ISSN 2347-1921

G-function Solutions for Diffusion and Laplace's Equations

Amir Pishkoo[1,2], Maslina Darus[2]

[1]Physics Department, Nuclear Science Research School (NSTRI)

P.O. Box 14395-836, Tehran, Iran

apishkoo@gmail.com (corresponding author)

[2]School of Mathematical Sciences, Faculty of Science and Technology

UniversitiKebangsaan Malaysia, 43600 Bangi, Selangor Darul.Ehsan, Malaysia

maslina@ukm.my

ABSTRACT

In this paper the Diffusion equation and Laplace's equation is solved by Modified separation of variables (MSV) method, suggested by Pishkoo and Darus. Using this method, Meijer's *G*-function solutions are derived in cylindrical coordinates system for two typical problems. These complex functions include all elementary functions and most of the special functions which are the solution of extensive problems in Physics and engineering.

Keywords: Meijer's G-function; Partial differential equation; Modified separation of variables; Diffusion equation; Laplace's equation.

Council for Innovative Research

Peer Review Research Publishing System

Journal: Journal of Advances in Mathematics

Vol 4, No 1

editor@cirworld.com

www.cirworld.com, member.cirworld.com

ISSN 2347-1921

INTRODUCTION

Meijer's *G*-functions are defined as Mellin-Barnes contour integrals which have been inexistence for over 60 years [1, 2, 3, 4, 5]. Evidence for the importance of the Meijer's *G*-function is given by the fact that the basic elementary functions and most of the special functions of mathematical physics, including the generalized hypergeometric functions, follow as its particular cases. Meijers *G*-function satisfies the linear ordinary differential equation (LODE) of the generalized hypergeometric type whose order isequal to max(p; q) [6, 7, 8]. This fact triggered us to verify the equality conditions between Meijer's *G*-function's LODE and some partial differential equations governingphysical phenomena [9, 10]. In physics, we have many ordinary and partial differential equations, in which their solutions are elementary functions, special functions or acombination of both of them. Thus, Meijer's *G*-functions can be the solution for manyphysical problems if the equality requirement between Meijers G-function's LODE those differential equations are verified. As such , we seek to deduce the solution of physical problems explicitly in terms of Meijer's *G*-functions.

Our previous works had focused on the introduction of the Modified separation of variables method (MSV), and applying it to solve partial differential equation related to the Reaction-Diffusion process [9], Laplace's, Diffusion and Schrodinger equations[10, 11] which led to representing its solution in terms of Meijer's *G*-functions. The Cartesian coordinates system is used to derive their solutions. However, In this paper we obtain *G*-function solutions for the same problems solved by "separation of variables (SV)", see [12], by using modified separation of variables (MSV) method, and incylindrical coordinates system as follows:

2 Meijer's G-function

We begin with the definition of Meijer's *G*-function as the following:

Definition 2.1 A definition of the Meijer's G-function is given by the following path integral in the complex plane, called Mellin-Barnes type integral [1, 2, 3, 4, 5]:

$$G_{p,q}^{m,n}\left({a_1,\dots,a_p \atop b_1,\dots,b_q}\middle| z\right) = \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j - s)\ \prod_{j=1}^{n}\Gamma(1-a_j+s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+s)\ \prod_{j=n+1}^{p}\Gamma(a_j - s)} z^s ds. \tag{2.1}$$

Here, the integers, *m; n; p* and *q* are called the orders of the G-function, or the components ofthe order *(m; n; p; q).* Here, both a_p and b_q are called "parameters" which are generally complex numbers. The definition holds under the following assumptions: $0 \le m \le q$ and $0 \le n \le p$, where *m; n; p* and *q* are integer numbers. $a_j - b_k \neq 1, 2, 3, \dots$ for$k = 1,2,\dots,n$ and $j = 1,2,\dots,m$ imply that no pole of any $\Gamma(b_j - s), j = 1, \dots, m$ coincides with anypole of any $\Gamma(1 - a_k + s), k = 1, \dots, n.$

