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Probabilistic Representation of a Normal Generalized Inverse Gaussian Integral: Application to Option Pricing

Werner Hürlimann
Wolters Kluwer Financial Services
Seefeldstrasse 69, CH-8008 Zürich
whurlimann@bluewin.ch

ABSTRACT

An analytical probabilistic integral representation for the European call option price in the Hurst – Platen -- Rachev subordinated asset price model with generalized inverse Gaussian subordinator is obtained. For the limiting gamma mixing case, the representation yields simpler closed-form formulas for the European risk-neutral call option price in the exponential variance-gamma process by Madan, Carr and Chang. An elementary state-price deflator derivation of the Hurst-Platen-Rachev option pricing formula is also included.

Keywords and phrases

Subordinated Gaussian process; Lévy process; Generalized hyperbolic; Variance gamma; Skew hyperbolic T; Hurst-Platen-Rachev option pricing model; State-price deflator

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1. INTRODUCTION

The Hurst-Platen-Rachev subordinated asset price model, introduced by Hurst et al. (1997/99), leads to a remarkable extension of the option pricing formula by Black and Scholes (1973) that besides the subordinator only depends upon the volatility parameter. In this model European call option prices depend upon a two-parameter normal mixed integral of the form

$$\Psi_G(a,b)=\int_0^\infty \Phi(a\sqrt{x^{-1}}+b\sqrt{x})f_G(x)dx, \tag{1.1}$$

where $f_G(x)$ is the probability density function of a non-negative mixing random variable G, and $\Phi(x)$ is the standard normal distribution. For the class of generalized inverse Gaussian mixing distributions, we derive a probabilistic integral representation for this normal integral Ψ -function.

For a standardized gamma mixing distribution, the Ψ -function in (1.1) is of the form stated in Kotz et al. (2001), p.296. It is related to the European risk-neutral call option price for an exponential variance-gamma (VG) price process initially derived by Madan et al. (1998), Theorem 2 and Appendix (see also Madan (2001), Section 6.3.1, equation (14)). These authors express (1.1) for the gamma mixing density in terms of the Macdonald function (modified Bessel function of the second kind) and the degenerate hyper-geometric function of two variables. As an important application, equation (4.12) yields a closed-form expression for the gamma Ψ -function as a sum of an incomplete beta function and an integrated Macdonald function, which is much simpler than the original representation. A more detailed account of the content follows.

Section 2 recalls the Hurst-Platen-Rachev subordinated asset price model together with its call option pricing formula, which involves the normal integral (1.1). For the class of generalized inverse Gaussian subordinators, we derive in Theorem 2.1 a probabilistic integral representation of the normal integral (1.1) in terms of two different generalized hyperbolic densities. Section 3 contains an elementary probabilistic derivation of the Hurst-Platen-Rachev option pricing formula. It is obtained through generalization of a previous result by the author in the state-price deflator option pricing framework. Applications to specific examples follow in Section 4. We discuss several sub-cases of the generalized hyperbolic distribution, in particular the hyperbolic, normal inverse Gaussian, and the normal harmonic distributions. Detailed formulas for the limiting gamma and reciprocal gamma mixing distributions associated to the variance-gamma and skew hyperbolic Student T distributions are also obtained.

2. Probabilistic Representation

In Mathematical Finance, the *subordinated Gaussian process* is defined as a drifted Brownian motion time changed by an independent *mixing process*. Viewed from the initial time 0 it is defined by

$$X_t=\theta\cdot G_t+\sigma\cdot W_{G_t},\quad t>0,\quad \sigma>0,-\infty<\theta<\infty, \tag{2.1}$$

where W_t is a standard Wiener process and G_t is an independent *subordinator*, that is an increasing, positive Lévy process. Hurst et al. (1999) consider the following *subordinated asset price model*. Given the current price of a risky asset at time 0 its future price at time $t>0$ is described by an exponential subordinated Gaussian process with drift μ and volatility σ of the type

$$S_t=S_0\exp(\mu t+X_t),\quad X_t=-\tfrac{1}{2}\sigma^2\cdot G_t+\sigma\cdot W_{G_t}. \tag{2.2}$$

Through application of the equivalent martingale measure method they derive the price of a European call option with maturity date T and exercise price K as

$$C(S_0,T,K,r,\sigma)=E^*[(S_T-K)_+]=S_0\cdot\Psi^-_{G_T}(d,\sigma)-Ke^{-rT}\cdot\Psi^+_{G_T}(d,\sigma), \tag{2.3}$$

$$\Psi^{\mp}_{G_T}(d,\sigma)=\int_0^\infty \Phi\left(\frac{d\pm\frac{1}{2}\sigma^2 w}{\sigma\sqrt{w}}\right)f_{G_T}(w)dw,\quad d=\ln(S_0/K)+rT, \tag{2.4}$$

where $f_{G_T}(x)$ is the probability density function of the mixing random variable G_T, r is the risk-free interest rate and $\Phi(x)$ is the standard normal distribution. If $G_T=T$ with probability one, then (2.3)-(2.4) yields the famous formula by Black and Scholes (1973). The subordinated asset price model is a remarkable generalization of the Black-Scholes-Merton asset price model that besides the subordinator only depends upon the volatility parameter. It has been first discussed in Hurst et al. (1997) (see also Rachev and Mittnik (2000) and Rachev et al. (2011), Section 7.6 with correction of the misprint in the stock price model, however). An elementary probabilistic derivation of (2.3)-(2.4) based on the state-price deflator approach follows in Section 3.

