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Strongly Rickart Modules

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Abstract

In this paper we introduce and study the concept of strongly Rickart modules and strongly CS-Rickart modules as a stronger than of Rickart modules [8] and CS-Rickart modules[3] respectively. A module M is said to be strongly Rickart module if the right annihilators of each single element in $S = End_R(M)$ is generated by a left semicentral idempotent in S. A module M is said to be strongly CS- Rickart if for any $\varphi \in S$, $r_M(\varphi)$ is an essential in fully invariant direct summand of M. Properties, results, characterizations and relation of these concepts with others known concepts of modules are studied.

Key word

Strongly Rickart module; Rickart module; strongly CS-Rickart module; CS-Rickart module

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1. Introduction

Throughout this paper R is an associative ring with identity and all module will be unitary right R-modules. Kaplansky in introduced a Baer ring R as the right annihilator of every non empty subset of a ring R is generated by an idempotent element [9]. In [17] Clark introduced a quasi-Baer ring as the right annihilator of every two sided ideal of a ring R is generated by an idempotent element in R. Rickart rings introduced by A. Hattori[2] and then studied by many authors. A ring R is Rickart if the right annihilator i of any single element of R is generated by an idempotent element of R. G.F. Birkenmeier, J. Y. Kim, J.K. Park, introduced a p.q.-Baer ring R as a generalization of a quasi-Baer ring[6]. A ring R is said to be p.q.-Baer if the right annihilator of every principle right ideal of a ring R is generated as an R-module by an idempotent element.

C.S. Roman in (2004) introduced a (quasi-)Baer modules in general module theoretic settings. A module M is (quasi-)Baer module if the right annihilator in M of every (two sided ideal) non empty subset of a ring $_SS= End_R(M)$ is generated by an idempotent element of S [4]. G. Lee in (2010) generalized Rickart rings [8] as a module M is Rickart if the right annihilator in M of any single element of S is generated by an idempotent element of S.L.Qiong, O.Bai, W.Tong in (2009) introduced a p.q.-Baer modules as a generalization of a p.q-Baer ring[14]. A module M is p.q.-Baer if the right annihilator in M of every principle right ideal of a ring S is generated by an idempotent element of S. Recently, the authers in[17] introduced the concept of strongly Rickart rings as stronger concepts of Rickart rings. A ring R is strongly Rickart if the right annihilator of each single element in R is generated by left semicentral idempotent of R.

For a ring R and a module M, recall that a module M is said to be satisfy the IFP (insertion factor property) if $r_M(\varphi)$ is a fully invariant submodule of M for each $\varphi \in S= End_R(M)$ [11]. A module M is said to be abelian if for each $f \in S$, $e^2=e \in S$, $m \in M$, fem = efm [14]. A module M is an abelian if and only if $S=End_R(M)$ is an abelian ring [14]. Follows[14] a module M is reduced if for each $m \in M$ and $f \in S$, if fm=0 implies $\mathrm{Im}\, f \cap Sm =0$. From [14] a module M is symmetric if for each $m \in M$ and $f, g \in S$, if fgm=0 implies gfm =0.

Notations: R is a ring and S is the endomorphism ring of a module M. For a ring S and $\varphi \in S$, the set $r_M(\varphi) = \{m \in M: \varphi m = 0\}$ (resp. $\ell_M(\varphi) =\{m\in M : m\varphi = 0\}$) is said to be the right (resp. left) annihilator in M of φ in S. An idempotent $e \in S$ is called left (resp. right) semicentral if fe=efe (resp. ef=efe), for all $f \in S$. An idempotent $e \in S= End_R(M)$ is called central if it commute with each $g\in S$.The sets $S_\ell(S)$, $S_r(S)$ and B(S) are the set of all left semicentral, right semicentral and central idempotent of S respectively. The samples$\leq$, $\trianglelefteq$, $\leq^{\oplus}$, $\trianglelefteq^{\oplus}$, $\leq^e$ and ◘ refer to submodule, fully invariant submodule,direct summand, fully invariant direct summand, essential submodule and end the proof.

1. Basic structure of strongly Rickart modules

In this paper we introduce the strongly Rickart modules as stronger than of Rickart modules [8] and as generalization of strongly Rickart rings [17].

Definition 1.1. A module M is said to be *strongly Rickart* if the right annihilators of each single element in $S =End_R(M)$ is generated by a left semicentral idempotent in S.

Remarks and Examples 1.2.

1. A module M is strongly Rickart if and only if $\mathrm{Ker}\varphi = r_M(\varphi)$ is fully invariant direct summand in M for any $\varphi \in S = End_R(M)$.

Proof. Since for any $e^2 =e \in S$, $eM \trianglelefteq M$ if and only if $e^2 =e \in S_\ell(S)$[7, Lemma 1.9], then the proof is obvious. ◘

2. A ring R is strongly Rickart if and only if R_R is strongly Rickart module.

3. Every strongly Rickart module is Rickart module, but the converse is not true in general. In fact, the Z-module $Z\oplus Z$ is not strongly Rickart. If $\alpha : Z\oplus Z \to Z\oplus Z$ is defined by $\alpha(a,b) = (a,0)$ then $\mathrm{Ker}\alpha = 0\oplus Z$ is not fully invariant submodule. For that: let $\beta \in End_R(Z\oplus Z)$ defined by $\beta(a, b) = (b ,a)$. $\beta(\mathrm{Ker}\alpha) = Z\oplus 0 \nleq \mathrm{Ker}\alpha$. So $Z\oplus Z$ is not strongly Rickart Z-module. But the $Z\oplus Z$ is Rickart Z-module by [8, Theorem(2.6.3)].

Furthermore there is another example shows that the converse is not true in general (see Examples 1.21(1)).

