ISSN 2347-1921

Nonlinear Fredholm-Volterra Integral Equation and its Numerical Solutions with Quadrature Methods

M. A. Abdou, A. M. Al-Bugami
Department of Mathematics Faculty of Education, Alexandria University, Egypt
Department of Mathematics, Faculty of Sciences, Taif University, Saudi Arabia.
abdella_777@yahoo.com, abeer101aa@yahoo.com

Abstract

In this work, the existence and uniqueness of solution of the nonlinear Fredholm-Volterra integral equation (**NF-VIE**), with continuous kernels, are discussed and proved in the space $L_2(\Omega)\times C(0,T)$. The Fredholm integral term (**FIT**) is considered in position while the Volterra integral term (VIT) is considered in time. Using a numerical technique we have a nonlinear system of Fredholm integral equations (SFIEs). This system of integral equations can be reduced, using quadrature methods, to a nonlinear algebraic system (**NAS**). Then, the **NAS** can be solved, using two numerical methods. These methods are: Trapezoidal rule method and Simpson's rule method. Finally, some numerical examples are considered and the error estimate, in each case, is computed.

Keywords: Nonlinear Fredholm-Volterra integral equation; nonlinear algebraic system; Trapezoidal rule; Simpson's rule.

Council for Innovative Research
Peer Review Research Publishing System
Journal: Journal of Advances in Mathematics

Vol 4, No 2
editor@cirworld.com
www.cirworld.com, member.cirworld.com

ISSN 2347-1921

1. Introduction

The integral equation is a mathematical model of many evolutionary problems with memory arising from biology, chemistry, physics, contact problems and engineering. In recent years, many different basic functions have been used to estimate the solution of integral equations, such as orthogonal bases and wavelets. At the same time the numerical methods take an important place in solving the integral equations numerically. The references Backer [1], Atkinson [2], Delves and Mohamed [3], and Jerri [4] contain many different methods for solving the integral equations numerically. In [5], Farshid Mirzaee solved Volterra integral equations of the first kind numerically by using quadrature rule. The discussions of F-VIE with its applications in contact problems in the theory of elasticity started by Abdou [6]. Abdou in [6] discussed the solution of F-VIE of the first kind in one, two and three dimensions, using separation of variable method. Also, the same author, in [7-9], used some different methods to obtain the solution of F–VIE of the first kind and of the second kind. Hendi and Albugami, in [10], obtained numerically, the solution of F-VIE of the second kind, using collocation and Galerkin methods. In [11, 12], the authors used two numerical methods to obtain the solution of F–VIE of the second kind when the kernel takes a logarithmic form and Hilbert kernel, respectively. In the references [13-16] the authors considered many different methods to solve the mixed integral equation numerically.

In this work, we consider the following mixed integral equation of the second kind:

$$\mu\phi(x,t)=f(x,t)+\lambda\int_{\Omega}k(x,y)\gamma(t,y,\phi(y,t))dy+\lambda\int_{0}^{t}F(t,\tau)\phi(x,\tau)d\tau \tag{1}$$

The formula (1) is called the **NF-VIE** in the space $L_2(\Omega)\times C(0,T)$, T < 1. Here, the **FIT** is considered in position with a continuous kernel $k(x, t)$; Ω is the domain of integration with respect to position. While, the **VIT** is considered in time with a positive continuous kernel $F(t,\tau)$ for all t, $\tau\in[0,T]$,T <1. The free term $f(x, t)$ is known continuous function in the space $L_2(\Omega)\times C(0,T)$, while $\phi(x, t)$ is unknown function to be determined in the same space. The numerical coefficient λ is called the parameter of the integral equation, may be complex, and has physical meaning, while the constant parameter μ defines the kind of the integral equation (1).