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From an analytical point of view the formula (2.4) depends upon the two-parameter normal mixed integral of the form

$$\Psi_G(a,b)=\int_0^\infty \Phi(a\sqrt{x^{-1}}+b\sqrt{x})f_G(x)dx. \quad (2.5)$$

where $f_G(x)$ is the probability density function of the non-negative mixing random variable G.

The focus is now on the three-parameter *generalized inverse Gaussian* (GIG) mixing random variable $G\sim GIG(\lambda,\delta,\gamma)$ with density function

$$f_{GIG}(x)=f_{GIG}(x;\lambda,\delta,\gamma)=\frac{(\gamma/\delta)^\lambda}{2K_\lambda(\delta\gamma)}x^{\lambda-1}e^{-\frac{1}{2}(\delta^2x^{-1}+\gamma^2x)},\quad x>0, \quad (2.6)$$

where $K_\lambda(x)$ is the Macdonald function defined by the integral representation

$$K_\lambda(x)=\frac{1}{2}\int_0^\infty y^{\lambda-1}e^{-\frac{1}{2}(y^{-1}+y)}dy. \quad (2.7)$$

Used as subordinator the GIG leads to the *generalized hyperbolic Lévy motion* (see Eberlein (2001), Section 5, for stochastic process justification, and Section 4 below for some important examples). The five parameter *generalized hyperbolic* (GH) random variable $X\sim GH(\lambda,\alpha,\beta,\delta,\mu)$ is defined by the density function

$$\begin{aligned} &f_{GH}(x)=f_{GH}(x;\lambda,\alpha,\beta,\delta,\mu)\\ &=C(\lambda,\alpha,\beta,\delta)e^{\beta(x-\mu)}(\delta^2+(x-\mu)^2)^{(\lambda-\frac{1}{2})/2}K_{\lambda-\frac{1}{2}}(\alpha\sqrt{\delta^2+(x-\mu)^2}),\\ &C(\lambda,\alpha,\beta,\delta)=\frac{(\alpha^2-\beta^2)^{\lambda/2}}{\sqrt{2\pi}\alpha^{\lambda-\frac{1}{2}}\delta^\lambda K_\lambda(\delta\sqrt{\alpha^2-\beta^2})}. \end{aligned} \quad (2.8)$$

The use of the GIG as mixing distribution for the GH follows from the normal variance-mean mixture representation $X=\mu+\beta\cdot G+W_G\sim GH(\lambda,\alpha,\beta,\delta,\mu)$ expressed in integral form by

$$f_{GH}(x;\lambda,\alpha,\beta,\delta,\mu)=\int_0^\infty\sqrt{y^{-1}}\varphi\left(\sqrt{y^{-1}}\{x-\mu-\beta y\}\right)f_{GIG}\left(x;\lambda,\delta,\gamma=\sqrt{\alpha^2-\beta^2}\right)dx, \quad (2.9)$$

with $\varphi(x)=\Phi'(x)$ the density of the standard normal. Now, the partial derivative with respect to a of the integrand in (2.5) is related to the integrand in (2.9) with $\mu=0,\beta=-b$, which suggests a close relationship between the GIG Ψ-function and GH distributions. Filling out the details one obtains the following main probabilistic representation.

Theorem 2.1. (*Probabilistic GIG Ψ-function representation*) The normal GIG mixed integral

$\Psi_{GIG}(a,b)=\int_0^\infty\Phi(a\sqrt{x^{-1}}+b\sqrt{x})f_{GIG}(x;\lambda,\delta,\gamma)dx$ is determined by

$$\Psi_{GIG}(a,b)=\frac{1}{2}+\operatorname{sgn}(b)\cdot\int_0^{|b|}f_{GH(-\lambda,\delta,0,\gamma,0)}(x)dx+\operatorname{sgn}(a)\cdot\int_0^{|a|}f_{GH(\lambda,\sqrt{\gamma^2+b^2},-\operatorname{sgn}(a)b,\delta,0)}(x)dx. \quad (2.10)$$

Proof. In the special case $a=0,b\ge 0$ set $J(z)=\Psi_{GIG}(0,z)=\int_0^\infty\Phi(z\sqrt{x})f_{GIG}(x)dx,\ z\ge 0$. One has $J(0)=\frac{1}{2}$, $J(b)=J(0)+\int_0^b J'(z)dz$. A calculation shows that

$$J'(z)=\int_0^\infty\sqrt{x}\varphi(z\sqrt{x})f_{GIG}(x)dx=\frac{1}{\sqrt{2\pi}}\frac{(\gamma/\delta)^\lambda}{2K_\lambda(\delta\gamma)}\cdot\int_0^\infty x^{(\lambda+\frac{1}{2})-1}e^{-\frac{1}{2}\{\delta^2x^{-1}+(\gamma^2+z^2)x\}}dx.$$

The integrand is related to a GIG density with changed parameters, namely to $GIG(\bar{\lambda},\bar{\delta},\bar{\gamma})$ with

$$\bar{\lambda} = \lambda + \tfrac{1}{2}, \quad \bar{\delta} = \delta, \quad \bar{\gamma} = \sqrt{\gamma^2 + z^2}.$$