Example 2.1 Using (2.1), we obtain the follows $G_{0,2}^{1,0}; G_{1,2}^{1,1}; G_{1,1}^{1,1}$: see [13]

$$G_{0,2}^{1,0} = \frac{1}{2\pi i}\int_L \frac{\Gamma(b_1 - s) z^s}{\Gamma(1 - b_2 + s)} ds. \tag{2.2}$$

$$G_{1,2}^{1,1} = \frac{1}{2\pi i}\int_L \frac{\Gamma(b_1 - s)\Gamma(1 - a_1 + s) z^s}{\Gamma(1 - b_2 + s)} ds. \tag{2.3}$$

$$G_{1,1}^{1,1} = \frac{1}{2\pi i}\int_L \Gamma(b_1 - s)\Gamma(1 - a_1 + s) z^s ds. \tag{2.4}$$

The Meijer's *G*-function $y(z) = G_{p,q}^{m,n}(z|_{b_k}^{a_j})$ satisfies the linear ordinary differential equation of the generalized hypergeometric type

$$[(-1)^{p-m-n} z \prod_{j=1}^{p}(z\frac{d}{dz} - a_j + 1) - \prod_{k=1}^{q}(z\frac{d}{dz} - b_k)]y(z) = 0. \tag{2.5}$$

whose order is equal to max(*p; q*), see [6, 7, 8].

Choosing appropriate values for *m; n; p; q;* orders of *G*-functions, Equation (2.5) can bechanged to complex first and second order linear differential equations. The following section discusses the properties of the solutions of complex first and second order differential equations, and then studies the properties of coefficient functions of these differential equations in the complex plane.

3 Results and Discussion

We start with using Modified separation of variables method (MSV) in cylindrical coordinates system as follows:

ISSN 2347-1921

3.1 The G-function Solutions for the Laplace's Equation

Step 1: In this section the separation of variables is employed while the Laplace's equation is solved in cylindrical coordinates system.

Step 2: Writing $\Phi(\rho,\varphi,Z)$ as a product of three functions, $\Phi(\rho,\varphi,Z)=R(\rho)S(\varphi)Z(z)$; Laplace's equation, $\nabla^2\Phi(\rho,\varphi,Z)=0$, is separated, into three ODEs:

$$\frac{d}{d\rho}(\rho\frac{dR}{d\rho})+(k^2\rho-\frac{m^2}{\rho})R=0, \qquad (3.1)$$

$$\frac{d^2S}{d\varphi^2}+m^2S=0, \qquad (3.2)$$

$$\frac{d^2Z}{dz^2}-k^2Z=0, \qquad (3.3)$$

where in anticipation of the correct BCs, we have written the constants as k^2 and $-m^2$ with m an integer.

Step 3:

1. ρ-component: for $m = 1; n = 0; p = 0; q = 2$, equation (2.5) reduces to

$$[-z-(z\frac{d}{dz}-b_2)(z\frac{d}{dz}-b_1)]G_{0,2}^{1,0}(z|_{b_1,b_2}^{-})=0.$$

By changing $z\to\alpha z^2$, we have

$$[-\alpha z^2-(\frac{z}{2}\frac{d}{dz}-b_2)(\frac{z}{2}\frac{d}{dz}-b_1)]G_{0,2}^{1,0}(\alpha z^2|_{b_1,b_2}^{-})=0.$$

By multiplying both sides of the equation by -4, we have

$$[z^2\frac{d^2}{dz^2}+(1-2(b_1+b_2))z\frac{d}{dz}+4\alpha z^2+4b_1b_2]G_{0,2}^{1,0}(\alpha z^2|_{b_1,b_2}^{-})=0.$$

On the other hand, let Bessel equation (3.1) with $z = k\rho$ is

$$[z^2\frac{d^2}{dz^2}+z\frac{d}{dz}+z^2-\nu^2]J_\nu(z)=0.$$

The conditions for equivalence or these two differential equations are

$$b_1+b_2=0, 4\alpha=1, 4b_1b_2=-\nu^2\Rightarrow b_1=\frac{\nu}{2}, b_2=\frac{-\nu}{2}, \alpha=\frac{1}{4}$$

and its solution is

$$R(\rho)=\begin{Bmatrix} J_\nu(\rho) & & SV \quad method \\ G_{0,2}^{1,0}(\frac{1}{4}\rho^2|_{\frac{\nu}{2},\frac{-\nu}{2}}^{-}) & & MSV \quad method \end{Bmatrix},$$

where $G_{0,2}^{1,0}$ is the first basic univalent G-function, see [13].

