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Schwarschild metric in six dimensions - a topological study

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Abstract: In this article we introduce some types of the deformtion retracts of the 6D Schwarzchild making use of Lagrangian equations. The retraction of this space into itself and into geodesics has been presented. The relation between folding and deformation retract of this space has been achieved. A relation for energy conservation similar to the one obtained in four dimensions has been obtained for the six dimensional case.

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1 Introduction

The real revolution in mathematical physics in the second half of twentieth century (and in pure mathematics itself) was algebraic topology and algebraic geometry [1]. In the nineteenth century, Mathematical physics was essentially the classical theory of ordinary and partial differential equations. The variational calculus, as a basic tool for physicists in theoretical mechanics, was seen with great reservation by mathematicians until Hilbert set up its rigorous foundation by pushing forward functional analysis. This marked the transition into the first half of twentieth century, where under the influence of quantum mechanics and relativity, mathematical physics turned mainly into functional analysis, complemented by the theory of Lie groups and by tensor analysis. All branches of theoretical physics still can expect the strongest impacts from use of the unprecedented wealth of results of algebraic topology and algebraic geometry of the second half of twentieth century [1].

Today, the concepts and methods of topology and geometry have become an indispensable part of theoretical physics. They have led to a deeper understanding of many crucial aspects in condensed matter physics, cosmology, gravity, and particle physics. Moreover, several intriguing connections between only apparently disconnected phenomena have been revealed based on these mathematical tools [2].

Topology enters General Relativity through the fundamental assumption that spacetime exists and is organized as a manifold. This means that spacetime has a well-defined dimension, but it also carries with it the inherent possibility of modified patterns of global connectivity, such as distinguish a sphere from a plane, or a torus from a surface of higher genus. Such modifications can be present in the spatial topology without affecting the time direction, but they can also have a genuinely spacetime character in which case the spatial topology changes with time [4]. The topology change in classical general relativity has been discussed in [7]. See [9] for some applications of differential topology in general relativity.

In general relativity, boundaries that are S^1-bundles over some compact manifolds arise in gravitational thermodynamics [19]. The trivial bundle $\sum = S^1 \times S^2$ is a classic example. Manifolds with complete Ricci-flat metrics admitting such boundaries are known; they are the Euclideanised Schwarzschild metric and the flat metric with periodic identification. York [21] shows that there are in general two or no Schwarzschild solutions depending on whether the squashing (the ratio of the radius of the S^1-fibre to that of the S^2-base) is below or above a critical value. York's results in 4-dimension extend readily to higher dimensions.

The simplest example of non-trivial bundles arises in quantum cosmology in which the boundary is a compact S^3, i.e., a non-trivial S^1 bundle over S^2. In the case of zero cosmological constant, regular 4-metrics admitting such an S^3 boundary are the Taub-Nut [22] and Taub-Bolt [23] metrics having zero and two-dimensional (regular) fixed point sets of the $U(1)$ action respectively.

1.1 Deformation Retract – Definitions

The theory of deformation retract is very interesting topic in Euclidean and non-Euclidean spaces. It has been investigated from different points of view in many branches of topology and differential geometry. A retraction is a continuous mapping from the entire space into a subspace which preserves the position of all points in that subspace [8].

(i) Let M and N be two smooth manifolds of dimensions m and n respectively. A map $f : M \to N$ is said to be an isometric folding of M into N if and only if for every piecewise geodesic path $\gamma : J \to M$, the induced path $fo\gamma : J \to N$ is a piecewise geodesic and of the same length as γ [3]. If f does not preserve the lengths, it is called topological folding. Many types of foldings are discussed in [11, 12, 13, 14, 15, 16]. Some applications are discussed in [6, 10].

(ii) A subset A of a topological space X is called a retract of X, if there exists a continuous map $r : X \to A$ such that [17]

(a) X is open

(b) $r(a) = a$, $\forall a \in A$.

(iii) A subset A of a topological space X is said to be a deformation retract if there exists a retraction $r : X \to A$, and a homotopy $f : X \times I \to X$ such that [17]

$$f(x,0) = x, \ \forall x \in X,$$

$$f(x,1) = r(x), \ \forall r \in X,$$

$$f(a,t) = a, \ \forall a \in A, t \in [0,1].$$

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The deformation retract is a particular case of homotopy equivalence, two spaces are homotopy equivalent if and only if they are both deformation retracts of a single larger space.

Deformation retracts of Stein spaces has been studied in [5] .The deformation retract of the 4D Schwarzchild metric has been discussed in [25] where it was found that the retraction of the Schwarzchild space is spacetime geodesic. The 5 dimensional case has been discussed in [18]. in this paper we are going to discuss the retraction for the six dimensional case.

1.2 Schwarzchild metric in 6 dimensions

For the Schwarzchild metric in $(n+1)$ dimensions we can write [19]

$$ds^2 = -\left(1-\frac{\mu}{r^{n-2}}\right)dt^2 + \left(1-\frac{\mu}{r^{n-2}}\right)^{-1} dr^2 + r^2 d\Omega_{n-1}^2 \tag{1}$$

where μ gives the black hole mass m which for $M_{n-1} \equiv S^{n-1}$ is [20]

$$\mu = \frac{16\pi G m}{(n-1)Vol(S^{n-1})}. \tag{2}$$

The bolt singularity at $r^{n-2} = \mu$ can be removed by periodically identifying the coordinate t with a period

$$\beta_\tau = \frac{4\pi}{n-2}\mu^{\frac{1}{n-2}}. \tag{3}$$

the coordinates r then takes values from $\mu^{\frac{1}{n-3}}$ to infinity and defines a complete metric over a manifold with $R^2 \times M_{n-1}$ tolpology possessing an $(n-2)$-dimensional fixed point set of the Killing vector d/dt, i.e., a bolt. For $M_{n-1} = S^{n-1}$ the metric is asymptotically Euclidean [19]. $d\Omega_{n-1}^2$ represents all the angular parts of an S_{n-1} sphere and defined as

$$d\Omega_{n-1}^2 = d\chi_2^2 + \sin^2\chi_2 d\chi_3^2 + \ldots + \prod_{m=2}^{n-1}\sin^2\chi_m d\chi_n^2 \tag{4}$$

in arbitrary dimensions for a boundary $\Sigma \equiv S^1 \times M_{n-1}$, the ratio of the two radii (of the bundle and the base) as a function of $\rho \equiv r/\mu^{\frac{1}{n-2}} \in [1,\infty)$ is:

$$\frac{\beta^2}{\alpha^2} = \frac{16}{n-2}\rho^{-2}\left(1-\frac{1}{\rho^{n-2}}\right). \tag{5}$$

