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Two Inequalities and Two Means

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ABSTRACT

The paper presents geometric derivations of Jensen's and Hermite-Hadamard's inequality. Jensen's inequality is further involved to a concept of quasi-arithmetic means. Hermite-Hadamard's inequality is applied to compare the basic quasi-arithmetic means, power and logarithmic means.

Keywords and phrases

Jensen's inequality; Hermite-Hadamard's inequality; power means; generalized logarithmic means.

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1. INTRODUCTION

Some research fields close to mathematics in the basis of their work use estimations between the two superior states expressed by the numerical values a and b. So we are talking about the means ranging between a and b, concerning inequalities in mathematics and entropies in applied sciences.

The branch of mathematical inequalities is focused to convex sets and convex functions of a real linear space $\mathcal{X}$. A set $\mathcal{A} \subseteq \mathcal{X}$ is convex if it contains the line segments connecting all pairs of its points (all binomial convex combinations $\alpha_1 A_1 + \alpha_2 A_2$ of points $A_1, A_2 \in \mathcal{A}$ and non-negative coefficients $\alpha_1, \alpha_2 \in \mathbb{R}$ satisfying $\alpha_1 + \alpha_2 = 1$). A function $f : \mathcal{A} \to \mathbb{R}$ is convex if the inequality

$$f(\alpha_1 A_1 + \alpha_2 A_2) \le \alpha_1 f(A_1) + \alpha_2 f(A_2) \qquad (1)$$

holds for all binomial convex combinations in $\mathcal{A}$.

Within the concept of convexity, we also use an affinity. A set $\mathcal{A} \subseteq \mathcal{X}$ is affine if it contains the lines passing through all pairs of its points (all binomial affine combinations $\alpha_1 A_1 + \alpha_2 A_2$ of points $A_1, A_2 \in \mathcal{A}$ and coefficients $\alpha_1, \alpha_2 \in \mathbb{R}$ satisfying $\alpha_1 + \alpha_2 = 1$). A function $f : \mathcal{A} \to \mathbb{R}$ is affine if the equality holds in equation (1) for all binomial affine combinations in $\mathcal{A}$.

Using the mathematical induction, it can be proved that a convex set contains all finite convex combinations of its points, and that every convex function satisfies the inequality in (1) for all finite convex combinations in $\mathcal{A}$. Similar is true for affine sets and functions.

2. JENSEN'S INEQUALITY

We want to present the famous Jensen's inequality (see[9]) by using the convex polygon.

Let numbers $a_1, \ldots, a_n$ belong to the domain of a real convex function f. Take the corresponding graph points $A_i = (a_i, f(a_i))$. Their convex hull

$$\mathcal{A} = \mathrm{conv}\{A_1, \ldots, A_n\} \qquad (2)$$

is convex polygon inscribed in the function epigraph

$$\mathrm{epi}f = \{(x, y) : y \ge f(x)\}. \qquad (3)$$

If $\alpha_1, \ldots, \alpha_n$ are non-negative coefficients satisfying $\sum_{i=1}^n \alpha_i = 1$, then the planar convex combination $\sum_{i=1}^n \alpha_i A_i$, that is, its center

$$A = \sum_{i=1}^n \alpha_i A_i = \left(\sum_{i=1}^n \alpha_i a_i, \sum_{i=1}^n \alpha_i f(a_i)\right)$$

belongs to the polygon $\mathcal{A}$. Therefore, it must be that

$$f\left(\sum_{i=1}^n \alpha_i a_i\right) \le \sum_{i=1}^n \alpha_i f(a_i), \qquad (4)$$

as can be seen in Figure 1. The inequality in equation (4) represents the discrete form of Jensen's inequality.

Remark 2.1. 1 *If the function f is strictly convex, then all the points A_i are vertices of the convex polygon $\mathcal{A}$.*

Example 2.2.2 *The application of Jensen' inequality to the function $f(x) = -\ln x$ and the arithmetic mean $(1/2)a^{-1} + (1/2)b^{-1}$ where a and b are positive numbers, yields the harmonic-geometric mean inequality*

$$\frac{2ab}{a+b} \le \sqrt{ab} \tag{5}$$

because

$$f\left(\frac{1}{2}a^{-1}+\frac{1}{2}b^{-1}\right)=\ln\frac{2ab}{a+b},\ \frac{1}{2}f(a^{-1})+\frac{1}{2}f(b^{-1})=\ln\sqrt{ab}.$$