2. φ-component: for $m = 1; n = 0; p = 0; q = 2,$ the equation (2.5) is reduced to:

By changing $z\to\alpha z^2$, the following is obtained:

$$[-\alpha z^2-(\frac{z}{2}\frac{d}{dz}-b_2)(\frac{z}{2}\frac{d}{dz}-b_1)]G_{0,2}^{1,0}(\alpha z^2|_{b_1,b_2}^{-})=0.$$

Dividing it by z^2, the following is retrieved:

$$[\frac{d^2}{dz^2}+(1-2(b_1+b_2))z\frac{d}{dz}+4\alpha+4\frac{b_1b_2}{z^2}]G^{1,0}_{0,2}(\alpha z^2|^{-}_{b_1,b_2})=0. \quad (3.4)$$

On the other hand let the equation (3.2)

$$[\frac{d^2}{dz^2}+m^2]Z=0.$$

The conditions for equality of these two differential equations are as follows:

$$1-2(b_1+b_2)=0, 4\alpha=m^2, b_1b_2=0.$$

Consequently, if $b_1 = 1/2$; $b_2 = 0$; $\alpha = m_2/4$; the first independent solution is:

$$Z_1(z)=\begin{Bmatrix} \sin mz & SV & method \\ G^{1,0}_{0,2}(\frac{m^2}{4}z^2|^{-}_{\frac{1}{2},0}) & MSV & method \end{Bmatrix}.$$

Furthermore, if $b_1 = 0$; $b_2 = 1/2$;$\alpha = m_2/4$, and the second independent solution is:

$$Z_2(z)=\begin{Bmatrix} \cos mz & SV & method \\ G^{1,0}_{0,2}(\frac{m^2}{4}z^2|^{-}_{0,\frac{1}{2}}) & MSV & method \end{Bmatrix}.$$

As it can be seen from above, the function $G^{1,0}_{0,2}$ is the solution again but its parametersare different.

3. *z*-component: likewise from the equation (2.5), and just like the φ-component, for the *z*-component equation (3.3) we have:

$$1-2(b_1+b_2)=0, 4\alpha=-k^2, b_1b_2=0.$$

Consequently, if $b_1 = 1/2$; $b_2 = 0$; $\alpha = -k^2/4$; the first independent solution is:

$$Z_1(z)=\begin{Bmatrix} \sinh kz & SV & method \\ G^{1,0}_{0,2}(\frac{-k^2}{4}z^2|^{-}_{\frac{1}{2},0}) & MSV & method \end{Bmatrix}.$$

Furthermore, if $b_1 = 0$; $b_2 = 1/2$;$\alpha = -k^2/4$, and the second independent solution is:

$$Z_2(z)=\begin{Bmatrix} \cosh kz & SV & method \\ G^{1,0}_{0,2}(\frac{-k^2}{4}z^2|^{-}_{0,\frac{1}{2}}) & MSV & method \end{Bmatrix}.$$

Thus, $G^{1,0}_{0,2}$ is appeared again as the solution, but it with different values of parametersis equivalence of *sinh z* and *cosh z* functions. Combining the solutions for ρ, φ and *z*components the overall solution is:

$$\Phi(\rho,\varphi,z)=[EG^{1,0}_{0,2}(\frac{1}{4}\rho^2|^{-}_{\frac{\nu}{2},\frac{-\nu}{2}})+FG^{1,0}_{0,2}(\frac{1}{4}\rho^2|^{-}_{\frac{-\nu}{2},\frac{\nu}{2}})G^{1,0}_{0,2}](\frac{1}{4}\rho^2|^{-}_{\frac{\nu}{2},\frac{-\nu}{2}})[AG^{1,0}_{0,2}(\frac{1}{4}\varphi^2|^{-}_{\frac{1}{2},0})+ BG^{1,0}_{0,2}(\frac{1}{4}\varphi^2|^{-}_{0,\frac{1}{2}})][CG^{1,0}_{0,2}(-\frac{1}{4}z^2|^{-}_{\frac{1}{2},0})+DG^{1,0}_{0,2}(-\frac{1}{4}z^2|^{-}_{0,\frac{1}{2}})]. \quad (3.5)$$