This ratio is called squashing, it generally starts from zeroand grows monotonically to a a maximum value and then decreases and approaches zero at infinity. The exact shape of the curve of $\frac{\beta^2}{\alpha^2}$ against ρ depends on dimension. In six dimensions $n=5$ and we have:

$$ds^2 = -\left(1-\frac{\mu}{r^3}\right)\frac{16}{9}\pi^2\mu^{\frac{2}{3}}d\tau^2 + \left(1-\frac{\mu}{r^3}\right)^{-1} dr^2 + r^2 d\Omega_4^2 \tag{6}$$

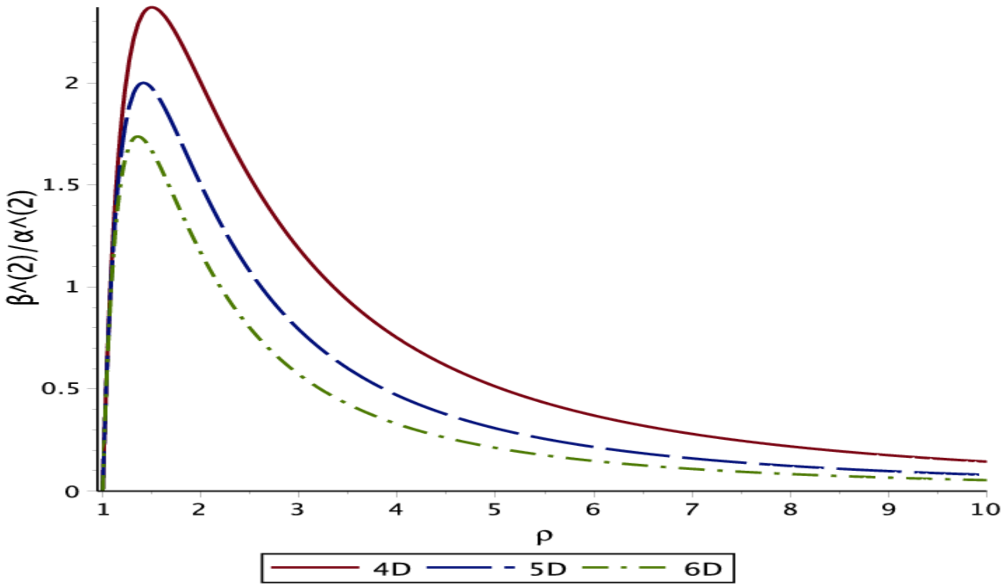

Figure 1: The ratio $\frac{\beta^2}{\alpha^2}$ against ρ for 4,5 and 6 dimensions.

Where

$$d\Omega_4^2 = d\theta^2 + \sin^2\theta d\phi^2 + \sin^2\theta \sin^2\phi dl^2 + \sin^2\theta \sin^2\phi \sin^2 l d\eta^2 \quad (7)$$

The coordinate t has been identified with a period $\frac{4}{3}\pi\mu^{\frac{1}{3}}\tau$ to remove the bolt singularity at $r^3 = \mu$. The squashing

$$\frac{\beta^2}{\alpha^2} = \frac{16}{3}\left(\frac{1}{\rho^2} - \frac{1}{\rho^5}\right) \quad (8)$$

increases monotonically from zero and approaches unity as $\rho \to \infty$ (Figure 1).

A $6D$ flat metric can be written as

$$ds^2 = -dx_o^2 + \sum_{i=1}^{5} dx_i^2 \quad (9)$$

So the coordinates of the $6D$ Schwarzchild space can be written as

$$x_o = \pm \frac{4\pi}{3}\sqrt{\left(1 - \frac{\mu}{r^3}\right)\mu^{\frac{2}{3}}\tau^2 + C_o} \quad (10)$$

$$x_1 = \pm\left(\frac{1}{2}r^2 + \frac{1}{3}\mu^{\frac{2}{3}} + \frac{1}{3}\mu^{\frac{1}{3}}\ln(1-\mu^{\frac{1}{3}})r - \frac{1}{3}\ln(1-\mu^{\frac{1}{3}})\mu^{\frac{2}{3}} - \frac{1}{6}(\mu^{\frac{1}{3}}r + \frac{1}{2}\mu^{\frac{2}{3}})\times\right.$$

$$\ln(r^2 + r\mu^{\frac{1}{3}} + \mu^{\frac{2}{3}}) - \frac{1}{\sqrt{3}}(\mu^{\frac{2}{3}} + r\mu^{\frac{1}{3}})\tanh^{-1}\left(\frac{2r+\mu^{\frac{1}{3}}}{\sqrt{3}\mu^{\frac{1}{3}}}\right) + \frac{1}{4}\mu^{\frac{2}{3}}\ln\left(\frac{4(r^2 + r\mu^{\frac{1}{3}} + \mu^{\frac{2}{3}})}{3\mu^{\frac{2}{3}}}\right) + C_1\Bigg)^{\frac{1}{2}} \quad (11)$$

$$x_2 = \pm\sqrt{r^2\theta^2 + C_2} \quad (12)$$

$$x_3 = \pm\sqrt{r^2\phi^2 \sin^2\theta + C_3} \quad (13)$$

$$x_4 = \pm\sqrt{r^2 l^2 \sin^2\theta \sin^2\phi + C_4} \quad (14)$$

$$x_5 = \pm\sqrt{r^2\eta^2 \sin^2\theta \sin^2\phi \sin^2 l + C_5} \quad (15)$$

where C_o, C_1, C_2, C_3, C_4 and C_5 are constants of integration.

