha \pi_M : MG \to KG$ where i_K is the inclusion $i_K : K \to KG$ and π_M is natural epimorphism

ISSN 2347-1921

$\pi_M : MG \to M$.

Proposition 2. Let G be a group and M be an R-module. Then $Z_R(M)G \subseteq Z_{RG}(MG)$.

Proof. Firstly we show that if K is essential submodule of M, then KG is essential in MG. Suppose $0 \neq \sum_{g\in G} m_g g \in MG$ such that $KG \bigcap \left(\sum_{g\in G} m_g g\right) RG = 0$ but,

$$\left(\sum_{g\in G} m_g g\right) RG = \left\{\left(\sum_{g\in G} m_g g\right)\left(\sum_{g\in G} r_g g\right) : \left(\sum_{g\in G} r_g g\right) \in RG\right\} =$$
$$\left\{\sum_{g\in G} l_g g : l_g = \sum_{h_1 h_2 = g} m_{h_1} r_{h_2} \text{ and } \left(\sum_{g\in G} r_g g\right) \in RG\right\} =$$
$$\left\{\sum_{g\in G} l_g g : l_g \in \sum_{g\in G} m_g R\right\} = \left(\sum_{g\in G} m_g R\right) G.$$

So $KG \bigcap \left(\sum_{g\in G} m_g R\right) G = 0$ and $K \bigcap \sum_{g\in G} m_g R = 0$ which is a contradiction. Now if $0 \neq \sum_{i=1}^{n} a_i g_i \in Z_R(M)G$, then $r(a_i) \subseteq^{ess} R$ for every i. Thus $r(a_i)G \subseteq^{ess} RG$ for every i and $\bigcap_{i=1}^{n} r(a_i)G \subseteq^{ess} RG$. But

$$r_{RG}\left(\sum_{i=1}^{n} a_i g_i\right) = \left\{\sum_{g\in G} r_g g \in RG : \left(\sum_{i=1}^{n} a_i g_i\right)\left(\sum_{g\in G} r_g g\right) = 0\right\} =$$
$$\left\{\sum_{g\in G} r_g g \in RG : \sum_{g\in G} k_g g = 0, k_g = \sum_{h_1 h_2 = g} a_{h_1} r_{h_2}\right\} =$$
$$\left\{\sum_{g\in G} r_g g : \sum_{h_1 h_2 = g} a_{h_1} r_{h_2} = 0\right\}$$

and so $\bigcap_{i=1}^{n} r(a_i)G \subseteq r_{RG}\left(\sum_{i=1}^{n} a_i g_i\right)$. Hence $Z_R(M)G \subseteq Z_{RG}(MG)$.

The next example shows that the $Z_R(M)G \neq Z_{RG}(MG)$ in the general case.

Example 2. Let R be the field Z_2 and $G = \{e, a, \mathrm{b}, \mathrm{c}\} \cong Z_5^{\,*}$. Then RG is local Kasch ring $Z_r = 0$ and the proper ideals of RG are:

$$I_1 = (e+a)RG = \{0, e+a, e+b, e+c, a+b, a+c, b+c, e+a+b+c\}$$
$$I_2 = (e+c)RG = \{0, e+c, a+b, e+a+b+c\}$$
$$I_3 = (e+a+b+c)RG = \{0, e+a+b+c\}$$

We see that each of them is essential in R and $Z_{RG}(RG) = I_1 = J(RG)$. Also we see that $soc(RG) = I_3$.

Corollary 6. Let M be an R-module. If MG is s-retractable RG-module, then M_R is s-retractable.

Proof. Suppose that MG is s-retractable RG-module and K is a nonzero singular submodule of M. By above Proposition KG is a singular RG-submodule of MG and $\hom_{RG}(MG, KG) \neq 0$. Then by [9, Corollary 2.5] $\hom_R(M, K) \neq 0$. Hence M is s-retractable.

Definition 3. A ring R is called (finitely) s-mod-retractable if all (finitely gener-ated) right R-modules are s-retractable.

Example 3.

1. Semisimple rings are s-mod-retractable.
2. Let R be a right (left) SI ring (every singular right (left) R-module is injective). Then, for any right R-module M, every singular submodule of M is a summand of M. Hence R is s-mod-retractable ring.
3. Let R be a right (left) V-ring (every right (left) R-module has a zero radical). Then, every right R-module is rad-retractable. Hence R is rad-mod-retractable ring.

Theorem 1. (Finite) s-mod-retractability is Morita invariant.