Using (2.6) with changed parameters and the reflection relationship $K_\lambda(x) = K_{-\lambda}(x)$, one sees that

$$J'(z) = \tfrac{1}{\sqrt{2\pi}} \tfrac{(\gamma/\delta)^\lambda}{K_\lambda(\delta\gamma)} (\bar{\delta}/\bar{\gamma})^{\bar{\lambda}} K_{\bar{\lambda}}(\bar{\delta}\bar{\gamma}) = \tfrac{1}{\sqrt{2\pi}} \tfrac{\delta^{\frac{1}{2}}\gamma^\lambda}{K_{-\lambda}(\delta\gamma)} (\gamma^2 + z^2)^{-(\lambda+\frac{1}{2})/2} K_{-\lambda-\frac{1}{2}}(\delta\sqrt{\gamma^2 + z^2}).$$

Comparing with (2.8) shows that this is the density of a $GH(\bar{\lambda}, \bar{\alpha}, \bar{\beta}, \bar{\delta}, \bar{\mu})$ with parameters

$$\bar{\lambda} = -\lambda, \bar{\alpha} = \delta, \bar{\beta} = 0, \bar{\delta} = \gamma, \bar{\mu} = 0.$$

The formula (2.10) for $a = 0, b \ge 0$ is shown. The formula for $a = 0, b < 0$ follows by noting that $\Phi(b\sqrt{x}) = \Phi(-|b|\sqrt{x}) = 1 - \Phi(|b|\sqrt{x})$, hence $J(b) = 1 - J(|b|)$. In general, if $a \ge 0$ set $I(z) = \int_0^\infty \Phi(z\sqrt{x^{-1}} + b\sqrt{x}) f_{GIG}(x)dx, z \ge 0$. One has $I(0) = J(b)$, $I(a) = J(b) + \int_0^a I'(z)dz$, and

$$I'(z) = \int_0^\infty \sqrt{x^{-1}} \varphi(z\sqrt{x^{-1}} + b\sqrt{x}) f_{GIG}(x)dx = \tfrac{1}{\sqrt{2\pi}} \tfrac{(\gamma/\delta)^\lambda}{2K_\lambda(\delta\gamma)} e^{-bz} \cdot \int_0^\infty x^{(\lambda-\frac{1}{2})-1} e^{-\frac{1}{2}\{(\delta^2+z^2)x^{-1} + (\gamma^2+b^2)x\}} dx.$$

The integrand is related to the density of a $GIG(\bar{\lambda}, \bar{\delta}, \bar{\gamma})$ with transformed parameters

$$\bar{\lambda} = \lambda - \tfrac{1}{2}, \quad \bar{\delta} = \sqrt{\delta^2 + z^2}, \quad \bar{\gamma} = \sqrt{\gamma^2 + b^2}.$$

Using again (2.6) one sees that

$$I'(z) = \tfrac{1}{\sqrt{2\pi}} \tfrac{(\gamma/\delta)^\lambda}{K_\lambda(\delta\gamma)} e^{-bz} (\bar{\delta}/\bar{\gamma})^{\bar{\lambda}} K_{\bar{\lambda}}(\bar{\delta}\bar{\gamma}) = \tfrac{1}{\sqrt{2\pi}} \frac{\gamma^\lambda}{(\gamma^2+b^2)^{\frac{\lambda-\frac{1}{2}}{2}} \delta^\lambda K_\lambda(\delta\gamma)} e^{-bz} (\delta^2 + z^2)^{(\lambda-\frac{1}{2})/2} K_{\lambda-\frac{1}{2}}(\sqrt{\gamma^2 + b^2}\sqrt{\delta^2 + z^2}).$$

A comparison with (2.8) shows that this is the density of a $GH(\bar{\lambda}, \bar{\alpha}, \bar{\beta}, \bar{\delta}, \bar{\mu})$ with parameters

$$\bar{\lambda} = \lambda, \bar{\alpha} = \sqrt{\gamma^2 + b^2}, \bar{\beta} = -b, \bar{\delta} = \delta, \bar{\mu} = 0.$$

The formula (2.10) for $a \ge 0$ and arbitrary b follows. If $a < 0$ one notes that

$$\Phi(a\sqrt{x^{-1}} + b\sqrt{x}) = \Phi(-|a|\sqrt{x^{-1}} + b\sqrt{x}) = 1 - \Phi(|a|\sqrt{x^{-1}} - b\sqrt{x}),$$

hence $\Psi_{GIG}(a,b) = 1 - \Psi_{GIG}(|a|, -b)$, and (2.10) follows immediately. ◊

3. Hurst-Planten-Rachev Option Pricing Formula

A generalization of Theorem 4.1 in Hürlimann (2013a) is formulated and used to derive the option pricing formula (2.3)-(2.4) by Hurst et al. (1999).