2 Euler-Lagrange equations of $6D$ Schwarzchild field

In general relativity, the geodesic equation is equivalent to the Euler-Lagrange equations

$$\frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{x}^\alpha}\right) - \frac{\partial L}{\partial x^\alpha} = 0, \quad i = 1,2,3,4 \quad (16)$$

associated to the Lagrangian

$$L(x^\mu, \dot{x}^\mu) = \frac{1}{2} g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu \quad (17)$$

To find a geodesic which is a subset of the 6D Schwarzchild space, the Lagrangian of the $6D$ Schwarzchild field can be written as

$$L = -\frac{16}{9}\pi^2\left(1-\frac{\mu}{r^3}\right)\mu^{\frac{2}{3}}\dot{\tau}^2 + \left(1-\frac{\mu}{r^3}\right)^{-1}\dot{r}^2 + r^2\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta + \dot{l}^2\sin^2\theta\sin^2\phi + \dot{\eta}^2\sin^2\theta\sin^2\phi\sin^2 l\right) \quad (18)$$

No explicit dependence on either τ or η, and thus $\frac{\partial L}{\partial \dot{\tau}}$ and $\frac{\partial L}{\partial \dot{\eta}}$ are constants of motion, i.e.

$$\left(1-\frac{\mu}{r^3}\right)\pi\mu^{\frac{1}{3}}\dot{\tau} = k, \qquad r^2(\sin^2\theta\sin^2\phi\sin^2 l)\dot{\eta} = h \quad (19)$$

with k and h are constants. h can be regarded as an equivalent of angular momentum per unit mass in the $6D$. using the Euler-Lagrange equations we get the full set of components as

$$\frac{d}{d\lambda}\left(\frac{\dot{r}}{1-\frac{\mu}{r^3}}\right) + \left[\frac{8\mu^{\frac{5}{3}}}{3r^4}\pi^2\dot{\tau}^2 + \frac{3\mu\dot{r}^2}{2r^4\left(1-\frac{\mu}{r^3}\right)^2} - \right.$$

(20)

$$r\left(\dot{\theta}^2+\dot{\phi}^2\sin^2\theta+\dot{l}^2\sin^2\theta\sin^2\phi+\dot{\eta}^2\sin^2\theta\sin^2\phi\sin^2 l\right)\Big]=0$$

$$\frac{d}{d\lambda}(2r^2\dot{\theta})-r^2\sin 2\theta(\dot{\phi}^2+\dot{l}^2\sin^2\phi+\dot{\eta}^2\sin^2\phi\sin^2 l)=0 \qquad (21)$$

$$\frac{d}{d\lambda}(2r^2\dot{\phi}\sin^2\theta)-r^2\sin^2\theta(\dot{l}^2\sin 2\phi+\dot{\eta}^2\sin^2 l\sin 2\phi)=0 \qquad (22)$$

$$\frac{d}{d\lambda}(2r^2\dot{l}\sin^2\theta\sin^2\phi)-r^2\dot{\eta}^2\sin^2\theta\sin^2\phi\sin 2l=0 \qquad (23)$$

$$\frac{d}{d\lambda}\left(1-\frac{\mu}{r^3}\right)\pi\mu^{\frac{1}{3}}\dot{\tau}=0 \qquad (24)$$

$$\frac{d}{d\lambda}r^2\dot{\eta}\sin^2\theta\sin^2\phi\sin^2 l=0. \qquad (25)$$

From $\left(1-\frac{\mu}{r^3}\right)\pi\mu^{\frac{1}{3}}\dot{\tau}=k$, setting $k=0$ gives two cases: (1) $\dot{\tau}=0$ or $\tau=A$. If $A=0$ we get the following coordinates

$$x_o=\pm\sqrt{C_o} \qquad (26)$$

$$x_1=\pm\left(\frac{1}{2}r^2+\frac{1}{3}\mu^{\frac{2}{3}}+\frac{1}{3}\mu^{\frac{1}{3}}\ln(1-\mu^{\frac{1}{3}})r-\frac{1}{3}\ln(1-\mu^{\frac{1}{3}})\mu^{\frac{2}{3}}-\frac{1}{6}(\mu^{\frac{1}{3}}r+\frac{1}{2}\mu^{\frac{2}{3}})\times\right.$$

$$\left.\ln(r^2+r\mu^{\frac{1}{3}}+\mu^{\frac{2}{3}})-\frac{1}{\sqrt{3}}(\mu^{\frac{2}{3}}+r\mu^{\frac{1}{3}})\tanh^{-1}\left(\frac{2r+\mu^{\frac{1}{3}}}{\sqrt{3}\mu^{\frac{1}{3}}}\right)+\frac{1}{4}\mu^{\frac{2}{3}}\ln\left(\frac{4(r^2+r\mu^{\frac{1}{3}}+\mu^{\frac{2}{3}})}{3\mu^{\frac{2}{3}}}\right)+C_1\right)^{\frac{1}{2}} \qquad (27)$$

$$x_2=\pm\sqrt{r^2\theta^2+C_2} \qquad (28)$$

$$x_3=\pm\sqrt{r^2\phi^2\sin^2\theta+C_3} \qquad (29)$$

$$x_4=\pm\sqrt{r^2 l^2\sin^2\theta\sin^2\phi+C_4} \qquad (30)$$

$$x_5=\pm\sqrt{r^2\eta^2\sin^2\theta\sin^2\phi\sin^2 l+C_5} \qquad (31)$$

Since $x_1^2+x_2^2+x_3^2+x_4^2+x_5^2-x_o^2>0$ which is the great circle S_1 in the $6D$ Schwarzchild spacetime S. These geodesic is a retraction in Schwarzchild space. $ds^2>0$. $\mu=0$ is not allowed as it leads to undefined coordinate.

From $r^2(\sin^2\theta\sin^2\phi\sin^2 l)\dot{\eta}=h$, if $h=0$ we have four cases:(1) $\dot{\eta}=0$ or $\eta=B$. If $B=0$ we get the coordinates as:

$$x_o=\pm\frac{4\pi}{3}\sqrt{\left(1-\frac{\mu}{r^3}\right)\mu^{\frac{2}{3}}\tau^2+C_o} \tag{32}$$

$$x_1=\pm\left(\frac{1}{2}r^2+\frac{1}{3}\mu^{\frac{2}{3}}+\frac{1}{3}\mu^{\frac{1}{3}}\ln(1-\mu^{\frac{1}{3}})r-\frac{1}{3}\ln(1-\mu^{\frac{1}{3}})\mu^{\frac{2}{3}}-\frac{1}{6}(\mu^{\frac{1}{3}}r+\frac{1}{2}\mu^{\frac{2}{3}})\times\right.$$