4. Baer module and strongly Rickart module are different concepts. In fact [6, Example (1.5)(i)] is a commutative regular ring which is not Baer ring. So is strongly Rickart ring which is not Baer ring, while the Z-module $Z\oplus Z$ is Baer which is not strongly Rickart module.

5. Every simple module is strongly Rickart module.

Proof. Since the endomorphism ring (say S) of every simple module (say M) is division ring, then for each $g \in S$ either kerg = R or kerg= 0 respectively, and in both case kerg is fully invariant direct summand. ◘

In the following result we show that the class of modules with IFP contains as a proper the class of strongly Rickart modules**.**

Proposition 1.3. Every strongly Rickart module satisfies the IFP.

Proof . Obvious, from definition of strongly Rickart module. ◘

Follows [11] if a module M satisfies the IFP then $S= End_R(M)$ (and hence M) is an abelian ring (module). Also, recall that $B(S) =S_\ell(S) \cap S_r(S)$. That's lead us to the following results.

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Proposition 1.4. A module M is strongly Rickart M if and only if M is an abelian and Rickart module.

Corollary 1.5. A module M is strongly Rickart module M if and only if the right annihilator of each endomorphism of M is generated by central idempotent in S.

Corollary 1.6. A module M is strongly Rickart M if and only if M is Rickart module satisfies the IFP.

It's well known that every reduced module is abelian module and every reduced module is a symmetric module. The three concepts are equivalent under Rickart module [14, Lemma 2.16, Lemma 2.18], so we have the following:

Corollary 1.7. A module M is strongly Rickart if and only if M is symmetric(and hence is reduced) and Rickart module.

One may think that in general, for f ∈S, $r_M(f) = r_M(f^2) = \ldots$,the following example shows that is not true in general.

Example 1.8. Let M = Z⊕Z_2 as Z-module. It's easy to check that S=$End_Z(M)$ = $\begin{pmatrix} Z & 0 \\ Z_2 & Z_2 \end{pmatrix}$. Let φ = $\begin{pmatrix} n & 0 \\ \bar{1} & \bar{0} \end{pmatrix}$ ∈ S such that n ≠ 0. Then $r_M(\varphi) = 2Z\oplus Z_2$ while $r_M(\varphi^2) = Z\oplus Z_2$.

The following corollary shows that in the strongly Rickart modules $r_M(f) = r_M(f^2)$.

Corollary 1.9. If M is a strongly Rickart module then for any f ∈ S= $End_R(M)$, $r_M(f) = r_M(f^2)$.

Proof. Suppose that M is a strongly Rickart module and f ∈S, then $r_M(f)$= eM for some central e^2=e ∈S (Corollary(1.5)). Firstly, it is clear that $r_M(f) \leq r_M(f^2)$. Now if x ∈ $r_M(f^2)$, then $f^2(x)$ = f (f(x)) = 0.Thus f(x) ∈ $r_M(f)$ = eM. Since e is central, and f(x) = ef(x) then f(x) = fe(x) = 0. Hence x ∈ $r_M(f)$ and so $r_M(f^2) = r_M(f)$ = eM for some central e^2 = e ∈S. ◘

Example 1.10. The concepts of Rickart modules and abelian modules are different.

1. Let Z be the ring of integers and $Mat_2(Z)$ the 2×2 full matrix ring over Z. We consider the ring. Let R = {$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ ∈ $Mat_2(Z)$| a≡d mod 2, b≡ 0 and c≡0 mod 2}. (note that you can consider M =R_R module). It is clear that 0 (zero matrix) and 1(identity matrix) are the only idempotent in R and hence R is an abelian ring. Now, let x= $\begin{pmatrix} 0 & 0 \\ -2 & 2 \end{pmatrix}$ ∈ R and y = $\begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix}$ ∈ $r_R(x)$. So xy =$\begin{pmatrix} 0 & 0 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix}$ = 0. If R is Rickart, then $r_R(x)$ = eR for some e^2 = e ∈R and hence $r_R(x)$ either 0 or R. If $r_R(x)$ = 0 then y=0 that a contradiction. If $r_M(x)$ = R, so x=0 which is a contradiction where x = $\begin{pmatrix} 0 & 0 \\ -2 & 2 \end{pmatrix}$ ≠ 0. Therefore R is not Rickart.

2. The 2×2 upper triangular matrix R = $\begin{pmatrix} Z/2Z & Z/2Z \\ 0 & Z/2Z \end{pmatrix}$ is left Rickart ring which is not abelian (where there is an idempotent a = $\begin{pmatrix} 0 & \bar{1} \\ 0 & \bar{1} \end{pmatrix}$ ∈R and b = $\begin{pmatrix} \bar{1} & \bar{1} \\ 0 & \bar{1} \end{pmatrix}$ ∈ R such that ab ≠ba. So a is not central idempotent in R.

It's known that Baer and quasi-Baer modules are p.q.-Baer modules while Rickart module and p.q.-Baer module are different concepts.

Example 1.11 [5, Example 2]

1. Let R be in example (1.10-1), then R is neither right nor left Rickart ring. Now, to show that R is a right p.q.-Baer ring, let (0≠) u = $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ ∈ R. So x = $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2a & 0 \\ 2c & 0 \end{pmatrix}$ ∈ uR. If y = $\begin{pmatrix} d & m \\ n & z \end{pmatrix}$ ∈ $r_R(uR)$, then xy = $\begin{pmatrix} 2ad & 2am \\ 2cd & 2cm \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. If a ≠ 0 or c ≠ 0, then d=0 and m=0. Also, let $w = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2a \\ 0 & 2c \end{pmatrix}$ ∈ uR. Then wy = $\begin{pmatrix} 2an & 2az \\ 2cn & 2cz \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. If a ≠ 0 or c ≠ 0, then z = 0 and n = 0. Hence if a ≠ 0 or c ≠ 0, then $y = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. If one replace $\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$ by $\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$ respectively, in x and w, by the same way, the result is y = $\begin{pmatrix} d & m \\ n & z \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Hence $r_R(uR)$ = 0 for any u ∈ R. Therefore R is p.q.-Baer ring.