The equation (3.5) shows the solution of each of components (ρ,φ and, *z*-component) is $G^{1,0}_{0,2}$, namely the same orders *1,0,0,2* but with different parameters. Because Laplace's equation is the special case of heat equation, here the time part of heat equation is alsosolved by MSV

4. *t*-component: for $m = 1$; $n = 0$; $p = 0$; $q = 1$, the equation (2.5) is reduced to:

$$[-z-(z\frac{d}{dz}-b_1)]G^{1,0}_{0,1}(z|^{-}_{b_1})=0.$$

By changing *z* into $\delta^2 z$ and dividing it by *-z*, the following is obtained:

$$[\frac{d}{dz}-\delta^2-\frac{b_1}{z}]G^{1,0}_{0,1}(-z|^{-}_{b_1})=0.$$

On the other hand, $[\frac{d}{dz} - \gamma^2]T(z) = 0$; the condition for the equivalence of these two differential equations is $b_1 = 0$, $\delta^2 = \gamma^2$, and the solution is as follows:

$$T(z) = \left\{\begin{matrix} \exp -\gamma^2 z & SV \quad method \\ G^{1,0}_{0,1}\left(-\gamma^2 z|^{-}_{0}\right) & MSV \quad method \end{matrix}\right\}.$$

Thus we have

$$g(t) = G^{1,0}_{0,1}(-\gamma^2 t|^{-}_{0}).$$

Finally, we complete Table 1 and all results are summarized in Table 1.

Table 1: The general solutions in terms of Meijer's G-functions and Elementary and Special functions

Differential equation	Solutions
$[\frac{d}{dz} - 1]T(z) = 0$	$T(z) = e^z = G^{1,0}_{0,1}(-z|^{-}_{0})$
$[\frac{d^2}{dz^2} + 1]Z(z) = 0$	$Z(z) = A\sin z + B\cos z$ $= AG^{1,0}_{0,2}(\frac{1}{4}z^2|^{-}_{\frac{1}{2},0}) + BG^{1,0}_{0,2}(\frac{1}{4}z^2|^{-}_{0,\frac{1}{2}})$
$[\frac{d^2}{dz^2} - 1]Z(z) = 0$	$Z(z) = C\sinh z + D\cosh z$ $= CG^{1,0}_{0,2}(-\frac{1}{4}z^2|^{-}_{\frac{1}{2},0}) + DG^{1,0}_{0,2}(-\frac{1}{4}z^2|^{-}_{0,\frac{1}{2}})$
$[z^2\frac{d^2}{dz^2} + z\frac{d}{dz} + z^2 - \nu^2]Z_\nu(z) = 0$	$Z_\nu(z) = EJ_\nu(z) + FJ_{-\nu}(z)$ $= EG^{1,0}_{0,2}(\frac{1}{4}z^2|^{-}_{\frac{\nu}{2},\frac{-\nu}{2}}) + FG^{1,0}_{0,2}(\frac{1}{4}z^2|^{-}_{\frac{-\nu}{2},\frac{\nu}{2}})$

4 Conducting Cylindrical Can

Example 4.2 Consider a cylindrical conducting can of radius **a** and height **h**. The potential varies at the top face as $\mathbf{V(\rho, \varphi)}$), while the lateral surface and the bottom faceare held at zero potential. Let us find the electrostatic potential at all points inside thecan in terms of Meijer's G-functions.