$$\left.\ln(r^2+r\mu^{\frac{1}{3}}+\mu^{\frac{2}{3}})-\frac{1}{\sqrt{3}}(\mu^{\frac{2}{3}}+r\mu^{\frac{1}{3}})\tanh^{-1}\left(\frac{2r+\mu^{\frac{1}{3}}}{\sqrt{3}\mu^{\frac{1}{3}}}\right)+\frac{1}{4}\mu^{\frac{2}{3}}\ln\left(\frac{4(r^2+r\mu^{\frac{1}{3}}+\mu^{\frac{2}{3}})}{3\mu^{\frac{2}{3}}}\right)+C_1\right)^{\frac{1}{2}} \tag{33}$$

$$x_2=\pm\sqrt{r^2\theta^2+C_2} \tag{34}$$

$$x_3=\pm\sqrt{r^2\phi^2\sin^2\theta+C_3} \tag{35}$$

$$x_4=\pm\sqrt{r^2l^2\sin^2\theta\sin^2\phi+C_4} \tag{36}$$

$$x_5=\pm\sqrt{C_5} \tag{37}$$

This is the geodesic hyperspacetime S_2 of the Schwarzchild space S. This is a retraction. $ds^2>0$.

(2) $\phi=0$ and in this case we get

$$x_o=\pm\frac{4\pi}{3}\sqrt{\left(1-\frac{\mu}{r^3}\right)\mu^{\frac{2}{3}}\tau^2+C_o} \tag{38}$$

$$x_1=\pm\left(\frac{1}{2}r^2+\frac{1}{3}\mu^{\frac{2}{3}}+\frac{1}{3}\mu^{\frac{1}{3}}\ln(1-\mu^{\frac{1}{3}})r-\frac{1}{3}\ln(1-\mu^{\frac{1}{3}})\mu^{\frac{2}{3}}-\frac{1}{6}(\mu^{\frac{1}{3}}r+\frac{1}{2}\mu^{\frac{2}{3}})\times\right.$$

$$\left.\ln(r^2+r\mu^{\frac{1}{3}}+\mu^{\frac{2}{3}})-\frac{1}{\sqrt{3}}(\mu^{\frac{2}{3}}+r\mu^{\frac{1}{3}})\tanh^{-1}\left(\frac{2r+\mu^{\frac{1}{3}}}{\sqrt{3}\mu^{\frac{1}{3}}}\right)+\frac{1}{4}\mu^{\frac{2}{3}}\ln\left(\frac{4(r^2+r\mu^{\frac{1}{3}}+\mu^{\frac{2}{3}})}{3\mu^{\frac{2}{3}}}\right)+C_1\right)^{\frac{1}{2}} \tag{39}$$

$$x_2=\pm\sqrt{r^2\theta^2+C_2} \tag{40}$$

$$x_3=\pm\sqrt{C_3} \tag{41}$$

$$x_4=\pm\sqrt{C_4} \tag{42}$$

$$x_5=\pm\sqrt{C_5} \tag{43}$$

This is the geodesic hyperspacetime S_3 of the Schwarzchild space S. This is a retraction. $ds^2 > 0$.

(3) $\theta = 0$ and in this case we get

$$x_o = \pm\frac{4\pi}{3}\sqrt{\left(1-\frac{\mu}{r^3}\right)\mu^{\frac{2}{3}}\tau^2 + C_o} \tag{44}$$

$$x_1 = \pm\left(\frac{1}{2}r^2 + \frac{1}{3}\mu^{\frac{2}{3}} + \frac{1}{3}\mu^{\frac{1}{3}}\ln(1-\mu^{\frac{1}{3}})r - \frac{1}{3}\ln(1-\mu^{\frac{1}{3}})\mu^{\frac{2}{3}} - \frac{1}{6}(\mu^{\frac{1}{3}}r + \frac{1}{2}\mu^{\frac{2}{3}})\times\right.$$

$$\left.\ln(r^2 + r\mu^{\frac{1}{3}} + \mu^{\frac{2}{3}}) - \frac{1}{\sqrt{3}}(\mu^{\frac{2}{3}} + r\mu^{\frac{1}{3}})\tanh^{-1}\left(\frac{2r+\mu^{\frac{1}{3}}}{\sqrt{3}\mu^{\frac{1}{3}}}\right) + \frac{1}{4}\mu^{\frac{2}{3}}\ln\left(\frac{4(r^2 + r\mu^{\frac{1}{3}} + \mu^{\frac{2}{3}})}{3\mu^{\frac{2}{3}}}\right) + C_1\right)^{\frac{1}{2}} \tag{45}$$

$$x_2 = \pm\sqrt{C_2} \tag{46}$$

$$x_3 = \pm\sqrt{C_3} \tag{47}$$

$$x_4 = \pm\sqrt{C_4} \tag{48}$$

$$x_5 = \pm\sqrt{C_5} \tag{49}$$

This is the geodesic hyperspacetime S_4 of the Schwarzchild space S. This is a retraction. $ds^2 > 0$.

(4) $l = 0$ and in this case we get

$$x_o = \pm\frac{4\pi}{3}\sqrt{\left(1-\frac{\mu}{r^3}\right)\mu^{\frac{2}{3}}\tau^2 + C_o} \tag{50}$$

$$x_1 = \pm\left(\frac{1}{2}r^2 + \frac{1}{3}\mu^{\frac{2}{3}} + \frac{1}{3}\mu^{\frac{1}{3}}\ln(1-\mu^{\frac{1}{3}})r - \frac{1}{3}\ln(1-\mu^{\frac{1}{3}})\mu^{\frac{2}{3}} - \frac{1}{6}(\mu^{\frac{1}{3}}r + \frac{1}{2}\mu^{\frac{2}{3}})\times\right.$$

$$\left.\ln(r^2 + r\mu^{\frac{1}{3}} + \mu^{\frac{2}{3}}) - \frac{1}{\sqrt{3}}(\mu^{\frac{2}{3}} + r\mu^{\frac{1}{3}})\tanh^{-1}\left(\frac{2r+\mu^{\frac{1}{3}}}{\sqrt{3}\mu^{\frac{1}{3}}}\right) + \frac{1}{4}\mu^{\frac{2}{3}}\ln\left(\frac{4(r^2 + r\mu^{\frac{1}{3}} + \mu^{\frac{2}{3}})}{3\mu^{\frac{2}{3}}}\right) + C_1\right)^{\frac{1}{2}} \tag{51}$$

$$x_2 = \pm\sqrt{r^2\theta^2 + C_2} \tag{52}$$

$$x_3 = \pm\sqrt{r^2\phi^2\sin^2\theta + C_3} \tag{53}$$

$$x_4 = \pm\sqrt{C_4} \tag{54}$$

$$x_5 = \pm\sqrt{C_5} \tag{55}$$

This is the geodesic hyperspacetime S_5 of the Schwarzchild space S. This is a retraction. $ds^2 > 0$.