2. Existence and uniqueness solution of nonlinear F-VIE:

To prove the existence of a unique solution of NF-VIE of Eq. (1), using Banach Fixed Point Theorem, we write it in the integral operator form:

$$\bar{Q}\phi(x,t)=\frac{1}{\mu}f(x,t)+Q\phi(x,t) \tag{2}$$

$$Q\phi(x,t)=K\phi(x,t)+F\phi(x,t) \tag{3}$$

where,

$$K\phi=\frac{\lambda}{\mu}\int_{\Omega}k(x,y)\gamma(t,y,\phi(y,t))dy\ ,\ F\phi=\frac{\lambda}{\mu}\int_{0}^{t}F(t,\tau)\phi(x,\tau)d\tau \tag{4}$$

Then, we assume the following conditions:

1- The kernel of the **FIT** satisfies the condition $|k(x,y)|\le B$ (B is a constant)

2-The kernel $F(t,\tau)\in C[0,T]$; $T<1$, satisfies $|F(t,\tau)|\le C$, $\forall\ t,\tau\in[0,T]$, (C is a constant)

3- The given function $f(x,t)$ with its partial derivatives are continuous in $L_2(\Omega)\times C(0,T)$ where

$$\|f(x,t)\|=\max\int_{0}^{t}[\int_{\Omega}|f(x,\tau)|^2dx]^{\frac{1}{2}}d\tau=D,\quad (D \text{ is a constant})$$

4- The known continuous function $\gamma(x,t,\phi(x,t))$, for the constants $Q>P_1$ and $Q>Q_1$, satisfies:

(i) $$\max_{0\le t\le T}\le\left|\int_{0}^{t}\{|\gamma(\tau,x,\phi(x,\tau)|^2dx\}^{\frac{1}{2}}d\tau\right|\le Q_1\|\phi(x,t)\|_{L_2(\Omega)\times C[0,T]}$$

(ii) $$|\gamma(t,x,\phi_1(x,t)-\gamma(t,x,\phi_2(x,t)|\le N(t,x)|\phi_1(x,t)-\phi_2(x,t)|$$

ISSN 2347-1921

Where, $\|N(t,x)\|_{L_2(\Omega)\times C[0,T]} = \max_{0\le t\le T}\left|\int_0^t \{\int_\Omega |N(\tau,x)|^2\, dx\}^{\frac{1}{2}} d\tau\right| = P_1 < \infty$

Theorem 1: The integral equation (1) has an exact and unique solution in the space $L_2(\Omega)\times C(0,T)$ under the following condition:

$$|\mu| > |\lambda|(BQ + CKT) \tag{5}$$

The proof of this theorem depends on the following two lemmas:

Lemma 1: Under the conditions (1)-(4), the integral operator $\bar{Q}$ of (2), maps the space $L_2(\Omega)\times C(0,T)$ into itself.

Proof

In view of the formulas (3) and (4); after taking the norm of the formula (2), we found

$$\|\bar{Q}\phi(x,t)\| \le \left|\frac{1}{\mu}\right|\|f(x,t)\| + \left|\frac{\lambda}{\mu}\right|\left(\left\|\int_\Omega k(x,y)\gamma(t,y,\phi(y,t))dy\right\| + \left\|\int_0^t |F(t,\tau)|\phi(x,\tau)d\tau\right\|\right) \tag{6}$$

Also, we have $\|F\phi\| \le \left|\frac{\lambda}{\mu}\right|\left\|\int_0^t |F(t,\tau)|\phi(x,\tau)d\tau\right\| \le \left|\frac{\lambda}{\mu}\right| CKT\ \|\phi(x,t)\|$

and, for the integral operator $K\phi$, we get $\|K\phi\| \le \left|\frac{\lambda}{\mu}\right| BQ\ \|\phi\|$

Hence, the formula (6), yields

$$\|\bar{Q}\phi(x,t)\| \le \sigma\|\phi(x,t)\| + \left|\frac{D}{\mu}\right|, \quad (\sigma = \left|\frac{\lambda}{\mu}\right|(BQ+CKT)), \qquad T = \max_{0\le t\le T} t,\ K = (\int_\Omega dx)^{\frac{1}{2}} \tag{7}$$

The last inequality (7) shows that, the operator $\bar{Q}$ maps the ball $R_p \subset L_2(\Omega)\times C(0,T)$ into itself, where

$$\rho = \frac{D}{[|\mu| - |\lambda|(BQ+CKT)]} = \frac{D}{|\mu|}(1-\sigma)^{-1}, \quad \sigma = \left|\frac{\lambda}{\mu}\right|(BQ+CKT).$$

Since $\rho > 0,\ D > 0,$ therefore we have $\sigma < 1.$ Furthermore, the inequality (7) yields the boundedness of the operator Q defined by (3), where $\|Q\phi(x,t)\| \le \sigma\|\phi(x,t)\|.$

Also, the above inequality and (7) define the boundedness of the operator $\bar{Q}.$

Lemma 2: The integral operator (2), under the condition (5), is a continuous and a contraction operator.