2. For a ring $\prod_{n=1}^{\infty} Z_2$where Z is the ring of integers modulo 2. Let T= {$(a_n)_{n=1}^{\infty}$| a_n eventually constant} and I = ={$(a_n)_{n=1}^{\infty}$|a_n = 0 eventually}. Then R = $\begin{pmatrix} T/I & T/I \\ 0 & T \end{pmatrix}$ is left Rickart ring which is not right p.q.-Baer ring.

Its well known that if a module M satisfies the IFP then $r_M(\varphi) = r_M(\varphi S)$. From this fact and Proposition (1.3) we can show that the class of strongly Rickart modules is contained in the class of p.q.-Baer modules.

Proposition 1.12. Every strongly Rickart module is p.q–Baer module.

The converse of proposition (1.12) is not true in general see Example (1.11-1)

Proposition 1.13. A module M is strongly Rickart if and only if M is p.q.-Baer module satisfies IFP.

Remark 1.14. Although the IFP implies the abelian condition but in the previous proposition we cannot replaced the IFP by the abelian concept and we can see that in Example (1.10).

A module M is said to be strongly bounded if every nonzero submodule of M contain a nonzero fully invariant submodule [11]. By using [11, Proposition (3.3)] we have the following corollary.

Corollary 1.15. If a module M is p.q – Baer strongly bounded then M is strongly Rickart.

A submodule N of a module M is said to be stable in M if for any homomorphism $f : N \to M$, $f(N) \leq N$ [13]. Also, recall from [16] a module M is SS-module if every direct summand of M is stable. It's well known that every stable submodule is fully invariant but the converse is not true in general. Following [16] every fully invariant direct summand sunmodule is stable. The following proposition gives some properties for a module with kerφ is stable for all $\varphi \in S=End_R(M)$.

Proposition 1.16. Let M be a module and $S = End_R(M)$ then:

1. If kerφ is stable (for each $\varphi \in S$), then M is SS-module.

Proof. Let $N \leq^{\oplus} M$ and $\rho : M (= N \oplus L) \to L$ be a canonical projection map. It's clear that kerρ=N. By hypothesis, N = kerρ is stable and hence M is SS –module. ◘

2. If M has the property Kerφ is stable for all endomorphism φ of M then so is $S = End_R(M)$.

Proof. Let $\varphi \in S$. Then $r_M(\varphi) = Ker\varphi$ is stable in M. To show that $r_S(\varphi)$ is stable in S, suppose there is $f:r_S(\varphi)\to S$ such that $f(r_S(\varphi)) \nleq r_S(\varphi)$ .i.e there is $g \in r_S(\varphi)$ such that $f(g) \notin r_S(\varphi)$. Then φg=0 and so $g(m) \in r_M(\varphi)$. Since $r_M(\varphi)$ is stable in M, then $f(g(m)) \in r_M(\varphi)$. So $\varphi(f(g(m))) = 0$ for all $m \in M$. Thus φfg = 0 a contradiction. Therefore S has the property that $r_S(\varphi)$ is stable for all $\varphi \in S$.◘

3. If every endomorphism of a module M is monomorphism then M has the property kerφ is stable for all $\varphi \in S = End_R(M)$.

Proposition 1.17. A module M is strongly Rickart if and only if M is Rickart and SS-module.

Corollary 1.18. A module M is strongly Rickart if and only if M is Rickart with kerφ is a stable for each $\varphi \in S =End_R(M)$.

Corollary 1.19. A module M is strongly Rickart if and only if kerφ is a stable direct summand for each $\varphi \in S=End_R(M)$.

Following [16], every indecomposable module is SS –module, so we have the following result.

Corollary 1.20. An indecomposable module M is Rickart module if and only if M is strongly Rickart.

Examples 1.21.

1. The vector space $V = F^2$ over the field F is semisimple F-module and hence is Rickart. But V is not SS- F-module [16]. In fact, let $S= \{(\alpha, 0): \alpha \in F\}$ and $S' = \{(0, \beta): \beta \in F\}$ then S and S' are subspaces of V. It's clear S and S' are generated by (1,0) and (0,1) respectively. So dim (S) = dim (S') =1 and S∩S'= 0. That gives S + S'= S⊕ S'. Then dim (S ⊕ S') = dim (S) + dim (S') = 1+1=2. Hence S' is a direct summand which is not stable submodule where if f: S'→ V such that f (0, x) = (x, 0) for all $x \in F$. Then $f(S') \nleq S'$. Therefore V is not strongly Rickart module.

2. Q and Z are Rickart and SS –module, so they are strongly Rickart Z- modules.

3. Z_{p^∞} is SS- module which is not Rickart, so Z_{p^∞} is not strongly Rickart Z-module.

We needed to the following lemma which appears in [4, Lemma 3.1.3].

Lemma 1.22. [4, Lemma 3.1.3]. Let $N_i \trianglelefteq^{\oplus} M$ for i=1, …, n, n∈ N. Then $\cap_{i\leq n} N_i \trianglelefteq^{\oplus} M$.

Theorem 1.23. Let M be a module and $S = End_R(M)$ .Then the following statements are equivalent

1. M is strongly Rickart.

2. The right annihilator of every finitely generated left ideal I of S is generated by a left semicentral idempotent in S.