Theorem1:

The retraction of 6D Schwarzchild space is a 6D spacetime geodesic.

3 deformation retract of 6D Schwarzchild space

The deformation retract of the 6D Schwarzchild space is defined as

$$\phi : Sc \times I \to Sc \tag{56}$$

where Sc is the 6-dimensional Schwarzchild space and I is the closed interval $[0,1]$. The retraction of $6D$ Schwarzchild space Sc is defined as

$$R : Sc \to S_1, S_2, S_3, S_4 \text{ and } S_5. \tag{57}$$

The deformation retract of the 6D Schwarzchild space Sc into a geodesic $S_1 \subset Sc$ is defined as

$$\begin{aligned}\phi(m,t) = (1-t)\Bigg\{ & \pm\frac{4\pi}{3}\sqrt{\left(1-\frac{\mu}{r^3}\right)\mu^{\frac{2}{3}}\tau^2 + C_o}, \pm\Bigg(\frac{1}{2}r^2 + \frac{1}{3}\mu^{\frac{2}{3}} + \frac{1}{3}\mu^{\frac{1}{3}}\ln(1-\mu^{\frac{1}{3}})r - \\ & \frac{1}{3}\ln(1-\mu^{\frac{1}{3}})\mu^{\frac{2}{3}} - \frac{1}{6}(\mu^{\frac{1}{3}}r + \frac{1}{2}\mu^{\frac{2}{3}})\ln(r^2 + r\mu^{\frac{1}{3}} + \mu^{\frac{2}{3}}) - \frac{1}{\sqrt{3}}(\mu^{\frac{2}{3}} + r\mu^{\frac{1}{3}})\tanh^{-1}\left(\frac{2r+\mu^{\frac{1}{3}}}{\sqrt{3}\mu^{\frac{1}{3}}}\right) \\ & + \frac{1}{4}\mu^{\frac{2}{3}}\ln\left(\frac{4(r^2 + r\mu^{\frac{1}{3}} + \mu^{\frac{2}{3}})}{3\mu^{\frac{2}{3}}}\right) + C_1\Bigg)^{\frac{1}{2}}, \pm\sqrt{r^2\theta^2 + C_2}, \pm\sqrt{r^2\phi^2\sin^2\theta + C_3}, \\ & , \pm\sqrt{r^2 l^2 \sin^2\theta \sin^2\phi + C_4}, \pm\sqrt{r^2\eta^2\sin^2\theta\sin^2\phi\sin^2 l + C_5}\Bigg\} + \tan\frac{\pi t}{4}\Bigg\{\pm\sqrt{C_o}, \\ & \pm\Bigg(\frac{1}{2}r^2 + \frac{1}{3}\mu^{\frac{2}{3}} + \frac{1}{3}\mu^{\frac{1}{3}}\ln(1-\mu^{\frac{1}{3}})r - \frac{1}{3}\ln(1-\mu^{\frac{1}{3}})\mu^{\frac{2}{3}} - \frac{1}{6}(\mu^{\frac{1}{3}}r + \frac{1}{2}\mu^{\frac{2}{3}})\ln(r^2 + r\mu^{\frac{1}{3}} + \mu^{\frac{2}{3}}) \\ & - \frac{1}{\sqrt{3}}(\mu^{\frac{2}{3}} + r\mu^{\frac{1}{3}})\tanh^{-1}\left(\frac{2r+\mu^{\frac{1}{3}}}{\sqrt{3}\mu^{\frac{1}{3}}}\right) + \frac{1}{4}\mu^{\frac{2}{3}}\ln\left(\frac{4(r^2 + r\mu^{\frac{1}{3}} + \mu^{\frac{2}{3}})}{3\mu^{\frac{2}{3}}}\right) + C_1\Bigg)^{\frac{1}{2}}, \\ & \pm\sqrt{r^2\theta^2 + C_2}, \pm\sqrt{r^2\phi^2\sin^2\theta + C_3}, \pm\sqrt{r^2 l^2\sin^2\theta\sin^2\phi + C_4} \\ & , \pm\sqrt{r^2\eta^2\sin^2\theta\sin^2\phi\sin^2 l + C_5}\Bigg\}\end{aligned} \tag{58}$$

where

$$\phi(m,0) = (1-t)\Bigg\{\pm\frac{4\pi}{3}\sqrt{\left(1-\frac{\mu}{r^3}\right)\mu^{\frac{2}{3}}\tau^2 + C_o}, \pm\Bigg(\frac{1}{2}r^2 + \frac{1}{3}\mu^{\frac{2}{3}} + \frac{1}{3}\mu^{\frac{1}{3}}\ln(1-\mu^{\frac{1}{3}})r -$$

$$\frac{1}{3}\ln(1-\mu^{\frac{1}{3}})\mu^{\frac{2}{3}}-\frac{1}{6}(\mu^{\frac{1}{3}}r+\frac{1}{2}\mu^{\frac{2}{3}})\ln(r^2+r\mu^{\frac{1}{3}}+\mu^{\frac{2}{3}})-\frac{1}{\sqrt{3}}(\mu^{\frac{2}{3}}+r\mu^{\frac{1}{3}})\tanh^{-1}\left(\frac{2r+\mu^{\frac{1}{3}}}{\sqrt{3}\mu^{\frac{1}{3}}}\right)$$

$$+\frac{1}{4}\mu^{\frac{2}{3}}\ln\left(\frac{4(r^2+r\mu^{\frac{1}{3}}+\mu^{\frac{2}{3}})}{3\mu^{\frac{2}{3}}}\right)+C_1\Bigg)^{\frac{1}{2}},\pm\sqrt{r^2\theta^2+C_2},\pm\sqrt{r^2\phi^2\sin^2\theta+C_3},$$

$$\pm\sqrt{r^2l^2\sin^2\theta\sin^2\phi+C_4},\pm\sqrt{r^2\eta^2\sin^2\theta\sin^2\phi\sin^2 l+C_5}\Big\} \quad (59)$$