Proof: To prove the continuity of the integral operator $\bar{Q}$, we consider two functions $\phi_1(x,t),\ \phi_2(x,t) \in L_2(\Omega)\times C[0,T]$. Then the formula (2) with the aid of (4) after applying Cauchy-Schwarz inequality and using conditions (1) and (2), yields

$$\|\bar{Q}\phi_1(x,t) - \bar{Q}\phi_2(x,t)\| \le [\left|\frac{\lambda}{\mu}\right|(BQ+CKT)]\|\phi_1(x,t) - \phi_2(x,t)\| \tag{8}$$

Hence, $\bar{Q}$ is a continuous operator in the space $L_2(\Omega)\times C(0,T)$, and under the condition (5), $\bar{Q}$ is a contraction operator. So, from lemmas (1) and (2) and Banach Fixed Point Theorem we can decide that the operator $\bar{Q}$ has a unique fixed point which is the unique solution of integral equation (1), and theorem 1 is completely proved.

3. System of nonlinear Fredholm integral equations: To represent (1) as **SFIEs** we divide the interval [0,*T*] as: $0 = t_0 < t_1 < ... < t_N = T$, and let $t = t_n,\ n = 0,1,2,...,N$. Therefore Eq. (1) reduces to **SFIEs** of the second kind, in the form:

$$\eta_n\phi_n(x) = \psi_n(x) + \lambda\int_a^b k(x,y)\gamma(t,y,\phi_n(y))dy, \tag{9}$$

$$\eta_n = (\mu - \lambda w_n F_{n,n}), \quad \psi_n(x) = f_n(x) + \lambda\sum_{j=0}^{n-1} w_j F_{n,j}\phi_j(x);\ n = 0,1,2,\ldots,N.$$

Equation (9) can be solved using recurrence relations.

4. Qudrature methods for solving system of NFIEs: In this section, we discuss the numerical solution of **NF-VIE** using two different quadrature methods.

4.1. Trapezoidal Rule: In this section, we use Trapezoidal rule for solving Eq. (9). So we approximate the right-hand integral of (9) with the trapezoidal rule to get

$$\eta_n\phi_n(x) = \psi_n(x) + \lambda\frac{h}{2}[k(x,x_0)\gamma(\phi(x_0)) + 2\sum_{p=1}^{m-1} k(x,x_p)\gamma(\phi(x_p)) + k(x,x_m)\gamma(\phi(x_m))] + E \tag{10}$$

Here, the error term is $E = -\frac{b-a}{12}h^2.\frac{d^2}{d\mu^2}(k_{n,\mu}.\gamma(\phi_\mu)),\quad h = \frac{b-a}{n},\ x_j = a + jh,\ j = 0,1,\ldots,n,\ \mu\in(a,b)$. For easily simplification, we define in (10) $\psi_n(x) = \psi_n,\ k(x_n,x_p) = k_{n,p},\ \phi(x_n) = \phi_n$. Hence, for $x = x_0, x_1, x_2, \ldots, x_n;\quad n = 0,1,2,\ldots,m,$ in (10), and after negligiting the error, we have the following system of (m+1) equations:

$$\eta_n\phi_n = \psi_n + \lambda\frac{h}{2}[k_{n0}\gamma(\phi_0) + 2\sum_{p=1}^{m-1} k_{np}\gamma(\phi_p) + k_{nm}\gamma(\phi_m)] \tag{11}$$

4.2. Simpson's Rule: Using Simpson's rule, we can approximate the right-hand integral of (10) to get

$$\eta_n\phi_n(x) = \psi_n(x) + \lambda\frac{h}{3}[k(x,x_0)\gamma(\phi(x_0)) + 4\sum_{p=1}^{m/2} k(x,x_{2p-1})\gamma(\phi(x_{2p-1})) + 2\sum_{p=1}^{(m-2)/2} k(x,x_{2p})\gamma(\phi(x_{2p})) + k(x,x_m)\gamma(\phi(x_m))] + E, \tag{12}$$