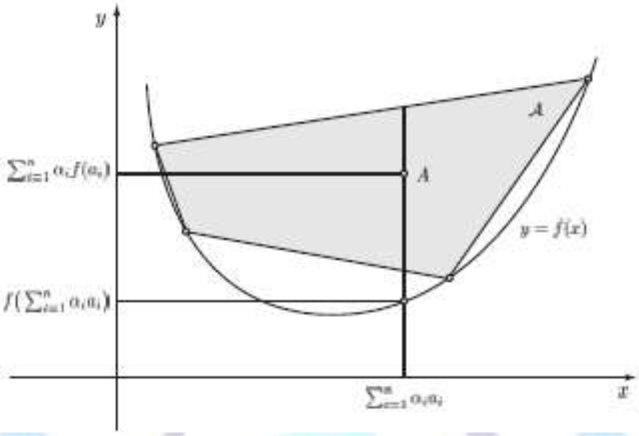

Figure 1. The geometric presentation of equation (4)

To get the integral form of Jensen's inequality inequality, we take an interval $[a,b]$ where $a<b$, and include an integrable function $g:[a,b]\to\mathbb{R}$. We assume that the image of g is contained in the domain of f, and $f(g)$ is integrable. Given the integer n, we take the points $a_{ni}=g(x_{ni})$ where $x_{ni}=a+(b-a)i/n$ assuming that $x_{n0}=a$, and the coefficients $\alpha_{ni}=1/n=[x_{ni}-x_{ni-1}]/(b-a)$. Substituting a_{ni} and α_{ni} in the inequality in equation (4), we get

$$f\left(\frac{1}{b-a}\sum_{i=1}^{n}\left[x_{ni}-x_{ni-1}\right]g\left(x_{ni}\right)\right)\le\frac{1}{b-a}\sum_{i=1}^{n}\left[x_{ni}-x_{ni-1}\right]f\left(g\left(x_{ni}\right)\right), \tag{6}$$

and letting n to infinity, we obtain the integral inequality

$$f\left(\frac{1}{b-a}\int_a^b g(x)dx\right)\le\frac{1}{b-a}\int_a^b f\left(g(x)\right)dx. \tag{7}$$

The importance and applicability of convex combinations dealing with inequalities can be seen in [13].

3. HERMITE-HADAMARD'S INEQUALITY

First we would like to derive the well-known Hermite-Hadamard's inequality (see [8] and [7]) in a simple way. For this purpose, two lines will be used.

Let a and b be real numbers such that $a<b$, and let $f:[a,b]\to\mathbb{R}$ be a convex function. Let $f_{\{a,b\}}^{\mathrm{cho}}$ be the function of the chord line passing through the points $A(a,f(a))$ and $B(b,f(b))$. Let $c\in(a,b)$ be an interior point, and let $f_{\{c\}}^{\mathrm{sup}}$ be the function of some support line passing through the point $C(c,f(c))$. Then the inequality

$$f_{\{c\}}^{\mathrm{sup}}(x)\le f(x)\le f_{\{a,b\}}^{\mathrm{cho}}(x) \tag{8}$$

holds for every $x\in[a,b]$. Integrating the above inequality on the interval $[a,b]$, we obtain

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$$(b-a)\, f_{\{c\}}^{\sup}\left(\frac{a+b}{2}\right) \le \int_a^b f(x)dx \le (b-a)\, f_{\{a,b\}}^{\text{cho}}\left(\frac{a+b}{2}\right), \tag{9}$$

as evidenced by Figure 2 (it is obvious that the area of the curvilinear trapezoid is between the areas of the support and chord trapeze). Applying the midpoint $c=(a+b)/2$ to the support line, and using the affinity of the chord line, we have

$$f_{\{\frac{a+b}{2}\}}^{\sup}\left(\frac{a+b}{2}\right) = f\left(\frac{a+b}{2}\right),$$

and

$$f_{\{a,b\}}^{\text{cho}}\left(\frac{a+b}{2}\right) = \frac{1}{2} f_{\{a,b\}}^{\text{cho}}(a) + \frac{1}{2} f_{\{a,b\}}^{\text{cho}}(b) = \frac{f(a)+f(b)}{2}.$$

Involving the above equalities to equation (9), and dividing with $b-a$, we achieve the Hermite-Hadamard inequality

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}. \tag{10}$$

The discrete form

$$f\left(\frac{a+b}{2}\right) \le \sum_{i=1}^{n} \alpha_i f(a_i) \le \frac{f(a)+f(b)}{2}. \tag{11}$$

holds for every convex combination $c=\sum_{i=1}^{n}\alpha_i a_i$ of points $a_i \in [a,b]$ with the center $c=(a+b)/2$.