$$\phi(m,1)=\Big\{\pm\sqrt{C_o},\pm\Bigg(\frac{1}{2}r^2+\frac{1}{3}\mu^{\frac{2}{3}}+\frac{1}{3}\mu^{\frac{1}{3}}\ln(1-\mu^{\frac{1}{3}})r-\frac{1}{3}\ln(1-\mu^{\frac{1}{3}})\mu^{\frac{2}{3}}$$

$$-\frac{1}{6}(\mu^{\frac{1}{3}}r+\frac{1}{2}\mu^{\frac{2}{3}})\ln(r^2+r\mu^{\frac{1}{3}}+\mu^{\frac{2}{3}})-\frac{1}{\sqrt{3}}(\mu^{\frac{2}{3}}+r\mu^{\frac{1}{3}})\tanh^{-1}\left(\frac{2r+\mu^{\frac{1}{3}}}{\sqrt{3}\mu^{\frac{1}{3}}}\right)$$

$$+\frac{1}{4}\mu^{\frac{2}{3}}\ln\left(\frac{4(r^2+r\mu^{\frac{1}{3}}+\mu^{\frac{2}{3}})}{3\mu^{\frac{2}{3}}}\right)+C_1\Bigg)^{\frac{1}{2}},\pm\sqrt{r^2\theta^2+C_2},\pm\sqrt{r^2\phi^2\sin^2\theta+C_3},$$

$$\pm\sqrt{r^2l^2\sin^2\theta\sin^2\phi+C_4},\pm\sqrt{r^2\eta^2\sin^2\theta\sin^2\phi\sin^2 l+C_5}\Big\} \quad (60)$$

The deformation retract of the 6D Schwarzchild space Sc into a geodesic $S_2 \subset Sc$ is defined as

$$\phi(m,t)=\cos\frac{\pi t}{2}\Big\{\pm\frac{4\pi}{3}\sqrt{\left(1-\frac{\mu}{r^3}\right)\mu^{\frac{2}{3}}\tau^2+C_o},\pm\Bigg(\frac{1}{2}r^2+\frac{1}{3}\mu^{\frac{2}{3}}+\frac{1}{3}\mu^{\frac{1}{3}}\ln(1-\mu^{\frac{1}{3}})r-$$

$$\frac{1}{3}\ln(1-\mu^{\frac{1}{3}})\mu^{\frac{2}{3}}-\frac{1}{6}(\mu^{\frac{1}{3}}r+\frac{1}{2}\mu^{\frac{2}{3}})\ln(r^2+r\mu^{\frac{1}{3}}+\mu^{\frac{2}{3}})-\frac{1}{\sqrt{3}}(\mu^{\frac{2}{3}}+r\mu^{\frac{1}{3}})\tanh^{-1}\left(\frac{2r+\mu^{\frac{1}{3}}}{\sqrt{3}\mu^{\frac{1}{3}}}\right)$$

$$+\frac{1}{4}\mu^{\frac{2}{3}}\ln\left(\frac{4(r^2+r\mu^{\frac{1}{3}}+\mu^{\frac{2}{3}})}{3\mu^{\frac{2}{3}}}\right)+C_1\Bigg)^{\frac{1}{2}},\pm\sqrt{r^2\theta^2+C_2},\pm\sqrt{r^2\phi^2\sin^2\theta+C_3},$$

$$\pm\sqrt{r^2l^2\sin^2\theta\sin^2\phi+C_4},\pm\sqrt{r^2\eta^2\sin^2\theta\sin^2\phi\sin^2 l+C_5}\Big\}+\sin\frac{\pi t}{2}\times$$

$$\left\{\pm\frac{4\pi}{3}\sqrt{\left(1-\frac{\mu}{r^3}\right)\mu^{\frac{2}{3}}\tau^2+C_o},\pm\left(\frac{1}{2}r^2+\frac{1}{3}\mu^{\frac{2}{3}}+\frac{1}{3}\mu^{\frac{1}{3}}\ln(1-\mu^{\frac{1}{3}})r-\frac{1}{3}\ln(1-\mu^{\frac{1}{3}})\mu^{\frac{2}{3}}\right.\right.$$

$$-\frac{1}{6}(\mu^{\frac{1}{3}}r+\frac{1}{2}\mu^{\frac{2}{3}})\ln(r^2+r\mu^{\frac{1}{3}}+\mu^{\frac{2}{3}})-\frac{1}{\sqrt{3}}(\mu^{\frac{2}{3}}+r\mu^{\frac{1}{3}})\tanh^{-1}\left(\frac{2r+\mu^{\frac{1}{3}}}{\sqrt{3}\mu^{\frac{1}{3}}}\right)+$$

$$\left.\frac{1}{4}\mu^{\frac{2}{3}}\ln\left(\frac{4(r^2+r\mu^{\frac{1}{3}}+\mu^{\frac{2}{3}})}{3\mu^{\frac{2}{3}}}\right)+C_1\right)^{\frac{1}{2}},\pm\sqrt{r^2\theta^2+C_2},\pm\sqrt{r^2\phi^2\sin^2\theta+C_3},$$

$$\left.\pm\sqrt{r^2l^2\sin^2\theta\sin^2\phi+C_4},\pm\sqrt{C_5}\right\} \tag{61}$$

The deformation retract of the 6D Schwarzchild space Sc into a geodesic $S_3 \subset Sc$ is defined as

$$\phi(m,t)=(1-t)\left\{\pm\frac{4\pi}{3}\sqrt{\left(1-\frac{\mu}{r^3}\right)\mu^{\frac{2}{3}}\tau^2+C_o},\right.$$

$$\pm\left(\frac{1}{2}r^2+\frac{1}{3}\mu^{\frac{2}{3}}+\frac{1}{3}\mu^{\frac{1}{3}}\ln(1-\mu^{\frac{1}{3}})r-\frac{1}{3}\ln(1-\mu^{\frac{1}{3}})\mu^{\frac{2}{3}}-\frac{1}{6}(\mu^{\frac{1}{3}}r+\frac{1}{2}\mu^{\frac{2}{3}})\ln(r^2+r\mu^{\frac{1}{3}}+\mu^{\frac{2}{3}})\right.$$

$$\left.-\frac{1}{\sqrt{3}}(\mu^{\frac{2}{3}}+r\mu^{\frac{1}{3}})\tanh^{-1}\left(\frac{2r+\mu^{\frac{1}{3}}}{\sqrt{3}\mu^{\frac{1}{3}}}\right)+\frac{1}{4}\mu^{\frac{2}{3}}\ln\left(\frac{4(r^2+r\mu^{\frac{1}{3}}+\mu^{\frac{2}{3}})}{3\mu^{\frac{2}{3}}}\right)+C_1\right)^{\frac{1}{2}},$$