Consider the following subordinated asset price model. Given the current price of a risky asset at time 0, its future price at time $t > 0$ is described by an exponential subordinated Gaussian process

$$S_t = S_0 \exp((\mu - \omega)t + X_t), \quad X_t = \theta \cdot G_t + \sigma \cdot W_{G_t}, \tag{3.1}$$

where μ represents the mean logarithmic rate of return of the risky asset per time unit. Using the defining relationship $E[S_t] = S_0 \exp(\mu t)$ at unit time, one sees that $\omega = C_X(1)$, where one assumes that the cgf of $X = X_1$ exists over some open interval, which contains one. Suppose that the *subordinated Gaussian deflator* has the same form as the price process in (3.1). For some parameters α, β (both to be determined) one sets for it (an Esscher transform measure)

$$D_t = \exp(-\alpha t - \beta X_t), \quad t > 0. \tag{3.2}$$

A simple cgf calculation shows that the state-price deflator martingale conditions

$$E[D_t] = e^{-rt}, \quad E[D_t S_t] = S_0, \quad t > 0, \tag{3.3}$$

are equivalent with the system of two non-linear equations in the three unknowns α, β, ω (use that X_t is a Lévy process, hence $C_{X_t}(u) = t \cdot C_X(u)$):

$$r - \alpha + C_X(-\beta) = 0, \quad \mu - \omega - \alpha + C_X(1-\beta) = 0. \tag{3.4}$$

Inserting the first equation into the second ones yields the necessary relationship

$$\mu - r - \omega + C_X(1-\beta) - C_X(-\beta) = 0. \tag{3.5}$$

Since the system (3.4) has one degree of freedom, the unknown ω can be chosen arbitrarily, say

$$\omega = \mu - r, \tag{3.6}$$

which is interpreted as the (time-independent) *subordinated market price of the risky asset*. With the made restriction on the cgf this value is always finite. Inserted into (3.5) shows that the parameter β is determined by the equation

$$C_X(1-\beta) = C_X(-\beta). \tag{3.7}$$

We are ready to show the following subordinated Gaussian deflator representation.

Theorem 3.1. (*Subordinated Gaussian deflator*) Given is a risk-free asset with constant return r and a risky asset with real-world price (3.1), where one assumes that the cgf of $X = X_1$ exists over some open interval, which contains one. Then, the subordinated Gaussian deflator of the exponential subordinated Gaussian process is determined by

$$D_t = \exp(-\alpha t - \beta X_t), \quad \alpha = r + C_X(-\beta), \quad \beta\sigma^2 = \theta + \tfrac{1}{2}\sigma^2. \tag{3.8}$$

Proof. The first equation in (3.4) yields the expression for α. Since X_t, G_t are Lévy processes, one has $C_{X_t}(u) = t \cdot C_X(u), C_{G_t} = t \cdot C_G(u), G = G_1$. Therefore, the known relationship $C_{X_t}(u) = C_{G_t}(\theta u + \frac{1}{2}\sigma^2 u^2)$ (e.g. Feller (1971), Section II.5) is equivalent with the equation $C_X(u) = C_G(\theta u + \frac{1}{2}\sigma^2 u^2)$. It follows that the condition (3.7) is equivalent with the equation

$$\theta(1-\beta) + \tfrac{1}{2}(1-\beta)^2\sigma^2 + \theta\beta - \tfrac{1}{2}\beta^2\sigma^2 = 0,$$

which implies the stated condition for β. ◊

As an immediate application, the special choice $\beta = 0$ implies that

$$C_X(-\beta) = 0, \quad \alpha = r, \quad \theta = -\tfrac{1}{2}\sigma^2.$$

It follows that the subordinated Gaussian deflator degenerates to the risk-free discount factor $D_t = e^{-rt}$. In this *simple subordinated market* the risky asset follows the price process (insert (3.6) into (3.1))

$$S_t = S_0 \exp(rt - \tfrac{1}{2}\sigma^2 \cdot G_t + \sigma \cdot W_{G_t}). \tag{3.9}$$

Moreover, the pricing of the European call option with maturity date T and exercise price K reduces to the Hurst-Platen-Rachev option pricing formula (2.3)-(2.4). A justification of the deflator approach and its relationship with the equivalent martingale measure method is found in Hürlimann (2013a), Remarks 4.1.

Theorem 3.2. (*European call option formula in the simple subordinated market*) Given is the asset price model (3.9) subject to the risk-free discount factor $D_t = e^{-rt}$. Then one has

$$C(S_0, T, K, r, \sigma) = E[D_T(S_T - K)_+] = S_0 \cdot \Psi^-_{G_T}(d, \sigma) - Ke^{-rT} \cdot \Psi^+_{G_T}(d, \sigma), \tag{3.10}$$

$$\Psi^{\mp}_{G_T}(d, \sigma) = \int_0^\infty \Phi\left(\frac{d \pm \frac{1}{2}\sigma^2 w}{\sigma\sqrt{w}}\right) f_{G_T}(w)dw, \quad d = \ln(S_0/K) + rT, \tag{3.11}$$

Proof. Through conditioning rewrite $C = E[D_T(S_T - K)_+]$ as $C = \int_0^\infty C(w) f_{G_T}(w)dw$ with

$$C(w) = E[(S_0 \cdot \exp\{-\tfrac{1}{2}\sigma^2 \cdot G_T + \sigma \cdot W_{G_T}\} - Ke^{-rT})_+ | G_T = w].$$