Proof. Let R and S be Morita equivalent rings. Assume that $f: Mod-R \to Mod-S$ and $g: Mod-S \to Mod-R$ are two category equivalences. Let M be an s-retractable R-module. Then, M is an s-retractable object in $Mod-R.$ Let $0 \neq N$ be a singular submodule of $f(M)$. Then, $\hom_R(M, g(N)) \neq 0$ since $g(N)$ is isomorphic to a singular submodule of M. Thus, we have $0 \neq \hom_S(f(M), fg(N)) \cong \hom_S(f(M), N).$ This follows that M is an s-retractable object in $Mod-S.$

Let R be a ring, n a positive integer and the ring $M_n(R)$ of all $n \times n$ matrices with entries in R.

Corollary 7. If R is (finitely) s-mod-retractable, then $M_n(R)$ is (finitely) s-mod-retractable.

Proof. By above Theorem.

Proposition 3. The class of (finite) s-mod-retractable rings is closed under taking homomorphic images.

Proof. Suppose R is a (finite) s-mod-retractable ring. It is well-known that $\hom_R(M,N) = \hom_{R/I}(M,N)$ for each ideal I of R and $M, N \in Mod-R/I$. Now the proof is clear.

Proposition 4. R is a right s-mod-retractable ring if and if only for every non-zero module M and every $m \in Z(M)$ such that $mR \subseteq^{ess} M$ there exists a non-zero homomorphism $M \to mR.$

Proof. The direct implication is clear. For the converse, assume that M is a nonzero module and N is a nonzero submodule. Let $n \in Z(N)$ be a nonzero. Then the identity map $id_{nR}: nR \to nR$ on nR may be extended to a homomorphism $\alpha: M \to E(nR).$ Note that $nR \subseteq^{ess} \alpha(M) \subseteq E(nR)$. Hence, by the hypothesis there is a non-zero homomorphism $\alpha(M) \to nR \subseteq N.$

Recall that, a ring is called right max provided every non-zero right module contains a maximal submodule.

Proposition 5. If R is a right s-mod-retractable ring, then R is right max.

Proof. If M is semisimple, then M has a maximal submodule. Now assume that $0 \neq M$ is not semisimple and contains no maximal submodule, fix $0 \neq m \in M \setminus soc(M)$ and an arbitrary maximal essential submodule N of mR. Then, M/N contains no maximal submodule and so there exist no non-zero homomorphism M/N into a simple mR/N, i.e. M/N is not s-retractable.

Recall that a torsion theory $\tau = (T, F)$ is a pair of classes of modules closed under isomorphic images such that $T \cap F = 0$, T is closed under taking factors, F is closed under submodules and for every module M there exists a submodule $\tau(M)$ for which $\tau(M) \in T$ and $\mathrm{M}/\tau(M) \in F$. Moreover, a torsion theory is hereditary if T is closed under submodules.

We extend the hereditary torsion theory to s-hereditary, where a torsion theory $\tau = (T, F)$ is called s-hereditary if T is closed under singular submodules.

For a class of right R-modules C, we consider the following annihilator classes:

$${}^{\circ}C = \left\{ M \in Mod - R \,\middle|\, \hom_R(M, C) = 0 \right\} \text{ and}$$

$$C^{\circ} = \left\{ M \in Mod - R \,\middle|\, \hom_R(C, M) = 0 \right\}$$

We notice that the annihilator classes of the form ${}^{\circ}C$ for some $C \subseteq Mod - R$ coincide with the torsion classes of modules, and C° coincide with the torsionfree classes of modules.

Theorem 2. A ring R is s-mod-retractable if and if only every torsion theory on $Mod - R$ is s-hereditary.

Proof. Suppose that R is s-mod-retractable and $\tau = (T, F)$ is a torsion theory. For $M \in T$ and $N \subseteq Z(M)$, let $\tau(N)$ be the torsion part of N. Then, $M / \tau(N) \in T$, while $N / \tau(N) \in F$. Then, $\hom_R(M / \tau(N), N / \tau(N)) = 0$. Since $N / \tau(N)$ is a submodule of $M / \tau(N)$ and $M / \tau(N)$ is s-retractable, it follows that $N / \tau(N) = 0$. Hence, $N \in T$. Conversely, suppose that M is an R-module and $0 \neq N \subseteq Z(M)$. If $\hom_R(M, N) = 0$, then $N \notin {}^{\circ}(M^{\circ})$. This implies that the torsion theory $({}^{\circ}(M^{\circ}), M^{\circ})$ is not s-hereditary.

Proposition 6. Let M_R be s-retractable with $S = End_R(M)$. If $f \in S$ with $Kerf \subseteq Z(M)$, then f is a monomorphism if and if only f is right regular in S.

Proof. This follows from the s-retractable condition on M_R and the fact that $r \cdot ann(f) = \hom_R(M, Kerf)$.

Proposition 7. The ring $\prod_{i \in I} R_i$ is right s-mod-retractable if and if only each R_i is a right s-mod-retractable ring for each $i \in I$, where I is an arbitrary finite set.