$E = -\frac{b-a}{180}h^4.\frac{d^4}{d\mu^4}(k_{n,\mu}.\gamma(\phi_\mu))$, is the error term. Also, we define in (12) the notations $\psi_n(x) = \psi_n,\ k(x_n,x_p) = k_{n,p},\ \phi(x_n) = \phi_n$. And, for $x = x_0, x_1, x_2, \ldots, x_n,\quad 0 \le n \le m,$ we have:

$$\eta_n\phi_n = \psi_n + \lambda\frac{h}{3}[k_{n0}\gamma(\phi_0) + 4\sum_{p=1}^{m/2} k_{n,2p-1}\gamma(\phi_{2p-1}) + 2\sum_{p=1}^{(m-2)/2} k_{n,2p}\gamma(\phi_{2p}) + k_{nm}\gamma(\phi_m)] + E \tag{13}$$

5. Numerical Examples:

Example (1) : Consider the **NF-VIE**

$\phi(x,t) = f(x,t) + \lambda\int_0^1 e^{x+y}(\phi(y,t))^2dy + \lambda\int_0^t \tau^2\phi(x,\tau)d\tau$ (the exact solution is $\phi(x,t) = t^2e^x$). With λ = 0.01, and the time t = 0.03, 0.01, 0.6 for N =10, 20.

Case 1: $t = 0.03$

Table1. Numerical results and errors of example1

N	x	Exact sol.	Trapezoidal Rule	Error Tra.	Simpson's Rule	Error Sim.
10	0	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000
	0.1	0.99465326	0.99465325×10^{-5}	0.5695×10^{-11}	0.99465325×10^{-5}	0.5695×10^{-11}

	0.3	0.00010933	0.00010933	0.5601×10^{-9}	0.00010933	0.5611×10^{-9}
	0.5	0.00037096	0.00037095	0.51671×10^{-8}	0.00037095	0.51973×10^{-8}
	0.7	0.00088806	0.00088804	0.230106×10^{-7}	0.00088804	0.233215×10^{-7}
	0.9	0.00179305	0.00179298	0.67950×10^{-7}	0.00179298	0.70026×10^{-7}
20	0	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000
	0.15	0.000023527	0.0000235271	0.3024×10^{-10}	0.0000235	0.3026×10^{-10}
	0.30	0.000109338	0.0001093380	0.5569×10^{-9}	0.0001093	0.5558×10^{-9}
	0.45	0.000285824	0.0002858216	0.32129×10^{-8}	0.0002858	0.32248×10^{-8}
	0.60	0.000590366	0.0005903544	0.120084×10^{-7}	0.0005903	0.119780×10^{-7}
	0.75	0.001071731	0.0010716981	0.33084×10^{-7}	0.0010716	0.33205×10^{-7}
	0.90	0.001793050	0.0017929743	0.76327×10^{-7}	0.0017929	0.75836×10^{-7}

Case 2: $t = 0.01$

Table2. Numerical results and errors of example1

N	x	Exact sol.	Trapezoidal Rule	Error Tra.	Simpson's Rule	Error Sim.
10	0	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000
	0.1	0.1105170×10^{-5}	0.11051708×10^{-5}	0.70×10^{-13}	0.11051708×10^{-5}	0.70×10^{-13}
	0.3	0.0000121	0.00001214	0.691×10^{-11}	0.00001214	0.692×10^{-11}
	0.5	0.0000412	0.00004121	0.6378×10^{-10}	0.00004121	0.6416×10^{-10}
	0.7	0.0000986	0.00009867	0.28402×10^{-9}	0.00009867	0.28786×10^{-9}
	0.9	0.0001992	0.00019922	0.8385×10^{-9}	0.00019922	0.8642×10^{-9}
20	0	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000
	0.15	0.26141270×10^{-5}	0.26141266×10^{-5}	0.373×10^{-12}	0.26141266×10^{-5}	0.373×10^{-12}
	0.30	0.00001214	0.00001214	0.688×10^{-11}	0.00001214	0.686×10^{-11}
	0.45	0.00003175	0.00003175	0.3967×10^{-10}	0.00003175	0.3981×10^{-10}
	0.60	0.00006559	0.00006559	0.14819×10^{-9}	0.00006559	0.14782×10^{-9}
	0.75	0.00011908	0.00011908	0.4082×10^{-9}	0.00011908	0.4097×10^{-9}
	0.90	0.00019922	0.00019922	0.9418×10^{-9}	0.00019922	0.9357×10^{-9}