3. The right annihilator of every principle ideal of S is generated by a left semicentral idempotent in S.

4. The right annihilator of every finitely generated ideal I of S is generated by a left semicentral idempotent in S.

Proof.1⟹2) Suppose that M is a strongly Rickart module and let $I \leq_S S$ be any nonzero left ideal with a finite generators $\varphi_1, ..., \varphi_n$. whereby(1), $r_M(\varphi_i) = e_iM$ for some left semicentral idempotent $e_i \in S$, ∀ i = 1, … n, and since $r_M(I) = \cap_{i=1}^{n} e_iM$, then by (Remarks and examples 1.2(1)) and (Lemma (1.22)), there is a left semicentral idempotent $e \in S$ such that $r_M(I) = eM$.

2⟹1) Let $\mu \in S$, then by hypothesis $r_M(\mu) = r_M(S\mu) \trianglelefteq^{\oplus} M$ where Sμ is a left principal ideal in S with one generator. Hence M is a strongly Rickart module.

(1⟺3) If we have (1), then for each $\varphi \in S =End_R(M)$ from proposition (1.3), $r_M(\varphi S) = r_M(\varphi) = r_M(S\varphi) = eM$ left semicentral idempotent e in S. Conversely, also for every principle ideal, $r_M(\varphi S) = r_M(\varphi) = r_M(S\varphi)$. So by hypothesis M is strongly Rickart module.

(1⟺4) Follows from Proposition (1.3), and the equivalent (1⟺2). ◘

Corollary 1.24. A module M is strongly Rickart if and only if for each finite family $\{\varphi_i\}_{i=1}^{n}$ of endomorphism of M, $\cap_{i=1}^{n} \ker\varphi_i$ is stable direct summand in M.

Proposition 1.25. A module M is strongly Rickart if and only if for each family $\{\varphi_i\}_{i=1}^{n}$ of endomorphism of M for each homomorphism μ: $\cap_{i=1}^{n} \ker\varphi_i \to M$, kerμ is stable direct summand in M.

Proof. Since M is strongly Rickart module, then ker$\varphi_i \trianglelefteq^{\oplus}$ M for each φ_i, i ∈ I. hence $\cap_{i=1}^{n}$ kerφ $\trianglelefteq^{\oplus}$ M (Lemma 1.22). So there exist a submodule N of M such that M = $\cap_{i=1}^{n}$ker$\varphi_i \oplus$ N. Let μ: $\cap_{i=1}^{n}$ kerφ_i →M be a homomorphism. Then μ can be extended to a $\acute{\mu}$: M → M such that $\acute{\mu}$ (N) = 0 and so ker$\acute{\mu} \trianglelefteq^{\oplus}$ M where M is strongly Rickart module. We claim that ker$\acute{\mu}$ = kerμ ⊕ N. Since $\acute{\mu}|_{\ker\mu}$ = μ and $\acute{\mu}$(N) =0. Then it is clear that kerμ ⊕ N ≤ ker$\acute{\mu}$. Now, let m ∈ ker$\acute{\mu}$. So m = k+b for k ∈ $\cap_{i=1}^{n}$kerφ_i and b∈ N which implies that 0 = $\acute{\mu}$(m) = $\acute{\mu}$(k) +$\acute{\mu}$(b) =μ(k) for k∈kerμ. Then m ∈ kerμ ⊕ N and hence ker$\acute{\mu}$=kerμ ⊕ N. So kerμ $\leq^{\oplus}$ M. But M is SS-module, hence kerμ is stable direct summand in M. Conversely, let $\{\varphi_i\}_{i=1}^{n}$ be a finite family of endomorphism of M. Define μ :$\cap_{i=1}^{n} \ker\varphi_i$ → M by μ (x) = 0 for all x ∈ $\cap_{i=1}^{n} \ker\varphi_i$. So by hypotheses, Kerμ = $\cap_{i=1}^{n} \ker\varphi_i$ is stable direct summand in M, Then by Corollary (1.24) M is strongly Rickart module. ◘

Recall that a module M is said to be satisfy the SIP if the intersection of any two (and hence a finite) direct summands of M is direct summand [12]. G. Lee in [8] proved that every Rickart module satisfy the SIP. That led us to introduced a strongly concept to SIP.

Definition 1.26. A module M is said to be satisfies the *strictly SIP* if the intersection of any two direct summands of M is fully invariant direct summands. A ring R is said to be satisfies the strictly SIP if R_R satisfies the strictly SIP

The following remarks give some properties for the module with strictly SIP.

Remarks and examples1.27.

1. If a module M satisfy the strictly SIP, then M satisfies SIP. But the converse is not true in general, for example: the Z⊕Z has SIP but not strictly SIP (Remarks and examples (1.2) (3)).

2. If a module has the property that kerφ is stable for each φ ∈ S = End_R(M). Then M satisfies the strictly SIP.

3. A module M satisfies the strictly SIP if and only if M is an abelian module if and only if S = End_R(M) is an abelian ring if and only if S satisfies the strictly SIP.

Proof. It's clear from the fact every idempotent element in S is central if and only if every direct summand of M is fully invariant. ◘

4. A module M satisfies the strictly SIP if and only if M is SS- module.

Proof. Suppose that M is SS-module. Let L and N be arbitrary direct summands in M. Hence L and N are stable and so fully invariant in M. Then L∩N is fully invariant direct summand in M (Lemma 1.22). Conversely if M satisfies the strictly SIP and L $\leq^{\oplus}$ M, then L= L∩M is a fully invariant direct summand of M and so L is stable in M. ◘

5. Every direct summand of a module M satisfies the strictly SIP is satisfies the strictly SIP.

Proof. Let N $\leq^{\oplus}$ M. If A and B be summands in N, then A and B are summands in M. Since M has strictly SIP, then A∩B $\trianglelefteq^{\oplus}$ M and so A∩B $\leq^{\oplus}$ N. Consider the sequence M $\overset{\rho}{\to} A \cap B \overset{j_1}{\to}$ N $\overset{\mu}{\to}$ N $\overset{j_2}{\to}$ M, where j_1, j_2 are the canonical injection monomorphisms, ρ is the canonical projection map on A∩B and μ any endomorphism of N, we have A∩B ⊇ $j_2\mu j_1\rho$(A∩B) = μ (A∩B) So, N satisfies the strictly SIP. ◘

The following proposition prove that the class of strongly Rickart modules contains in the class of strictly SIP.