The distribution of the conditional random variable $(-\tfrac{1}{2}\sigma^2 \cdot G_T + \sigma \cdot W_{G_T} | G_T = w)$ is determined by the conditional mean

$$E[-\tfrac{1}{2}\sigma^2 \cdot G_T + \sigma \cdot W_{G_T} | G_T = w] = -\tfrac{1}{2}\sigma^2 w,$$

and the conditional variance

$$Var[-\tfrac{1}{2}\sigma^2 \cdot G_T + \sigma \cdot W_{G_T} | G_T = w] = \sigma^2 w.$$

It follows that

$$C(w) = \int_0^\infty (S_0 e^{-\frac{1}{2}\sigma^2 \cdot w + \sigma \cdot \sqrt{w}x} - Ke^{-rT})_+ \varphi(x)dx.$$

Now, the expression in the bracket is non-negative provided $x \ge d(w) = \frac{\ln(K/S_0) - rT}{\sigma\sqrt{w}} + \frac{1}{2}\sigma\sqrt{w}$, and one obtains (using e.g. Hürlimann (2013a), Lemma A1.1)

$$C(w) = \int_{d(w)}^\infty (S_0 e^{-\frac{1}{2}\sigma^2 \cdot w + \sigma \cdot \sqrt{w}x} - Ke^{-rT})\varphi(x)dx = S_0 \cdot \Phi\left(\sigma\sqrt{w} - d(w)\right) - Ke^{-rT} \cdot \Phi\left(-d(w)\right)$$
$$= S_0 \cdot \Phi\left(\frac{d + \frac{1}{2}\sigma^2 w}{\sigma\sqrt{w}}\right) - Ke^{-rT} \cdot \Phi\left(\frac{d - \frac{1}{2}\sigma^2 w}{\sigma\sqrt{w}}\right),$$

which implies the result. ◊

4. Application to some generalized inverse Gaussian Mixing Distributions

An important class of subordinated Gaussian processes is induced by the generalized hyperbolic (GH) distribution. It belongs to the generalized inverse Gaussian (GIG) mixing random variable $G \sim GIG(\lambda, \delta, \gamma)$ with cgf

$$C_G(t) = \tfrac{1}{2}\lambda \cdot \ln\left\{\frac{\gamma^2}{\gamma^2 - 2t}\right\} + \ln\left\{\frac{K_\lambda(\delta\sqrt{\gamma^2 - 2t})}{K_\lambda(\delta\gamma)}\right\}.$$

The domain of variation of the parameters depends upon three cases.

<u>**Case 1**</u>: generic GH distribution with $-\infty < \lambda < \infty, \quad \delta > 0, \quad \gamma > 0$

The main examples include the hyperbolic (HYP) for $\lambda = 1$, the normal inverse Gaussian (NIG) for $\lambda = -\frac{1}{2}$, and the normal harmonic (NH) for $\lambda = 0$.

<u>**Case 2**</u>: variance-gamma (VG) distribution for $\lambda > 0, \quad \delta = 0, \quad \gamma > 0$

In this limiting situation the mixing distribution degenerates to a gamma distribution.

<u>**Case 3**</u>: skew hyperbolic T (SHT) distribution for $\lambda < 0, \quad \delta > 0, \quad \gamma = 0$

In this limiting case the GIG degenerates to a reciprocal or inverse gamma distribution.

Based on Theorem 2.1 we evaluate the Ψ-function case by case. Recall that it depends upon the random variables $GH(-\lambda, \delta, 0, \gamma, 0)$ and $GH(\lambda, \sqrt{\gamma^2 + b^2}, -\mathrm{sgn}(a)b, \delta, 0)$ associated to the generalized inverse Gaussian mixing random variables $GIG(-\lambda, \gamma, \delta)$ and $GIG(\lambda, \delta, \gamma)$ respectively. In the following, the reciprocal of a random variable X is denoted by $RX = X^{-1}$. Using that the density of a reciprocal random variable is given by $f_{RX}(x) = x^{-2} f_X(x^{-1})$ one sees that $GIG(-\lambda, \gamma, \delta) = RGIG(\lambda, \delta, \gamma)$.

<u>**Case 1**</u>: generic case

Hyperbolic distribution (HYP) : $\lambda = 1$

One has the normal variance-mean mixture relationships

$$GH(-1,\delta,0,\gamma,0) = W_{GIG(-1,\gamma,\delta)} = W_{RGIG(1,\delta,\gamma)}, \tag{4.1}$$

$$GH(1,\sqrt{\gamma^2+b^2},-\mathrm{sgn}(a)b,\delta,0) = -\mathrm{sgn}(a)b \cdot GIG(1,\delta,\gamma) + W_{GIG(1,\delta,\gamma)} \tag{4.2}$$

Normal inverse Gaussian (NIG) : $\lambda = -\frac{1}{2}$

The special case $\lambda = -\frac{1}{2}$ of (2.6) is called inverse Gaussian. It belongs to the random variable denoted $IG(\delta,\gamma)$. First of all, one has $GIG(-\frac{1}{2},\gamma,\delta) = IG(\gamma,\delta) = RIG(\delta,\gamma) = RGIG(\frac{1}{2},\delta,\gamma)$. One obtains the normal variance reciprocal inverse Gaussian random variable

$$GH(\tfrac{1}{2},\delta,0,\gamma,0) = W_{RIG(\delta,\gamma)}, \text{ with density function} \tag{4.3}$$

$$f_{GH(\frac{1}{2},\delta,0,\gamma,0)}(x) = \tfrac{1}{\pi}\delta e^{\delta\gamma} K_0(\delta\sqrt{\gamma^2+x^2}). \tag{4.4}$$