$$\pm\sqrt{r^2\theta^2+C_2},\pm\sqrt{r^2\phi^2\sin^2\theta+C_3},\pm\sqrt{r^2l^2\sin^2\theta\sin^2\phi+C_4}$$

$$\left.,\pm\sqrt{r^2\eta^2\sin^2\theta\sin^2\phi\sin^2 l+C_5}\right\}+\sin\frac{\pi t}{2}\left\{\pm\frac{4\pi}{3}\sqrt{\left(1-\frac{\mu}{r^3}\right)\mu^{\frac{2}{3}}\tau^2+C_o},\right.$$

$$\pm\left(\frac{1}{2}r^2+\frac{1}{3}\mu^{\frac{2}{3}}+\frac{1}{3}\mu^{\frac{1}{3}}\ln(1-\mu^{\frac{1}{3}})r-\frac{1}{3}\ln(1-\mu^{\frac{1}{3}})\mu^{\frac{2}{3}}-\frac{1}{6}(\mu^{\frac{1}{3}}r+\frac{1}{2}\mu^{\frac{2}{3}})\ln(r^2+r\mu^{\frac{1}{3}}+\mu^{\frac{2}{3}})\right.$$

$$\left.-\frac{1}{\sqrt{3}}(\mu^{\frac{2}{3}}+r\mu^{\frac{1}{3}})\tanh^{-1}\left(\frac{2r+\mu^{\frac{1}{3}}}{\sqrt{3}\mu^{\frac{1}{3}}}\right)+\frac{1}{4}\mu^{\frac{2}{3}}\ln\left(\frac{4(r^2+r\mu^{\frac{1}{3}}+\mu^{\frac{2}{3}})}{3\mu^{\frac{2}{3}}}\right)+C_1\right)^{\frac{1}{2}},$$

$$\left.\pm\sqrt{r^2\theta^2+C_2},\pm\sqrt{C_3},\pm\sqrt{C_4},\pm\sqrt{C_5}\right\} \tag{62}$$

The deformation retract of the 6D Schwarzchild space Sc into a geodesic $S_4 \subset Sc$ is defined as

$$\phi(m,t) = \cos\frac{\pi t}{2}\left\{\pm\frac{4\pi}{3}\sqrt{\left(1-\frac{\mu}{r^3}\right)\mu^{\frac{2}{3}}\tau^2 + C_o}\,,\right.$$

$$\pm\left(\frac{1}{2}r^2+\frac{1}{3}\mu^{\frac{2}{3}}+\frac{1}{3}\mu^{\frac{1}{3}}\ln(1-\mu^{\frac{1}{3}})r-\frac{1}{3}\ln(1-\mu^{\frac{1}{3}})\mu^{\frac{2}{3}}-\frac{1}{6}(\mu^{\frac{1}{3}}r+\frac{1}{2}\mu^{\frac{2}{3}})\ln(r^2+r\mu^{\frac{1}{3}}+\mu^{\frac{2}{3}})\right.$$

$$\left.-\frac{1}{\sqrt{3}}(\mu^{\frac{2}{3}}+r\mu^{\frac{1}{3}})\tanh^{-1}\left(\frac{2r+\mu^{\frac{1}{3}}}{\sqrt{3}\mu^{\frac{1}{3}}}\right)+\frac{1}{4}\mu^{\frac{2}{3}}\ln\left(\frac{4(r^2+r\mu^{\frac{1}{3}}+\mu^{\frac{2}{3}})}{3\mu^{\frac{2}{3}}}\right)+C_1\right)^{\frac{1}{2}},$$

$$\pm\sqrt{r^2\theta^2+C_2},\pm\sqrt{r^2\phi^2\sin^2\theta+C_3},\pm\sqrt{r^2l^2\sin^2\theta\sin^2\phi+C_4}$$

$$\left.,\pm\sqrt{r^2\eta^2\sin^2\theta\sin^2\phi\sin^2 l+C_5}\right\}+\tan\frac{\pi t}{4}\left\{\pm\frac{4\pi}{3}\sqrt{\left(1-\frac{\mu}{r^3}\right)\mu^{\frac{2}{3}}\tau^2+C_o}\,,\right.$$

$$\pm\left(\frac{1}{2}r^2+\frac{1}{3}\mu^{\frac{2}{3}}+\frac{1}{3}\mu^{\frac{1}{3}}\ln(1-\mu^{\frac{1}{3}})r-\frac{1}{3}\ln(1-\mu^{\frac{1}{3}})\mu^{\frac{2}{3}}-\frac{1}{6}(\mu^{\frac{1}{3}}r+\frac{1}{2}\mu^{\frac{2}{3}})\ln(r^2+r\mu^{\frac{1}{3}}+\mu^{\frac{2}{3}})\right.$$

$$\left.-\frac{1}{\sqrt{3}}(\mu^{\frac{2}{3}}+r\mu^{\frac{1}{3}})\tanh^{-1}\left(\frac{2r+\mu^{\frac{1}{3}}}{\sqrt{3}\mu^{\frac{1}{3}}}\right)+\frac{1}{4}\mu^{\frac{2}{3}}\ln\left(\frac{4(r^2+r\mu^{\frac{1}{3}}+\mu^{\frac{2}{3}})}{3\mu^{\frac{2}{3}}}\right)+C_1\right)^{\frac{1}{2}},\pm\sqrt{C_2},$$

$$\left.\pm\sqrt{C_3},\pm\sqrt{C_4},\pm\sqrt{C_5}\right\} \qquad (63)$$

The deformation retract of the 6D Schwarzchild space Sc into a geodesic $S_5 \subset Sc$ is defined as

$$\phi(m,t) = \cos\frac{\pi t}{2}\left\{\pm\frac{4\pi}{3}\sqrt{\left(1-\frac{\mu}{r^3}\right)\mu^{\frac{2}{3}}\tau^2 + C_o}\,,\right.$$