Proof. Assume that $\prod_{i \in I} R_i$ is right s-mod-retractable. Since R_i is a homomorphic image of $\prod_{i \in I} R_i$, so the result follows from Proposition 3. Now Let each e_i denote the unit element of R_i for all $i \in I$. A module M over $\prod_{i \in I} R_i$ may be written as set direct product $\prod_{i \in I} M_i$, where $M_{iR_i} = Me_i$ and external operation defined as $(r_i)_{i \in I}(m_i)_{i \in I} = (r_i m_i)_{i \in I}$. Thus, s-retractability of M is given by s-retractability of each M_i $i \in I$. But, since each R_i is s-mod-retractable, this condition is satisfied.

Corollary 8. The class of s-mod-retractable rings is closed under taking finite direct products.

Proof. By above Proposition.

If R is a ring, $R[X]$ denotes the polynomial ring with X a set of commuting indeterminate over R. If $X = \{x\}$, then we use $R[x]$ in place of $R[\{x\}]$.

Proposition 8. If $R[x]$ is s-mod-retractable ring, then R is s-mod-retractable ring.

Proof. Since $R \cong R[x] / R[x]x$, the result is clear from Proposition 3.

Semi-coretractable Modules

Definition 4. A right R-module M is called semi-coretractable if $\hom_R(M / K, M) \neq 0$ for every maximal and essential submodule K of M.

Example 4. The Z-module Q is semi-coretractable but not coretractable.

Corollary 9. Let M is finitely generated R-module. If M is uniform R-module and semi-coretractable, then M is coretractable.

Proof. Let L be a nonzero submodule of M. Then, there exist a maximal submodule K of M such that $L \subset K$ (M is finitely generated). Thus, there exist a nonzero homomorphism $f \in End(M)$ such that $f(K)=0.$ Hence $f(L)=0$ and M is coretractable.

Corollary 10. If M is semi-coretractable and M has an essential maximal submodule, then $soc(M) \bigcap Z(M) \neq 0.$

Proof. Let K be maximal essential submodule of M. Then, $\hom_R(M/K, M) \neq 0.$ Since M/K is simple singular so $soc(M) \bigcap Z(M) \neq 0.$

Proposition 9. For a ring R, the following statements are equivalent:

(1) R is a right Kasch ring;

(2) R_R is a coretractable module;

(3) R_R is a semi-coretractable module.

Proof. $(1) \Rightarrow (2)$ Let K be a right ideal. There exist a maximal right ideal M such that $K \subseteq M$. Then, there exist a nonzero homomorphism $\alpha : R/M \to R$. Thus, there a nonzero map $\beta = \varphi\alpha : R/K \to R$ where φ is natural epimorphism $R/K \to R/M$. Hence, R is coretractable.

$(2) \Rightarrow (3)$ Clear.

$(3) \Rightarrow (1)$ If L is a nonessential maximal right ideal, then $L \bigcap aR = 0$ for some $0 \neq a \in R$ and $L \oplus aR = R.$ Thus, $L = eR$ where $e^2 = e$ and $l(L) = R(1-e) \neq 0.$ Now, if L is an essential maximal right ideal, then by

(3) $\hom(R/L, R) \neq 0.$ Hence, there exist an nonzero homomorphism

$f : R/L \to R$ and if $f(1+L) = a \neq 0$ so $a \in l(L).$ Therefore, R is right Kasch.

The following Proposition shows that if the ring R is right Kasch, then all free right R-module are semi-coretractable without any extra conditions.

Proposition 10. For a ring R the following statements are equivalent:

(1) R is right Kasch

(2) All free right R-modules are semi-coretractable.

Proof. $(1) \Rightarrow (2)$. Let F be a nonzero free right R-module and K be a maximal essential submodule of F. Then, F/K is singular simple right R-module and $\hom_R(F/K, R) \neq 0$. Hence, $\hom_R(F/K, F) \neq 0$.

$(2) \Rightarrow (1)$. Above Proposition.

Proposition 11. Let R be a ring. If all cyclic right R-modules are semi-coretractable, then R is a left perfect ring.

Proof. Suppose that every cyclic right R-module is semi-coretractable, so R/J is semi-coretractable as a right R-module. Thus, R/J is semi-coretractable as a right R/J-module and R/J is right Kasch ring (Proposition 9). But $J(R/J)=0$ so R/J is semisimple ring. Now if M be a nonzero cyclic right R-module then by the fact $rad(M) \neq M$, M has a maximal submodule. If M has no essential submodule, then M is semisimple. If M has an essential submodule L, then L contained in a maximal submodule of M and $soc(M) \neq 0$. Hence, R is right semi-artinian and J is left T-nilpotent [12, Corollary VIII 2.7]. Therefore, R is left perfect ring.

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