The special case of (2.8) for $\lambda = -\frac{1}{2}$, $\mu = 0$ is called normal inverse Gaussian. The corresponding random variable is denoted by $NIG(\alpha,\beta,\delta)$. One has the normal variance-mean mixture representation

$$\begin{aligned}&GH(-\tfrac{1}{2},\sqrt{\gamma^2+b^2},-\mathrm{sgn}(a)b,\delta,0)\\ &= NIG(\sqrt{\gamma^2+b^2},-\mathrm{sgn}(a)b,\delta) = -\mathrm{sgn}(a)b \cdot IG(\delta,\gamma) + W_{IG(\delta,\gamma)}\end{aligned} \tag{4.5}$$

Normal harmonic (NH) : $\lambda = 0$

The special case $\lambda = 0$ of (2.6) defines the harmonic law. It belongs to the random variable denoted $H(\delta,\gamma)$. One has $GIG(0,\gamma,\delta) = H(\gamma,\delta) = RH(\delta,\gamma) = RGIG(0,\delta,\gamma)$ and

$$GH(0,\delta,0,\gamma,0) = W_{RH(\delta,\gamma)}, \tag{4.6}$$

$$GH(0,\sqrt{\gamma^2+b^2},-\mathrm{sgn}(a)b,\delta,0) = -\mathrm{sgn}(a)b \cdot H(\delta,\gamma) + W_{H(\delta,\gamma)}. \tag{4.7}$$

Case 2 : variance-gamma (VG)

Paolella (2007) shows that the limiting case $\delta = 0$ of (2.6) with $\lambda > 0, \gamma > 0$ is a gamma random variable $GIG(\lambda,0,\gamma) = \Gamma(\lambda,\frac{1}{2}\gamma^2)$ with density

$$f_{\Gamma(\lambda,\frac{1}{2}\gamma^2)}(x) = \frac{(\frac{1}{2}\gamma^2)^\lambda}{\Gamma(\lambda)} x^{\lambda-1} e^{-\frac{1}{2}\gamma^2 x}. \tag{4.8}$$

One has $GIG(-\lambda,\gamma,0) = RGIG(\lambda,0,\gamma) = R\Gamma(\lambda,\frac{1}{2}\gamma^2)$ and thus $GH(-\lambda,0,0,\gamma,0) = W_{R\Gamma(\lambda,\frac{1}{2}\gamma^2)}$. To evaluate the latter distribution, one notes that the reciprocal gamma has density

$$f_{R\Gamma(\lambda,\frac{1}{2}\gamma^2)}(x) = \frac{(\frac{1}{2}\gamma^2)^\lambda}{\Gamma(\lambda)} x^{-\lambda-1} e^{-\frac{1}{2}\gamma^2 x^{-1}}. \tag{4.9}$$

With (2.9) and a change of variables it follows that

$$\begin{aligned}f_{GH(-\lambda,0,0,\gamma,0)}(x) &= \int_0^\infty \sqrt{y^{-1}}\varphi\left(x\sqrt{y^{-1}}\right) f_{R\Gamma(\lambda,\frac{1}{2}\gamma^2)}(y)dy = \frac{1}{\sqrt{2\pi}\Gamma(\lambda)}(\tfrac{1}{2}\gamma^2)^\lambda \cdot \int_0^\infty y^{-(\lambda+\frac{1}{2})} e^{-\frac{1}{2}(\gamma^2+x^2)y^{-1}} dy\\ &= \frac{1}{\sqrt{2\pi}\Gamma(\lambda)}(\tfrac{1}{2}\gamma^2)^\lambda \cdot \int_0^\infty z^{(\lambda+\frac{1}{2})-1} e^{-\frac{1}{2}(\gamma^2+x^2)z} dz = \frac{1}{B(\frac{1}{2},\lambda)} \frac{1}{\gamma}\left(\frac{\gamma^2}{\gamma^2+x^2}\right)^{\lambda+\frac{1}{2}}.\end{aligned}$$

where the last expression follows by noting that the integrand is related to the gamma density $\Gamma(\lambda+\frac{1}{2},\frac{1}{2}(\gamma^2+x^2))$ and using the fact that $B(\frac{1}{2},\lambda)=\frac{\sqrt{\pi}\Gamma(\lambda)}{\Gamma(\lambda+\frac{1}{2})}$ is the beta function. The rescaled random variable $Z=\gamma^{-1}\cdot GH(-\lambda,0,0,\gamma,0)$ has a Pearson type VII density, which is related to the Student t distribution. If $\lambda=\frac{1}{2}\upsilon, \upsilon=1,2,3,...$ a positive integer, then $\sqrt{\upsilon}\cdot Z$ has a Student t with υ degrees of freedom. Now, for $x\ge 0$ the substitution $z=\sqrt{t/(1-t)}$ shows the identity

$$\int_0^x \frac{dz}{(1+z^2)^{\lambda+\frac{1}{2}}}=\frac{1}{2}\cdot\int_0^{x^2/(1+x^2)} t^{-\frac{1}{2}}(1-t)^{\lambda-1}dt,$$