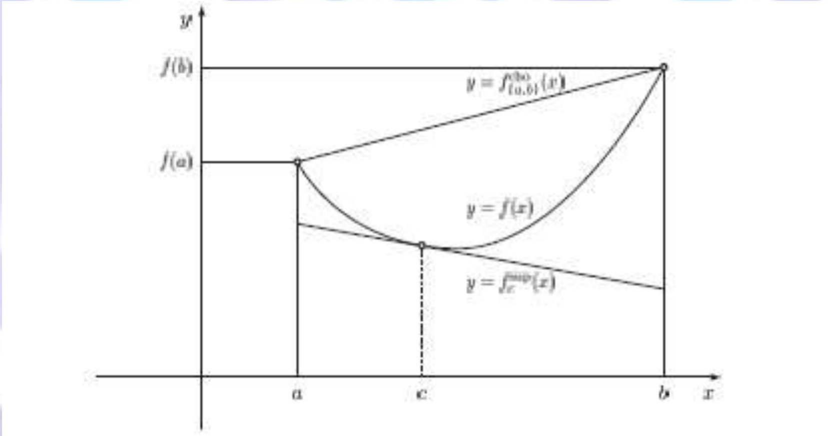

Figure 2. The geometric presentation of equation (9)

Remark 3.1. 3 *The support trapeze area on the left side of equation (9) attains the maximal value at the midpoint* $c=(a+b)/2$.

Example 3.2. 4 *The application of Hermite-Hadamard's inequality to the exponential function* $f(x)=e^x$ *on the interval* $[\ln a, \ln b]$ *where* $0<a<b$, *yields the geometric-logarithmic-arithmetic mean inequality*

$$\sqrt{ab} \le \frac{b-a}{\ln b - \ln a} \le \frac{a+b}{2} \tag{12}$$

because

$$f\left(\frac{\ln a+\ln b}{2}\right)=\sqrt{ab},\ \int_{\ln a}^{\ln b}e^{x}\,dx=b-a,\ \frac{f(\ln a)+f(\ln b)}{2}=\frac{a+b}{2}.$$

4. QUASI-ARITHMETIC MEANS

Every convex combination $c=\sum_{i=1}^{n}\alpha_i a_i$ of numbers a_i such that $a_{\min}=a<a_{\max}=b$ can be reduced to the binomial form $c=\alpha a+\beta b$, where $\alpha=(b-c)/(b-a)$ and $\beta=(c-a)/(b-a)$. So, the means between the two given numbers are preferred. To generalize a notion of the arithmetic mean of numbers a and b, we use a strictly monotone continuous function $\varphi:[a,b]\to\mathbb{R}$ assuming that $a<b$. Theorem 4.2 indicates the way in which two specific means can be compared, and it is the main section result.

The discrete quasi-arithmetic mean of the numbers a and b respecting the function φ is defined by the number

$$M_{\varphi}^{\mathrm{dis}}(a,b)=\varphi^{-1}\left(\frac{\varphi(a)+\varphi(b)}{2}\right). \tag{13}$$

The integral quasi-arithmetic mean of the numbers a and b respecting φ is the number

$$M_{\varphi}^{\mathrm{int}}(a,b)=\varphi^{-1}\left(\frac{1}{b-a}\int_a^b\varphi(x)dx\right). \tag{14}$$

The discrete or integral quasi-arithmetic mean $M_{\varphi}(a,b)$ is in $[a,b]$ because the numbers in parentheses are in $\varphi([a,b])$. These means satisfy the affinity property, that is, the equality

$$M_{\alpha\varphi+\beta}(a,b)=M_{\varphi}(a,b) \tag{15}$$

holds for every pair of real numbers $\alpha\neq 0$ and β.

According to the right inequality of the Hermite-Hadamard formula in equation (10), we have the inequality

$$M_{\varphi}^{\mathrm{int}}(a,b)\le M_{\varphi}^{\mathrm{dis}}(a,b) \tag{16}$$

if φ is either convex and increasing or concave and decreasing, and the reverse inequality if φ is either convex and decreasing or concave and increasing.