Proposition 1.28. Every strongly Rickart module M satisfies the strictly SIP. The converse is true if M is Rickart module.

Proof. The first statement follows from Proposition (1.4) and Remarks and examples (1.27-3). For the converse, if φ ∈ S then from Rickart property kerφ is a direct summand in M. But M has strictly SIP, so by Remarks and examples (1.27-4) kerφ $\trianglelefteq^{\oplus}$ M.◘

Now we can summarize the previous results in the following proposition

Proposition 1.29. Let M be a module and S = End $_R$(M). Then the following statements are equivalent.

1. M is a strongly Rickart.

2. M is a Rickart with Kerφ is stable for all φ ∈ S.

3. M is a Rickart satisfies the strictly SIP.

4. M is a Rickart and S is abelian

5. M is a Rickart and M is abelian.

6. M is a Rickart and M is symmetric.

7. M is a Rickart and M is SS -module.

8. M is a Rickart and M is reduced.

9. M is p.q.-Baer satisfies the IFP.

A submodule of strongly Rickart module needed not strongly Rickart in general. In fact the Z-module Q⊕Z_2 is strongly Rickart Z-module while the submodule N = Z⊕Z_2 is not, where from Example (1.8), there is φ ∈ End_R(N) such that r_N(φ) = 2Z⊕Z_2 is not direct summand in Z⊕Z_2. On the other hand if a module M contain a strongly Rickart submodule that's not mean the strongly Rickart property valid for M and we can see that in the Z-module Z_4 which is not strongly Rickart while the submodule $2Z_4 \cong Z_2$ is strongly Rickart.

The following results give us under which condition the submodule of strongly Rickart module is strongly Rickart

Proposition 1.30. Let M be a strongly Rickart module and $N \trianglelefteq M$. If every endomorphism $f \in End_R(N)$ can be extended to an endomorphism $g \in End_R(M)$, then N is a strongly Rickart module.

Proof. Suppose that M is a strongly Rickart module and $N \leq M$. If $f \in End_R(N)$,then by hypothesis there is $g \in End_R(M)$ such that $g|_N = f$. Since M is strongly Rickart module then there is a left semicentral idempotent $e \in End_R(M)$ such that Kerg = eM. So g(eN) = 0 where g(eM) = 0. Since $eN \leq N$ then $eN \leq N \cap \ker g = \ker f$, hence $\ker f = eN \trianglelefteq^{\oplus} N$ and so N is strongly Rickart submodule. ◘

Corollary 1.31. For any quasi–injective module M, if the injective hull of M is strongly Rickart, then so is M.

Recall that a module M is FI-quasi-injective if for each $N \trianglelefteq M$ and any homomorphism $f: N \to M$ can be extended to an endomorphism $g: M \to M$ [16]. We have the following result.

Corollary 1.32. Every fully invariant submodule of strongly Rickart FI–quasi –injective module is strongly Rickart.

As in a Baer, quasi-Baer, p.q.-Baer and Rickart module, the property of strongly Rickart is inherit by the direct summand.

Proposition 1.33. A direct summand of strongly Rickart module is strongly Rickart.

Proof. Let N be a direct summand of strongly Rickart module M. From [8, Theorem 2.1.6] N is Rickart submodule. Now for each $f \in End(N)$, $\ker f \leq^{\oplus} N \leq^{\oplus} M$. But M is strongly Rickart, so, kerf is fully invariant direct summand in M. Now, consider the sequence $M \overset{\rho}{\to} \ker f \overset{j_1}{\to} N \overset{g}{\to} N \overset{j_2}{\to} M$, $\ker f \supseteq j_2 g j_1 \rho(\ker f) = g(\ker f)$ where j_1, j_2 are the canonical injection map, ρ is the canonical projective map on kerf and g any endomorphism of N. So, N is strongly Rickart subodule. ◘

Unlike direct summand, a direct sum of strongly Rickart modules needed not be strongly Rickart for a prime number P. In fact the Z-module $Z \oplus Z_p$ is not strongly Rickart Z-modules although Z and Z_p is not strongly Rickart Z-module.

We need to the following result which appears in [1,Lemma 1.9].

Lemma 1.34. Let a module $M = M_1 \oplus M_2$ be a direct sum of submodules M_1 and M_2. Then M_1 is fully invariant submodule of M if and only if $Hom_R(M_1,M_2) = 0$.

Proposition 1.35. A module $M = M_1 \oplus M_2$ is strongly Rickart module if and only if the following conditions hold

1. M_i is strongly Rickart for each $i \in \{1, 2\}$.

2. $Hom_R(M_i, M_j) = 0$ for each $i \neq j$.