This is the step:

This is a three-dimensional problem involving Laplace's equation, a separation of variables for Laplace's equation, $\Phi(\rho, \varphi, Z) = R(\rho)S(\varphi)Z(z)$, leads to the three ODEs:

$$\frac{d}{d\rho}(\rho\frac{dR}{d\rho}) + (k^2\rho - \frac{m^2}{\rho})R = 0, \tag{4.1}$$

$$\frac{d^2S}{d\varphi^2} + m^2 S = 0, \qquad \frac{d^2Z}{dz^2} - k^2 Z = 0, \tag{4.2}$$

where in preparation for the correct BCs, we have chosen the constants as k^2 and $-m^2$ with m an integer. The first of these equations with $x = k\rho$ is Bessel equation, whose general solution as shown in Table 1 can be written as:

$$R(\rho) = AJ_m(k\rho) + BY_m(k\rho) = AG^{1,0}_{0,2}(\frac{k^2}{4}\rho^2|^{-}_{\frac{m}{2},\frac{-m}{2}}) + BG^{1,0}_{0,2}(\frac{1}{4}\rho^2|^{-}_{\frac{-\nu}{2},\frac{\nu}{2}}).$$

Using this fact that the potential must be finite everywhere inside the can (including at $\rho = 0$) causes B to vanish because the Neumann function $Y_m(k\rho)$ is not defined at $\rho = 0$.On the other hand, Φ must vanish at $\rho = a$. This gives $J_m(ka) = G^{1,0}_{0,2}(\frac{k^2}{4}a^2|^{-}_{\frac{m}{2},\frac{-m}{2}}) = 0$, which needs that ka be a root of the Bessel function of order m. Let x_{mn} denote the$n$th zero of the Bessel function of order m, then we have $k = x_{mn}/a$ for n =1,2,… .

Since the extra condition of periodicity is usually imposed on the potential for variable φ, the second DE, in terms of Meijer's G-functions, has the general solution (see Table1)

$$S(\varphi) = CG^{1,0}_{0,2}(\frac{m^2}{4}\varphi^2|^{-}_{\frac{1}{2},0}) + DG^{1,0}_{0,2}(\frac{m^2}{4}\varphi^2|^{-}_{0,\frac{1}{2}}).$$

Finally the third DE has a general solution of the form

$$Z(z) = EG^{1,0}_{0,2}(-\frac{x^2_{mn}}{4a^2}z^2|^{-}_{\frac{1}{2},0}) + FG^{1,0}_{0,2}(-\frac{x^2_{mn}}{4a^2}z^2|^{-}_{0,\frac{1}{2}}).$$

The vanishing of Φ at $z = 0$ yields that $Z(z) = EG^{1,0}_{0,2}(-\frac{x^2_{mn}}{4a^2}z^2|^{-}_{\frac{1}{2},0})$. If we multiply *R; S;*and *Z* and sum over all possible values of *m* and *n*, then we have the following solutionfor potential Φ:

$$\Phi(\rho,\varphi,z) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} G^{1,0}_{0,2}(\frac{x^2_{mn}}{4a^2}\rho^2|^{-}_{\frac{m}{2},\frac{-m}{2}})G^{1,0}_{0,2}(-\frac{x^2_{mn}}{4a^2}z^2|^{-}_{\frac{1}{2},0})[\qquad (4.3)$$

$$A_{mn}G^{1,0}_{0,2}(\frac{m^2}{4}\varphi^2|^{-}_{\frac{1}{2},0}) + B_{mn}G^{1,0}_{0,2}(\frac{m^2}{4}\varphi^2|^{-}_{0,\frac{1}{2}})].$$

5 Circular Heat-conducting Plate

Example 5.3 Consider a circular heat-conducting plate of radius a whose temperatureat **t = 0** has a distribution function f(ρ,φ). Let us find G-function solution for the variation of **T** for all points (ρ,φ) on the plate for time **t > 0** when the edge is kept at **T = 0**.

where in anticipation of the correct BCs, we have written the constants as k^2 and $-m^2$ with *m* an integer.