which implies the relationship

$$\int_0^{|b|} f_{GH(-\lambda,\delta,0,\gamma,0)}(x)dx=\frac{1}{2}\cdot F_{Be(\frac{1}{2},\lambda)}(\tfrac{b^2}{\gamma^2+b^2}), \tag{4.10}$$

where $Be(\frac{1}{2},\lambda)$ a beta random variable with distribution $F_{Be(\frac{1}{2},\lambda)}(x)=\frac{1}{B(\frac{1}{2},\lambda)}\cdot\int_0^x t^{-\frac{1}{2}}(1-t)^{\lambda-1}dt$. Similarly, the density of the variance gamma mixture

$$X=GH(\lambda,\sqrt{\gamma^2+b^2},c,0,0)=c\cdot\Gamma(\lambda,\tfrac{1}{2}\gamma^2)+W_{\Gamma(\lambda,\frac{1}{2}\gamma^2)},\quad c=-\operatorname{sgn}(a)b,$$

is by (2.9) given by

$$f_X(x)=\int_0^\infty \sqrt{y^{-1}}\varphi\left((x-cy)\sqrt{y^{-1}}\right)f_{\Gamma(\lambda,\frac{1}{2}\gamma^2)}(y)dy=\frac{1}{\sqrt{2\pi}\Gamma(\lambda)}(\tfrac{1}{2}\gamma^2)^\lambda\cdot\int_0^\infty y^{(\lambda-\frac{1}{2})-1}e^{-\frac{1}{2}\{(x-cy)^2y^{-1}+\gamma^2y\}}dy$$

$$=\frac{1}{\sqrt{2\pi}\Gamma(\lambda)}(\tfrac{1}{2}\gamma^2)^\lambda e^{cx}\cdot\int_0^\infty y^{(\lambda-\frac{1}{2})-1}e^{-\frac{1}{2}\{x^2y^{-1}+(c^2+\gamma^2)y\}}dy=\frac{1}{\sqrt{\pi}\Gamma(\lambda)}\gamma^{2\lambda}e^{-\operatorname{sgn}(a)bx}\cdot\left(\frac{|x|}{2\sqrt{b^2+\gamma^2}}\right)^{\lambda-\frac{1}{2}}K_{\lambda-\frac{1}{2}}(\sqrt{b^2+\gamma^2}|x|),$$

where the last equality is shown by using that the integrand is related to the density of the generalized inverse Gaussian $GIG(\lambda-\frac{1}{2},|x|,\sqrt{b^2+\gamma^2})$. Now, with the parameter transformation

$$\alpha=\sqrt{b^2+\gamma^2}+\operatorname{sgn}(a)b,\quad \beta=\sqrt{b^2+\gamma^2}-\operatorname{sgn}(a)b,$$

one sees that

$$f_X(x)=\frac{(\alpha\beta)^\lambda}{\sqrt{\pi}\Gamma(\lambda)}\left(\frac{|x|}{\alpha+\beta}\right)^{\lambda-\frac{1}{2}}\cdot\exp\left(-\tfrac{1}{2}(\alpha-\beta)x\right)\cdot K_{\lambda-\frac{1}{2}}(\tfrac{1}{2}(\alpha+\beta)|x|),\quad x\neq 0, \tag{4.11}$$

is the density of a variance-gamma $VG(\lambda,\alpha,\beta)$ (e.g. Hürlimann (2013b), equation (A4.21)). Inserting (4.10) and (4.11) into (2.10) one obtains the gamma Ψ -function representation

$$\Psi_{GIG}(a,b)=\frac{1}{2}\{1+\operatorname{sgn}(b)\cdot F_{Be(\frac{1}{2},\lambda)}(\tfrac{b^2}{\gamma^2+b^2})\}+\operatorname{sgn}(a)\cdot\int_0^{|a|} f_{VG(\lambda,\sqrt{b^2+\gamma^2}+\operatorname{sgn}(a)b,\sqrt{b^2+\gamma^2}-\operatorname{sgn}(a)b)}(x)dx. \tag{4.12}$$

<u>**Case 3**</u> **:** skew hyperbolic T (SHT)

Paolella (2007) shows that the limiting case $\gamma=0$ of (2.6) with $\lambda<0,\delta>0$ is a reciprocal gamma random variable $GIG(\lambda,\delta,0)=R\Gamma(-\lambda,\frac{1}{2}\delta^2)$ with density

$$f_{R\Gamma(-\lambda,\frac{1}{2}\delta^2)}(x)=\frac{(\frac{1}{2}\delta^2)^{-\lambda}}{\Gamma(-\lambda)}x^{\lambda-1}e^{-\frac{1}{2}\delta^2x^{-1}}. \tag{4.13}$$