Proof. The conditions (1) and (2) holds from Proposition (1.33), Proposition (1.17) and Lemma (1.34). Conversely, suppose that $S_i = End_R(M_i)$ for i = 1,2 and $S = End_R(M)$. Then $S = \begin{pmatrix} Hom(M_1, M_1) & Hom(M_2, M_1) \\ Hom(M_1, M_2) & Hom(M_2, M_2) \end{pmatrix}$. Since $Hom_R(M_i, M_j) = 0$ for all $i \neq j$, so $S = \begin{pmatrix} s_1 & 0 \\ 0 & s_2 \end{pmatrix}$. Let $g = \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix} \in S$ where $g_i \in S_i$ for i=1,2. Then from (1), $r_{M_1}(g_1) = e_1M_1$ and $r_{M_2}(g_2) = e_2M_2$ for some $e_i^2 = e_i \in S_\ell(S_i)$ for i=1, 2. Now, $\begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix}^2 = \begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix} \in S$, and $xe = \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix}\begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix} = \begin{pmatrix} x_1e_1 & 0 \\ 0 & x_2e_2 \end{pmatrix} = \begin{pmatrix} e_1x_1e_1 & 0 \\ 0 & e_2x_2e_2 \end{pmatrix} = \begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix}\begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix}\begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix} = exe$ where e_1 and e_2 are left semicentral idempotent. We claim that $r_M(g) = eM$. For that let $m = \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} \in r_M(g)$ for $m_i \in r_{M_i}(g_i) = e_iM_i$ for i = 1,2. So $m_i = e_im_i$ for i = 1,2. Hence $m = \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = \begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix}\begin{pmatrix} m_1 \\ m_2 \end{pmatrix} \in eM$. Thus $r_M(g) \leq eM$. Now, since $ge = \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix}\begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and if $x = em \in eM$ then gx = gem = 0. Hence $x \in r_M(g)$. Thus $eM \leq r_M(g)$. Therefore $r_M(g) = eM$. ◘

Example 1.36. Consider the modules $M_1 = Q$ and $M_2 = Z_p$ where P is a prime number. Each of Q and Z_p are strongly Rickart module. Now, $Hom_Z(Q, Z_p) = 0$ and $Hom_Z(Z_P, Q) = 0$ then $M = Q \oplus Z_p$ is strongly Rickart module. While if $M_1 = Z$ and $M_2 = Z_p$ then it's well known that $Hom_Z(Z, Z_p) \neq 0$ where p is a prime number. Then $Z \oplus Z_p$ as Z-module cannot be strongly Rickart module even though Z and Z_p are strongly Rickart Z-modules.

2. Endomorphism ring of strongly Rickart modules

In this section we investigate some properties of endomorphism ring of strongly Rickart modules. From [17], a strongly Rickart ring is left-right symmetric as well as known in Baer and quasi-Baer ring while Rickart and p.q.-Baer ring is not. Also, it's known that the endomorphism ring of a Baer, quasi-Baer and Rickart rings is Baer, quasi-Baer [4] and Rickart [8] respectively. The same result is available for strongly Rickart modules.

Proposition 2.1. The endomorphism ring of strongly Rickart module is strongly Rickart.

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Proof. Let M be a strongly Rickart Module with S = $End_R(M)$. Let φ ∈ S, so by hypothesis r_M (φ) = eM for some left semicentral idempotent e ∈ S. We claim that r_S(φ) = eS. For that 0 = φe(M).Thus e ∈ r_S(φ). Now, let (0 ≠) α ∈ r_S(φ). Hence α(M) ≤ r_M(φ) = eM. So α = eα ∈ eS. Therefore r_S(φ)= eS for some left semicentral idempotent e ∈S. ◘

Remark 2.2.The converse of the previous proposition is not true in general. In fact, the Z-module Z_{p^∞} is not strongly Rickart while it's well known that it's endomorphism ring S= $End_z (Z_{p^\infty})$ is commutative domain and so it is strongly Rickart ring.

Recall that a module M is said to be retractable if for all (0≠) N ≤ M, there is (0≠) φ ∈ S= $End_R(M)$ with Imφ ≤ N [4].

The following propositions give the necessary condition under which the converse of proposition (2.1) is true.

Proposition 2.3. Let M be a retractable module .Then M is strongly Rickart module if and only if S = $End_R(M)$ is strongly Rickart ring.

Proof . The necessary condition follows from Proposition (2.1). For the sufficient condition, let φ ∈ S. Since S is strongly Rickart ring, then r_S(φ) = eS for some e^2 = e ∈ $S_\ell(S)$. Hence eM ≤ r_M(φ) (where φe = 0). Now, if (0 ≠) m ∈ r_M(φ) and m ∉ eM, then φ(m) = 0. Since M = eM⊕(1-e) M. Then (0≠) m ∈ (1-e)M. But M is a retractable module and mR ≤ M. So there is (0≠ μ) ∈ S such that Imμ ≤ mR ≤ (1-e)M. Then Imμ ≤ Im(1-e) and hence μ ∈ (1-e)S. Also since Imμ ≤ mR, then we have φμ(M) ≤ φ(mR)= 0. Thus μ ∈ r_S(φ) = eS. That is μ ∈ eS ∩ (1-e)S = 0. Thus μ=0 which is a contradiction where μ ≠ 0. Hence r_M(φ) = eM for some e^2 = e ∈ $S_\ell(S)$. Therefore M is strongly Rickart. ◘

The following result is stronger than that in [8, lemma(2.3.4)].

proposition 2.4. If M is a strongly Rickart module, then every nonzero left annihilator I in S contains a nonzero central idempotent.

Proof. Suppose that M is strongly Rickart module and I = ℓ_S(A) ≠ 0 for some proper nonempty subset A of M. From [8, Lemma 2.3.4], we have every nonzero left annihilator (say I) in S conations a nonzero idempotent element. But M is an abelian (Proposition 1.4) then so is S. Hence I contain a nonzero central idempotent.◘

Proposition 2.5. Let M be a module and S = $End_R(M)$ have no infinite set of nonzero orthogonal idempotents. If M is strongly Rickart module then its endomorphism can decomposed into a finite direct product of domains.