Here are the steps:

For this two-dimensional problem involving heat equation, a separation of variables, $T(\rho,\varphi,t) = R(\rho)S(\varphi)g(t)$, results in the following ODEs:

$$\frac{dg}{dt} = k^2\lambda g, \qquad \frac{d^2S}{d\varphi^2} + \mu S = 0, \qquad \frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} - (\frac{\mu}{\rho^2} + \lambda)R = 0. \qquad (5.1)$$

If we assume that there is exponential decay for the temperature, then we need that $\lambda \equiv b^2 < 0.$ Thus we have

$$g(t) = AG^{1,0}_{0,1}(-k^2b^2t|^{-}_{0}).$$

when the extra condition of periodicity is imposed on the temperature, we must have $\mu = m^2$, where *m* is an integer. Then it can be written as the following:

$$S(\varphi) = BG^{1,0}_{0,2}(\frac{m^2}{4}\varphi^2|^{-}_{\frac{1}{2},0}) + CG^{1,0}_{0,2}(\frac{m^2}{4}\varphi^2|^{-}_{0,\frac{1}{2}}).$$

To have finite *T* at $\rho = 0$, no Neumann function $Y_m(k\rho)$ is to be presented. This leadsto the following solutions:

$$R(\rho) = DG^{1,0}_{0,2}(\frac{1}{4}b^2\rho^2|^{-}_{\frac{m}{2},\frac{-m}{2}}).$$

If the temperature is to be zero at $\rho = a$, another BC, then we must have $J_m(ba) = 0$,or $b = x_{mn}/a$. Now we can multiply *g(t), S(φ),* and *R(ρ)* and then the general solutioncan be written as

$$T(\rho,\varphi,t) = \sum_{m=0,n=1}^{\infty} G^{1,0}_{0,1}(-k^2(\frac{x_{mn}}{a})^2t|^{-}_{0})G^{1,0}_{0,2}(\frac{1}{4}(\frac{x_{mn}}{a})^2\rho^2|^{-}_{\frac{m}{2},\frac{-m}{2}})[\qquad (5.2)$$

$$A_{mn}G^{1,0}_{0,2}(\frac{m^2}{4}\varphi^2|^{-}_{\frac{1}{2},0}) + B_{mn}G^{1,0}_{0,2}(\frac{m^2}{4}\varphi^2|^{-}_{0,\frac{1}{2}})].$$

Thus the variation of temperature for all points is obtained in terms of G-functions.

Acknowledgments

This work was supported by MOHE with the grant number: ERGS/1/2013/STG06/UKM/01/2.

REFERENCES

[1] Kiryakova, V. 1994. Generalized fractional calculus and applications. , vol. 301. Chapman & Hall/CRC.

[2] Andrews, L. 1985. Special functions for engineers and applied mathematicians. Macmillan.

[3] Luck, Y. 1969. The Special Functions and Their Approximations. Complex Variables. New York: Academic press.

[4] Askey, R. 2010. Meijer G-function. Cambridge: Cambridge University Press.

[5] Klimyik, A. 2001. Meijer G-function. Berlin: Springer.

[6] Erdelyi, A., Magnus, W., Oberhettinger, F. & Tricomi, F. 1953. Higher transcendental functions. New York: McGraw Hill.

[7] Prudnikov, A., Brychkov, I. &Marichev, O. 1992. Integrals and Series: Some More Special Functions. Gordon & Breach.

[8] Kilbas, A., Srivastava, H. & Trujillo, J. 2006. Theory And Applications of Fractional Differential Equations. Elsevier.

[9] Pishkoo, A. &Darus, M. 2013. Meijers G-functions (MGFs) in Micro- and Nano-structures. Journal of Computational and Theoretical Nanoscience 10 (10): 2478-2483.

[10] Pishkoo, A. &Darus, M. 2013. Some Applications of Meijer G-functions as Solutions of Differential Equations in Physical Models Journal of Mathematical Physcis, Analysis, Geometry3 (9): 379-391.

[11] Pishkoo, A. & Darus, M. 2012. Meijers G-functions as the solution of Schrodinger equation. Proceeding of the 43rd Iranian Mathematics Conference. University of Tabriz, Tabriz, Iran.

[12] Hassani, S. 1998. Mathematical Physics: A Modern Introduction to Its Foundations. US: Springer.

[13] Pishkoo, A. &Darus, M. 2012. Fractional difference integral transformations of univalent Meijers G-functions. Journal of Inequalities and Applications (2012): Article 36.