Lemma 4.1. 5 *Let $f:[a,b]\to\mathbb{R}$ be a convex function. Let $[c,d]\subset[a,b]$ be a subinterval such that $c<d$ and*

$$\frac{c+d}{2}=\frac{a+b}{2}. \tag{17}$$

Then

$$\frac{f(c)+f(d)}{2}\le\frac{f(a)+f(b)}{2} \tag{18}$$

and

$$\frac{1}{d-c}\int_c^d f(x)dx\le\frac{1}{b-a}\int_a^b f(x)dx. \tag{19}$$

Proof. Let us prove the integral inequality in equation (19). Put $\mathcal{A}=[c,d]$, $|\mathcal{A}|=d-c$, $\mathcal{B}=[a,b]$ and $|\mathcal{B}|=b-a$. Assuming equation (17), and applying the affinity of the function $f_{\{c,d\}}^{\mathrm{cho}}$, we obtain the equalities

$$\frac{1}{|\mathcal{A}|}\int_{\mathcal{A}}f_{\{c,d\}}^{\mathrm{cho}}(x)dx=\frac{1}{|\mathcal{B}|}\int_{\mathcal{B}}f_{\{c,d\}}^{\mathrm{cho}}(x)dx=\frac{1}{|\mathcal{B}\setminus\mathcal{A}|}\int_{\mathcal{B}\setminus\mathcal{A}}f_{\{c,d\}}^{\mathrm{cho}}(x)dx. \tag{20}$$

Using equation (20), and the fact that $f(x) \le f^{\text{cho}}_{\{c,d\}}(x)$ for all $x \in \mathcal{A}$ as well as $f^{\text{cho}}_{\{c,d\}}(x) \le f(x)$ for all $x \in \mathcal{B} \setminus \mathcal{A}$, we get

$$\begin{aligned} \frac{1}{|\mathcal{A}|}\int_{\mathcal{A}} f(x)dx &\le \frac{1}{|\mathcal{A}|}\int_{\mathcal{A}} f^{\text{cho}}_{\{c,d\}}(x)dx = \frac{1}{|\mathcal{B}\setminus\mathcal{A}|}\int_{\mathcal{B}\setminus\mathcal{A}} f^{\text{cho}}_{\{c,d\}}(x)dx \\ &\le \frac{1}{|\mathcal{B}\setminus\mathcal{A}|}\int_{\mathcal{B}\setminus\mathcal{A}} f(x)dx. \end{aligned} \tag{21}$$

Now the binomial convex combination

$$\frac{1}{|\mathcal{B}|}\int_{\mathcal{B}} f(x)dx = \frac{|\mathcal{A}|}{|\mathcal{B}|}\left(\frac{1}{|\mathcal{A}|}\int_{\mathcal{A}} f(x)dx\right) + \frac{|\mathcal{B}\setminus\mathcal{A}|}{|\mathcal{B}|}\left(\frac{1}{|\mathcal{B}\setminus\mathcal{A}|}\int_{\mathcal{B}\setminus\mathcal{A}} f(x)dx\right) \tag{22}$$

provides the inequality in equation (19). □

Relying on the above lemma, we can prove the next rule for comparison of discrete or integral quasi-arithmetic means $M_\varphi(c,d)$ and $M_\varphi(a,b)$.

Theorem 4.2. 6 *Let $\varphi:[a,b] \to \mathbb{R}$ be a strictly monotone continuous function. Let $[c,d] \subset [a,b]$ be a subinterval such that $c < d$ and*

$$\frac{c+d}{2} = \frac{a+b}{2}. \tag{23}$$

If φ is either convex and increasing or concave and decreasing, then

$$M_\varphi(c,d) \le M_\varphi(a,b). \tag{24}$$

If φ is either convex and decreasing or concave and increasing, then the reverse inequality is valid in equation (24).

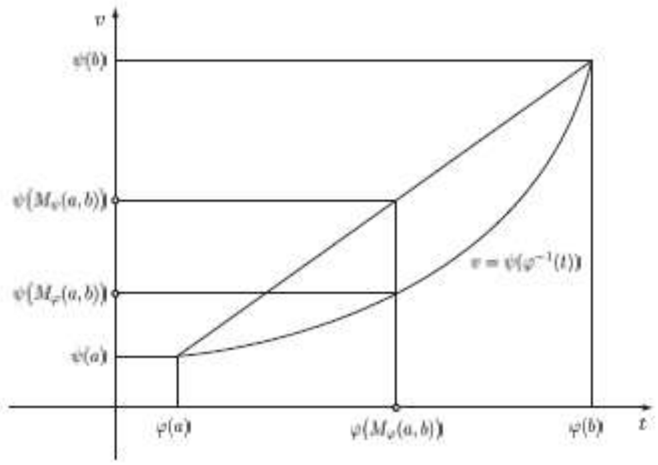

Figure 3. The graphic presentation of equation (25)

Having two strictly monotone continuous functions $\varphi, \psi:[a,b] \to \mathbb{R}$, we say that the function ψ is φ-convex if the composition function $\psi(\varphi^{-1})$ is convex. In order to compare quasi-arithmetic means M_φ and M_ψ, we rely on the details presented in the following theorem.