Proof. Since M is strongly Rickart module then by Proposition (2.1), S is strongly Rickart ring and so S is a reduced Rickart ring (Corollary 1.7) .By hypothesis S have no infinite set of nonzero orthogonal idempotents, and hence from [10, Proposition 3] S can be decomposed into a finite direct product of domains. ◘

A ring R is said to be Von Neumann regular if for each a∈R there is b ∈ R such that aba = a [15]. A ring R is said to be strongly regular if a^2b = a for each a∈R and some b ∈ R[15]. Its well known that a module M has C_2-condition if every submodule N of M isomorphic to a direct summand L of M is direct summand of M.

Proposition 2.6. A module M is strongly Rickart satisfying the C_2- condition if and only if S= $End_R(M)$ is Von Neumann regular ring satisfies the strictly SIP.

Proof. ⇐) Since S is Von Neumann regular ring then M is a right Rickart R-module with C_2-condition [8, Theorem (2.2.20)]. But S satisfies the strictly SIP so is abelian and hence M is an abelian (Remarks and examples(1.27-3)). Then from Proposition (1.4) M is strongly Rickart.

⇒) From [8, Corollary (2.2.20)] S is a regular ring. But M is strongly Rickart module then M is an abelian (Proposition 1.4). Hence S satisfy strictly SIP (Remarks and examples (1.27-3). ◘

Theorem 2.7. The following conditions are equivalent:

1. M is a strongly Rickart with C_2-condition.
2. S is a strongly regular ring.
3. S is a Von Neumann regular satisfies the strictly SIP.
4. kerφ and Imφ is fully invariant direct summand in M for all φ ∈S.

Proof. (1⟺3) Following Proposition 2.6.

(2⟺3) Since the strictly SIP and abelian property are equivalent then by [15, 3.11, p.21] the proof is complete.

(3⟺4) Suppose (3) then by [8, Theorem 2.2.20] kerφ and Imφ is direct summands in M for all φ ∈ S. So by Remarks and examples (1.27-4) we have kerφ and Imφ is fully invariant in M for all φ ∈ S. Now suppose (4) hold, so by [8, Theorem 2.2.20], S is a Von Neumann regular. Since kerφ is a stable submodule in M then by Remarks and examples(1.27-2), M satisfies the strictly SIP ◘

3. Strongly CS-Rickart modules

Recall that a module M is strongly extending if every submodule of M essential in a stable direct summand of M[16]. From [3], a module M is CS-Rickart if for any φ∈S, r_M(φ) essential in a direct summand of M. Recall that a module M is nonsingular if the singular submodule Z(M) = { m∈ M| $r_R(m) \leq^e R$} = 0. A module M is said to be K–nonsingular if for each φ

$\in$S, kerφ essential in M then $\varphi = 0$ [4]. Also, recall that a module M is multiplication if for each submodule of M is of the form IM for some ideal I of R.

In this section we introduce and study strongly CS-Rickart module as a stronger than CS-Rickart module and a generalization of strongly Rickart modules.

Definition 3.3. A module M is said to be *strongly CS- Rickart* if for any $\varphi \in S$, $r_M(\varphi)$ is an essential in fully invariant direct summand.

Remarks and examples 3.4.

1. Every strongly Rickart module is a strongly CS-Rickart.

Proof. Since $r_M(\varphi) \leq^e r_M(\varphi)$, so if M is a strongly Rickart module we have $r_M(\varphi) \leq^e r_M(\varphi) \trianglelefteq^{\oplus} M$ for all $\varphi \in S$. ◘

2. The converse of (1) is not true in general. In fact, the Z-module Z_{p^n}(p is prime number and n ≥1 be an integer) is strongly CS-Rickart : for that $\alpha : Z_{p^n} \to Z_{p^n}$ defined by $\varphi(\bar{x}) = \bar{x}p$ for each $\bar{x} \in Z_{p^n}$. Then $0 \neq \ker\alpha = Z_{p^{n-1}} \leq^e$ M and hence is not direct summand. So M is strongly CS-Rickart which is not strongly Rickart module. In particular, the Z-module Z_4 is strongly CS-Rickart as Z-module which is not strongly Rickart module.

3. Every uniform module M is strongly CS-Rickart.

4. The Z-module Z_6 is strongly CS-Rickart module which is not uniform Z-module.

5. Every strongly extending module is strongly CS-Rickart. The converse is true when every submodule of M is a right annihilator of some finitely generated left ideal of S =$End_R(M)$.

Proof. The first statement is clear. Conversely, let $N \leq M$, then $N = r_M(I)$ where I is a finitely generated ideal in S and generated by$\{\varphi_1,\varphi_2,\ldots,\varphi_n\}$for some φ_i ,1≤ i ≤n. Since M is strongly CS-Rickart module then N= $\bigcap_{i=1}^{n} r_M(\varphi_i) \leq^e \bigcap_{i=1}^{n} e_i M$ for $e_i^2=e_i \in S_\ell(S)$ [15, 17.3-4, p.138]. So, there is $e^2 = e \in S_\ell(S)$ such that $\bigcap_{i=1}^{n} e_i M$= eM and hence $N \leq^e eM \trianglelefteq^{\oplus} M$. Then M is strongly extending module. ◘

6. Every multiplication CS-Rickart is strongly CS-Rickart.

Proof. Let $\varphi \in S = End_R(M)$, and M be a multiplication CS-Rickart module. Then $r_M(\varphi) \leq^e B \leq^{\oplus} M$. Now, let $\mu \in S$. Since M is multiplication then B=IM for some ideal I in R. Hence $\mu(B) = \mu(IM) = I\mu(M) \leq IM = B$. Therefore M is strongly CS-Rickart module. ◘

7. In particular of (6) every cyclic CS-Rickart over commutative ring is strongly CS-Rickart. Moreover, a commutative ring R is strongly CS-Rickart if and only if CS-Rickart.

Proposition 3.5 A module M is strongly CS-Rickart if and only if M is CS-Rickart satisfies the strictly SIP (and hence SS-module).