One has $GIG(-\lambda,0,\delta)=RGIG(\lambda,\delta,0)=\Gamma(-\lambda,\frac{1}{2}\delta^2)$ and thus $GH(-\lambda,\delta,0,0,0)=W_{\Gamma(-\lambda,\frac{1}{2}\delta^2)}$. To obtain the distribution of the latter, one uses (2.9) to see that

$$f_{GH(-\lambda,\delta,0,0,0)}(x)=\int_0^\infty \sqrt{y^{-1}}\varphi\left(x\sqrt{y^{-1}}\right)f_{\Gamma(-\lambda,\frac{1}{2}\delta^2)}(y)dy=\frac{1}{\sqrt{2\pi}\Gamma(-\lambda)}(\tfrac{1}{2}\delta^2)^{-\lambda}\cdot\int_0^\infty y^{(-\lambda-\frac{1}{2})-1}e^{-\frac{1}{2}(x^2y^{-1}+\delta^2 y)}dy$$
$$=\frac{\delta^{-2\lambda}}{\sqrt{\pi}\Gamma(-\lambda)}\cdot(\tfrac{|x|}{2\delta})^{-\lambda-\frac{1}{2}}K_{-\lambda-\frac{1}{2}}(\delta|x|),$$

where the last equality is shown by using that the integrand is related to the density of the generalized inverse Gaussian $GIG(-\lambda-\frac{1}{2},|x|,\delta)$. With the parameter transformation $\alpha=\beta=\delta$, and (4.11), one sees that this is the density of a *generalized Laplace* $GL(-\lambda,\delta)=VG(-\lambda,\delta,\delta)$ studied previously by Mathai (1993a/b), Koponen (1995) and Chan (1998) (see Hürlimann (2013b), Notes 2.2). Similarly, the density of the variance reciprocal gamma mixture

$$X=GH(\lambda,|b|,c,\delta,0)=c\cdot R\Gamma(-\lambda,\tfrac{1}{2}\delta^2)+W_{R\Gamma(-\lambda,\frac{1}{2}\delta^2)},\quad c=-\text{sgn}(a)b,$$

is by (2.9) given by

$$f_X(x)=\int_0^\infty \sqrt{y^{-1}}\varphi\left((x-cy)\sqrt{y^{-1}}\right)f_{R\Gamma(-\lambda,\frac{1}{2}\delta^2)}(y)dy=\frac{1}{\sqrt{2\pi}\Gamma(-\lambda)}(\tfrac{1}{2}\delta^2)^{-\lambda}\cdot\int_0^\infty y^{(\lambda-\frac{1}{2})-1}e^{-\frac{1}{2}\{\delta^2+(x-cy)^2\}y^{-1}}dy$$
$$=\frac{1}{\sqrt{2\pi}\Gamma(-\lambda)}(\tfrac{1}{2}\delta^2)^{-\lambda}e^{cx}\cdot\int_0^\infty y^{(\lambda-\frac{1}{2})-1}e^{-\frac{1}{2}\{(\delta^2+x^2)y^{-1}+c^2y\}}dy=\frac{\sqrt{2}}{\sqrt{\pi}\Gamma(-\lambda)}(\tfrac{1}{2}\delta^2)^{-\lambda}e^{cx}\cdot(\tfrac{\sqrt{\delta^2+x^2}}{|b|})^{\lambda-\frac{1}{2}}K_{\lambda-\frac{1}{2}}(|b|\sqrt{\delta^2+x^2}),$$

where for the last equality one uses that the integrand is related to the density of the generalized inverse Gaussian $GIG(\lambda-\frac{1}{2},\sqrt{\delta^2+x^2},|b|)$. Comparing this with the density of a *skew hyperbolic T* random variable $SHT(-\lambda,\beta,\delta)$ defined by (e.g. Aas and Haff (2006), equation (8))

$$f_{SHT(-\lambda,\beta,\delta)}(x)=\frac{2^{\lambda+\frac{1}{2}}\delta^{-2\lambda}|\beta|^{-\lambda+\frac{1}{2}}}{\sqrt{\pi}\Gamma(-\lambda)}e^{\beta x}\cdot\frac{K_{-\lambda+\frac{1}{2}}(|\beta|\sqrt{\delta^2+x^2})}{(\sqrt{\delta^2+x^2})^{-\lambda+\frac{1}{2}}},\tag{4.14}$$

shows that $GH(\lambda,|b|,-\text{sgn}(a)b,\delta,0)=SHT(-\lambda,-\text{sgn}(a)b,\delta)$. Together, one obtains from Theorem 2.1 the reciprocal gamma Ψ -function representation

$$\Psi_{GIG}(a,b)=\frac{1}{2}+\text{sgn}(b)\cdot\int_0^{|b|}f_{GL(-\lambda,\delta)}+\text{sgn}(a)\cdot\int_0^{|a|}f_{SHT(-\lambda,-\text{sgn}(a)b,\delta)}(x)dx.\tag{4.15}$$

To conclude, let us mention that the naming “skew hyperbolic T” was proposed by Scott et al. (2011). Further applications of the SHT are found in Frecka and Hopwood (1983), Theodossiu (1998), Aas and Haff (2006), Hürlimann (2009), Ghysels and Wang (2011), etc.

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Author' biography

Werner Hürlimann has studied mathematics and physics at ETHZ, where he obtained his PhD in 1980 with a thesis in higher algebra. After postdoctoral fellowships at Yale University and at the Max Planck Institute in Bonn, he became 1984 an actuary at Winterthur Life and Pensions, a senior actuary at Aon Re and IRMG Switzerland 2003-06, a senior consultant at IRIS in Zürich 2006-2008, and is currently employed at FRSGlobal Switzerland (a Wolters Kluwer Company). He has been visiting associate professor in actuarial science at the University of Toronto during the academic year 1988-89. More information is found at his homepage https://sites.google.com/site/whurlimann/.