Theorem A. 7 *Let $\varphi, \psi:[a,b] \to \mathbb{R}$ be strictly monotone continuous functions.*

If ψ is either φ-convex and increasing or φ-concave and decreasing, then

$$M_\varphi(a,b) \le M_\psi(a,b). \tag{25}$$

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If ψ is either φ-convex and decreasing or φ-concave and increasing, then the reverse inequality is valid in equation (25).

The inequality in equation (25) is strict if ψ is either strictly φ-convex or strictly φ-concave. To prove Theorem A we apply Jensen's inequality to the convex or concave function $f=\psi(\varphi^{-1})$. Different forms of quasi-arithmetic means have been considered in [12].

The quasi-arithmetic mean inequality in equation (25) is graphically presented in Figure 3. The point $M(\varphi(M_{\varphi}(a,b)),\psi(M_{\psi}(a,b)))$ is located at the chord line in the discrete case, and M is located between the curve graph and chord line in the integral case.

5. POWER AND LOGARITHMIC MEANS

Power means are investigated and used as a special case of quasi-arithmetic means. These means apply the power function $\varphi(x)=x^r$. Logarithmic means arise by continuous extending of power means. The main result of the section is the presentation of Theorem 5.2 determining the order of power and logarithmic means.

Take positive numbers a and b, and consider the discrete mean

$$M^{\text{dis}}_{\varphi(x)=x^r}(a,b)=\left[\frac{a^r+b^r}{2}\right]^{\frac{1}{r}} \tag{26}$$

with the exponent $r\neq 0$. Calculating the limit as r approaches 0, we get the power means of order r in the form

$$M_r(a,b)=\begin{cases}\left[\dfrac{a^r+b^r}{2}\right]^{\frac{1}{r}} & , \quad r\neq 0\\ \sqrt{ab} & , \quad r=0\end{cases}. \tag{27}$$

The known members of the mean collection M_r are harmonic mean $H=M_{-1}$, geometric mean $G=M_0$, and arithmetic mean $A=M_1$.

Now take different positive numbers a and b, and consider the integral mean

$$M^{\text{int}}_{\varphi(x)=x^{r-1}}(a,b)=\left(\frac{1}{b-a}\int_a^b x^{r-1}dx\right)^{\frac{1}{r-1}}=\left[\frac{a^r-b^r}{r(a-b)}\right]^{\frac{1}{r-1}} \tag{28}$$

with the exponent $r\neq 0,1$. Calculating the limits as r approaches 0 and 1, and the limit as b approaches a, we get the generalized logarithmic means (see [17]) of order r as

$$L_r(a,b)=\begin{cases}\left[\dfrac{a^r-b^r}{r(a-b)}\right]^{\frac{1}{r-1}} & , \quad r\neq 0,1 \quad , \quad a\neq b\\ \dfrac{a-b}{\ln a-\ln b} & , \quad r=0 \quad , \quad a\neq b\\ \dfrac{1}{e}\left[\dfrac{a^a}{b^b}\right]^{\frac{1}{a-b}} & , \quad r=1 \quad , \quad a\neq b\\ a & , \quad\quad\quad\quad\quad a=b\end{cases}. \tag{29}$$

The important means of the collection L_r are geometric mean $G = L_{-1}$, logarithmic mean $L = L_0$, identric mean $I = L_1$, and arithmetic mean $A = L_2$.

Remark 5.1. 8 *The logarithmic and identric mean for different positive numbers a and b can be obtained by using integrals,*

$$L(a,b) = \left(\frac{1}{b-a}\int_a^b \frac{1}{x}dx\right)^{-1} = \frac{a-b}{\ln a - \ln b} \tag{30}$$

and

$$I(a,b) = \exp\left(\frac{1}{b-a}\int_a^b \ln x dx\right) = \frac{1}{e}\left[\frac{a^a}{b^b}\right]^{\frac{1}{a-b}}. \tag{31}$$

Assume that $0 < a < b$. Relying on Theorem A, we can prove that the mean functions $r \mapsto M_r(a,b)$ and $r \mapsto L_r(a,b)$ are strictly increasing on the whole domain $\mathbb{R}$, having the limit value a at negative infinity, and b at positive infinity. So, these two mean functions are continuous bijections of $\mathbb{R}$ to (a,b).