Case 3: $t = 0.6$

Table3. Numerical results and errors of example1

N	x	Exact sol.	Trapezoidal Rule	Error Tra.	Simpson's Rule	Error Sim.
10	0	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000
	0.1	0.00397861	0.00397770	0.9129×10^{-6}	0.00397770	0.9129×10^{-6}
	0.3	0.04373542	0.04364545	0.0000899	0.04364529	0.0000901
	0.5	0.14838491	0.14755418	0.0008307	0.14754939	0.0008355
	0.7	0.35522597	0.35151872	0.0037072	0.35146984	0.0037561
	0.9	0.71722026	0.70620897	0.0110112	0.70588621	0.0113340
20	0	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000
	0.15	0.00941085	0.00940600	0.4847×10^{-5}	0.00940600	0.4851×10^{-5}
	0.30	0.04373542	0.04364612	0.0000893	0.04364629	0.0000891
	0.45	0.11432995	0.11381440	0.0005155	0.11381251	0.0005174

	0.60	0.23614659	0.23420401	0.0019425	0.23420881	0.0019377
	0.75	0.42869250	0.42332267	0.0053698	0.42330372	0.0053887
	0.90	0.71722026	0.70478157	0.0124386	0.70485749	0.0123627

Example 2. Consider $\phi(x,t) = f(x,t) + \lambda\int_0^1 e^x(\phi(y,t))^2 dy + \lambda\int_0^t \tau^2\phi(x,\tau)d\tau$ (the exact solution is $\phi(x,t) = te^x$).

With λ = 0.01, and the time t = 0.03, 0.01, 0.6 for N =10, 20.

Case 1: $t = 0.03$

Table 4. Numerical results and errors of example 2

N	x	Exact sol.	Trapezoidal Rule	Error Tra.	Simpson's Rule	Error Sim.
10	0	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000
	0.1	0.00331551	0.00331519	0.31794×10^{-6}	0.00331519	0.31794×10^{-6}
	0.3	0.01214872	0.01214532	0.34088×10^{-5}	0.01214530	0.34280×10^{-5}
	0.5	0.02473081	0.02471983	0.000010	0.02471971	0.000011
	0.7	0.04228880	0.04226486	0.000023	0.04226440	0.000024
	0.9	0.06640928	0.06636875	0.000040	0.06636732	0.000041
20	0	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000
	0.15	0.00522825	0.00522751	0.74408×10^{-6}	0.00522750	0.74573×10^{-6}
	0.30	0.01214872	0.01214536	0.33685×10^{-5}	0.01214537	0.33555×10^{-5}
	0.45	0.02117221	0.02116376	0.84499×10^{-5}	0.02116371	0.84991×10^{-5}
	0.60	0.03279813	0.03277982	0.000018	0.03277992	0.000018
	0.75	0.04763250	0.04760157	0.000030	0.04760136	0.000031
	0.90	0.06640928	0.06636221	0.000047	0.06636272	0.000046

Case 2: $t = 0.01$

Table 5. Numerical results and errors of example 2

N	x	Exact sol.	Trapezoidal Rule	Error Tra.	Simpson's Rule	Error Sim.
10	0	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000
	0.1	0.00110517	0.00110513	0.35312×10^{-7}	0.00110513	0.35312×10^{-7}
	0.3	0.00404957	0.00404919	0.37851×10^{-6}	0.00404919	0.38065×10^{-6}
	0.5	0.00824360	0.00824238	0.12196×10^{-5}	0.00824237	0.12328×10^{-5}
	0.7	0.01409626	0.01409361	0.26571×10^{-5}	0.01409356	0.27078×10^{-5}
	0.9	0.02213642	0.02213193	0.44940×10^{-5}	0.02213177	0.46529×10^{-5}
20	0	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000
	0.15	0.00174275	0.00174266	0.82651×10^{-7}	0.00174266	0.82834×10^{-7}
	0.30	0.00404957	0.00404920	0.37412×10^{-6}	0.00404920	0.37268×10^{-6}
	0.45	0.00705740	0.00705646	0.93835×10^{-6}	0.00705646	0.94381×10^{-6}
	0.60	0.01093271	0.01093068	0.20290×10^{-5}	0.01093069	0.20186×10^{-5}
	0.75	0.01587750	0.01587407	0.34255×10^{-5}	0.01587405	0.34482×10^{-5}
	0.90	0.02213642	0.02213121	0.52089×10^{-5}	0.02213127	0.51532×10^{-5}