Proof. ⟹) It's clear that if a module M is strongly CS-Rickart then M is CS-Rickart module. Now, let N = eM and L= fM be summands of M for some $e^2 = e$ and $f^2 = f \in S = End_R(M)$. Since $r_M(1-e)$ = eM and $r_M(1-f)$ = fM. Then by strongly CS-Rickart property, $r_M(1-e) \leq^e B \trianglelefteq^{\oplus} M$. But eM is closed in M, then N = eM= $r_M(1-e) = B \trianglelefteq^{\oplus} M$. In the same way L= fM = $r_M(1-f) \trianglelefteq^{\oplus} M$. So by Lemma (1.22), $eM \cap fM \trianglelefteq^{\oplus} M$.

⟸)Suppose that M is CS-Rickart module, so $r_M(\varphi) \leq^e B \leq^{\oplus} M$ for each $\varphi \in S$. Now, by strictly SIP we have $r_M(\varphi) \leq^e B = B\cap M \trianglelefteq^{\oplus} M$. So M is strongly CS-Rickart module. ◘

Corollary 3.6. A module M is strongly CS-Rickart if and only if M is CS-Rickart and abelian module.

Proposition 3.7. Every direct summand of a strongly CS-Rickart module M is strongly CS-Rickart.

Proof. Let N be a direct summand of strongly CS-Rickart module M and $\varphi \in End_R(N)$. Then there is $\varphi^* \in End_R(M)$ such that $\varphi^* = \varphi \oplus 0$. But M is strongly CS-Rickart module. Then $r_M(\varphi^*) \leq^e B \trianglelefteq^{\oplus} M$. Now $r_M(\varphi^*) = r_M(\varphi) \cap r_M(0) = r_M(\varphi) \cap M = r_M(\varphi) \leq^e B \trianglelefteq^{\oplus} M$. But $r_M(\varphi) \leq N \leq^{\oplus} M$, then $r_M(\varphi) \leq B\cap N \trianglelefteq^{\oplus} M$, where M is strictly SIP by (5). So $B\cap N \leq^{\oplus} N$. Now, $r_M(\varphi) \leq^e B\cap N \leq^{\oplus} N \leq^{\oplus} M$. By Remark and examples (3.4-5) and (Remarks and examples(1.27-5)), then $B\cap N \trianglelefteq^{\oplus} N$. Therefore N is strongly CS-Rickart submodule. ◘

Proposition 3.8. The following statements are equivalent for a module M.

1. M is strongly CS-Rickart module.

2.The right annihilator in M of any finitely generated left ideal $I = \langle\varphi_1,\varphi_2,\ldots,\varphi_n\rangle$ of S is essential in fully invariant direct summand.

Proof. 1⟹2) Suppose that $I = \langle\varphi_1,\varphi_2,\ldots,\varphi_n\rangle$ be a finite generated ideal in S. Then $r_M(I) = \bigcap_{i=1}^{n} r_M(\varphi_i)$ whereby (1) each of $r_M(\varphi_i) \leq^e B_i = e_i M \trianglelefteq^{\oplus} M$(i.e $e_i \in S_\ell(S)$). Hence from [15, 17.3-4, p.138] $r_M(I) = \bigcap_{i=1}^{n} r_M(\varphi_i) \leq^e \bigcap_{i=1}^{n} e_i M$. Now since $e_i \in S_\ell(S)$, then there is $e \in S_\ell(S)$ such that $\bigcap_{i=1}^{n} e_i M$ = eM (Lemma (1.22). So $r_M(I) = \bigcap_{i=1}^{n} r_M(\varphi_i) \leq^e eM \trianglelefteq^{\oplus} M$.

2⟹1) Let $\varphi \in S$, since $r_M(\varphi) = r_M(S\varphi)$, then from (2) the proof is complete. ◘

We have now reached to give the basic conclusion in this section

Theorem 3.9. A module M is a strongly Rickart if and only if M is K-nonsigular strongly CS-Rickart module.

Proof. ⟹) Since every strongly Rickart is a Rickart module, then from [8, Proposition 2.1.12] M is K– nonsingular and from Remarks and examples (3.4-1) M is strongly CS-Rickart module.

⟸) By [3, Lemma 3], $r_M(\varphi) \leq^{\oplus} M$ for all $\varphi \in S$. Hence $r_M(\varphi)$ is closed in M. But $r_M(\varphi) \leq^{e} B \trianglelefteq^{\oplus} M$. So $r_M(\varphi) = B$. That gives $r_M(\varphi) \trianglelefteq^{\oplus} M$. Hence M is strongly Rickart module. ◘

It's well known that every nonsingular module is K– nonsingular but the converse is not true in general. In fact the Z-module Z_p is K–nonsingular which is not nonsingular for each prime number p. But when $M= R_R$, then nonsingular and K–nonsingular concepts are coincide.

Corollary 3.10. A ring R_R is a strongly Rickart if and only if R_R is nonsingular strongly CS-Rickart.

Corollary3.11. A module M is strongly Rickart if M is strongly extending K–nonsingular module.

Example 3.2. The Z-module $M = Z_{p^\infty}$ is strongly extending module where M is uniform module [16]. But M is not K–nonsingular, where if $\alpha(\bar{x}) = \bar{x}p$ for each $(0\neq)\ \bar{x} \in M$ (clear that α is an epimorphism) then $(0 \neq) \ker\alpha \leq^{e} M$. Hence $\ker\alpha \ntrianglelefteq^{\oplus} M$. Therefore M is not strongly Rickart module. ◘

We can summarize the previous results by the following proposition

Proposition 3.11. For a module M the following statements are equivalent

1. M is a strongly Rickart
2. M is K-nonsigular strongly CS-Rickart module.
3. M is K-nonsigular CS-Rickart module satisfies the strictly SIP.

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