Applying the convexity and concavity of the power function $f(x) = x^{r-1}$ to the Hermite-Hadamard inequality, we can determine the mutual order of power and logarithmic means.

Theorem 5.2. 9 *If $0 < a < b$ and $r < 2 < s$, then we have the series of inequalities*

$$a < M_{r-1}(a,b) < L_r(a,b) < A(a,b) < L_s(a,b) < M_{s-1}(a,b) < b. \tag{32}$$

Proof. Taking into account what has been said above, we need to prove the inequalities $M_{r-1}(a,b) < L_r(a,b) < A(a,b)$ and $A(a,b) < L_s(a,b) < M_{s-1}(a,b)$. Prove the inequality referring to r by using the power function $f(x) = x^{r-1}$ in the cases $r < 1$ and $1 < r < 2$. If $r = 1$, the accompanying inequality $M_0(a,b) < L_1(a,b) < A(a,b)$ holds.

If $r < 1$, then applying the Hermite-Hadamard's inequality to the strictly convex function f on the interval $[a,b]$, we get

$$\left[\frac{a+b}{2}\right]^{r-1} < \frac{a^r - b^r}{r(a-b)} < \frac{a^{r-1}+b^{r-1}}{2}, \tag{33}$$

and raising to the negative power $1/(r-1)$ yields $A(a,b) > L_r(a,b) > M_{r-1}(a,b)$.

If $1 < r < 2$, then applying the Hermite-Hadamard's inequality to the strictly concave function f on $[a,b]$, we get the reverse inequality in equation (33), and raising to the positive power $1/(r-1)$ yields the required inequality.

Power and logarithmic means play an important role in representing and solving the problems of thermodynamics, quantum mechanics and information theory. In these direct applications, the aforementioned means are usually called entropies. For more details, see [6].

6. A SHORT LIST OF MEAN INEQUALITIES

In addition to the previously mentioned means, we also use the centroidal mean of two positive numbers a and b, defined by

$$C(a,b) = \frac{2(a^2+ab+b^2)}{3(a+b)}. \tag{34}$$

Let a and b be different positive numbers. With the well-known inequality

$$H(a,b) < G(a,b) < L(a,b) < I(a,b) < A(a,b) < C(a,b), \qquad (35)$$

we point out the following inequalities:

$$M_0(a,b) < L(a,b) < M_{1/3}(a,b) \qquad (36)$$

$$M_{2/3}(a,b) < I(a,b) < M_{\ln 2}(a,b) \qquad (37)$$

$$M_0^2(a,b) < L(a,b)I(a,b) < M_{1/2}^2(a,b) \qquad (38)$$

$$I^{\frac{1}{2}}(a,b)G^{\frac{1}{2}}(a,b) < L(a,b) < \frac{1}{2}I(a,b) + \frac{1}{2}G(a,b) \qquad (39)$$

$$A^{\frac{1}{3}}(a,b)G^{\frac{2}{3}}(a,b) < L(a,b) < \frac{1}{3}A(a,b) + \frac{2}{3}G(a,b) \qquad (40)$$

$$\alpha A(a,b) + (1-\alpha)G(a,b) < I(a,b) < \beta A(a,b) + (1-\beta)G(a,b) \qquad (41)$$

$$\alpha \le \frac{2}{3}, \beta \ge \frac{2}{e}$$

$$\alpha C(a,b) + (1-\alpha)H(a,b) < L(a,b) < \beta C(a,b) + (1-\beta)H(a,b) \qquad (42)$$

$$\alpha \le 0, \beta \ge \frac{1}{2}$$

$$\alpha C(a,b) + (1-\alpha)H(a,b) < I(a,b) < \beta C(a,b) + (1-\beta)H(a,b) \qquad (43)$$

$$\alpha \le \frac{3}{2e}, \beta \ge \frac{5}{8}$$

For the derivation of the inequalities in (36)-(38) see [2, 11, 14, 15, 18], for the inequality in (39) see [1], for the inequality in (40) see [4, 10, 16], for the inequality in (41) see [3], and for the inequalities in (42)-(43) see [5].

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