ISSN 2347-1921

Case 3: $t = 0.6$

Table 6. Numerical results and errors of example 2

N	x	Exact sol.	Trapezoidal Rule	Error Tra.	Simpson's Rule	Error Sim.
10	0	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000
	0.1	0.06631025	0.06618156	0.00012868	0.06618156	0.00012868
	0.3	0.24297458	0.24158543	0.00138914	0.24157781	0.00139676
	0.5	0.49461638	0.49010652	0.00450986	0.49005990	0.00455647
	0.7	0.84577613	0.83582640	0.00994973	0.83564800	0.01012813
	0.9	1.32818568	1.31095689	0.01722878	1.31039980	0.01778587
20	0	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000
	0.15	0.10456508	0.10426487	0.00030020	0.10426421	0.00030086
	0.30	0.24297458	0.24161146	0.00136312	0.24161661	0.00135796
	0.45	0.42344429	0.42000896	0.00343532	0.41998958	0.00345470
	0.60	0.65596276	0.64810542	0.00785734	0.64814225	0.00782051
	0.75	0.95265000	0.93915079	0.01349920	0.93071141	0.01357886
	0.90	1.32818568	1.30725173	0.02093395	1.30744593	0.02073974

6. Conclusion

1-When the values of N is increasing and the values of the time T kept fixed, the error is increasing, where the atomic bond between the particles of the material, in the applied science, is increasing.

2-When the value of time t is increasing and N kept fixed, the error is increasing.

3- The error results for Simpson's rule is less than the error results for Trapezoidal rule.

4- The Trapezoidal rule and Simpson's rule methods are the efficient numerical methods, for solving the NF-VIE with continuous kernels, compared to the other methods.

References

[1] C.T.H. Baker, Geoffrey.F. Miller, Treatment of Integral Equations by Numerical Methods, Acadmic Press, 1982.

[2] K.E. Atkinson, The Numerical Solution of Integral Equation of the Second Kind, Cambridge, 1997.

[3] L.M. Delves and J.L. Mohamed,Computational Methods for Integral Equations, University press, 1985.

[4] A. J . Jerri , Introduction to Integral Equations with Applications , Clarkson University , Potsdam , New York , 1985 .

[5] Farshid Mirzaee, Numerical solution for Volterra integral equations of the first kind via Quadrature rule, Applied Mathematical Sciences, Vol. 6, 2012, No. 20, 969-974.

[6] M. A. Abdou, Fredholm – Volterra integral equation of the first kind and the contact problem, Apple. Math. Comput .125 (2002) 177-193.

[7] M .A. Abdou, Fredholm – Volterra integral equation and generalized potential kernel, Apple. Math. Comput, 13 (2002) 81 – 94.

[8] M. A. Abdou, On asymptotic method for Fredholm–Volterra integral equation of the second kind in contact problems. J. Comp. Appl. Math. 154 (2003) 431–446.

[9] M. A. Abdou, Fredholm–Volterra integral equation with singular kernel, Apple. Math. Comput. 137 (2003) 231–243.

[10] F.A. Hendi, A. M. Albugami, Numerical solution for Fredholm–Volterra integral equation of the second kind by using collocation and Galerkin methods , J. KSU(Science) (2010) 22, 37–40.

[11] M.A. Abdou, K.I. Mohamed and A.S. Ismail, On the numerical Solutions of Fredholm–Volterra integral equation, J. Appl. Math. Comput, 146 (2003) 713–728.

[12] M.A. Abdou and F.A .Hendi, Numerical solution for Fredholm integral equation with Hilbert kernel, J.KSIAM. Vol (9), No.1, 111– 123, 2005.

[13] H. Guoqiang, Asymptotic error expansion for the Nyström method for a nonlinear Volterra-Fredholm integral equation, J. Comput. Math.Appl.59 (1955) 49-59

[14] H. Du, M. Cui, A method of solving nonlinear mixed Volterra-Fredholm integral equation, Appl. Math. Sci., 1 (2007) 2505-2516.

[15] K. Maleknejad, B. Rahimi, Medication of block pulse functions and their application to solve numerically Volterra integral equation of the first kind. Commun. Nonlin. Sci. Numer. Simu., 16 (2011) 2469-2477.

[16] A. Cardone, E. Messina, E. Russo, A fast iterative method for discretized Volterra-Fredholm integral equations, J. Comput. Appl. Math., 189 (2006) 568-579.