$$\pm\left(\frac{1}{2}r^2+\frac{1}{3}\mu^{\frac{2}{3}}+\frac{1}{3}\mu^{\frac{1}{3}}\ln(1-\mu^{\frac{1}{3}})r-\frac{1}{3}\ln(1-\mu^{\frac{1}{3}})\mu^{\frac{2}{3}}-\frac{1}{6}(\mu^{\frac{1}{3}}r+\frac{1}{2}\mu^{\frac{2}{3}})\ln(r^2+r\mu^{\frac{1}{3}}+\mu^{\frac{2}{3}})\right.$$

$$\left.-\frac{1}{\sqrt{3}}(\mu^{\frac{2}{3}}+r\mu^{\frac{1}{3}})\tanh^{-1}\left(\frac{2r+\mu^{\frac{1}{3}}}{\sqrt{3}\mu^{\frac{1}{3}}}\right)+\frac{1}{4}\mu^{\frac{2}{3}}\ln\left(\frac{4(r^2+r\mu^{\frac{1}{3}}+\mu^{\frac{2}{3}})}{3\mu^{\frac{2}{3}}}\right)+C_1\right)^{\frac{1}{2}},$$

$$\pm\sqrt{r^2\theta^2+C_2},\pm\sqrt{r^2\phi^2\sin^2\theta+C_3},\pm\sqrt{r^2l^2\sin^2\theta\sin^2\phi+C_4}$$

$$,\pm\sqrt{r^2\eta^2\sin^2\theta\sin^2\phi\sin^2 l+C_5}\Big\}+\tan\frac{\pi t}{4}\Big\{\pm\frac{4\pi}{3}\sqrt{\left(1-\frac{\mu}{r^3}\right)\mu^{\frac{2}{3}}\tau^2+C_o},$$

$$\pm\left(\frac{1}{2}r^2+\frac{1}{3}\mu^{\frac{2}{3}}+\frac{1}{3}\mu^{\frac{1}{3}}\ln(1-\mu^{\frac{1}{3}})r-\frac{1}{3}\ln(1-\mu^{\frac{1}{3}})\mu^{\frac{2}{3}}-\frac{1}{6}(\mu^{\frac{1}{3}}r+\frac{1}{2}\mu^{\frac{2}{3}})\ln(r^2+r\mu^{\frac{1}{3}}+\mu^{\frac{2}{3}})\right.$$

$$\left.-\frac{1}{\sqrt{3}}(\mu^{\frac{2}{3}}+r\mu^{\frac{1}{3}})\tanh^{-1}\left(\frac{2r+\mu^{\frac{1}{3}}}{\sqrt{3}\mu^{\frac{1}{3}}}\right)+\frac{1}{4}\mu^{\frac{2}{3}}\ln\left(\frac{4(r^2+r\mu^{\frac{1}{3}}+\mu^{\frac{2}{3}})}{3\mu^{\frac{2}{3}}}\right)+C_1\right)^{\frac{1}{2}},$$

$$\pm\sqrt{r^2\theta^2+C_2},\pm\sqrt{r^2\phi^2\sin^{\theta}+C_3},\pm\sqrt{C_4},\pm\sqrt{C_5}\Big\} \tag{64}$$

Now we are going to discuss the folding f of the 6D Schwarzchild space Sc. Let $f:Sc\to Sc$ where

$$f(x_o,x_1,x_2,x_3,x_4,x_5)=(x_o,x_1,x_2,|x_3|,x_4,x_5)$$

An isometric folding of the 6D Schwarzchild space into itself may be defined as:

$$f:\Big\{\pm\frac{4\pi}{3}\sqrt{\left(1-\frac{\mu}{r^3}\right)\mu^{\frac{2}{3}}\tau^2+C_o},\pm\left(\frac{1}{2}r^2+\frac{1}{3}\mu^{\frac{2}{3}}+\frac{1}{3}\mu^{\frac{1}{3}}\ln(1-\mu^{\frac{1}{3}})r-\frac{1}{3}\ln(1-\mu^{\frac{1}{3}})\mu^{\frac{2}{3}}\right.$$

$$-\frac{1}{6}(\mu^{\frac{1}{3}}r+\frac{1}{2}\mu^{\frac{2}{3}})\ln(r^2+r\mu^{\frac{1}{3}}+\mu^{\frac{2}{3}})-\frac{1}{\sqrt{3}}(\mu^{\frac{2}{3}}+r\mu^{\frac{1}{3}})\tanh^{-1}\left(\frac{2r+\mu^{\frac{1}{3}}}{\sqrt{3}\mu^{\frac{1}{3}}}\right)+$$

$$\left.\frac{1}{4}\mu^{\frac{2}{3}}\ln\left(\frac{4(r^2+r\mu^{\frac{1}{3}}+\mu^{\frac{2}{3}})}{3\mu^{\frac{2}{3}}}\right)+C_1\right)^{\frac{1}{2}},\pm\sqrt{r^2\theta^2+C_2},\pm\sqrt{r^2\phi^2\sin^2\theta+C_3},$$

$$\pm\sqrt{r^2l^2\sin^2\theta\sin^2\phi+C_4},\pm\sqrt{r^2\eta^2\sin^2\theta\sin^2\phi\sin^2 l+C_5}\Big\}\to\Big\{\pm\frac{4\pi}{3}\sqrt{\left(1-\frac{\mu}{r^3}\right)\mu^{\frac{2}{3}}\tau^2+C_o},$$

$$\pm\left(\frac{1}{2}r^2+\frac{1}{3}\mu^{\frac{2}{3}}+\frac{1}{3}\mu^{\frac{1}{3}}\ln(1-\mu^{\frac{1}{3}})r-\frac{1}{3}\ln(1-\mu^{\frac{1}{3}})\mu^{\frac{2}{3}}-\frac{1}{6}(\mu^{\frac{1}{3}}r+\frac{1}{2}\mu^{\frac{2}{3}})\ln(r^2+r\mu^{\frac{1}{3}}+\mu^{\frac{2}{3}})\right.$$

$$-\frac{1}{\sqrt{3}}(\mu^{\frac{2}{3}}+r\mu^{\frac{1}{3}})\tanh^{-1}\left(\frac{2r+\mu^{\frac{1}{3}}}{\sqrt{3}\mu^{\frac{1}{3}}}\right)+\frac{1}{4}\mu^{\frac{2}{3}}\ln\left(\frac{4(r^2+r\mu^{\frac{1}{3}}+\mu^{\frac{2}{3}})}{3\mu^{\frac{2}{3}}}\right)+C_1\Bigg)^{\frac{1}{2}},\pm\sqrt{r^2\theta^2+C_2},$$

$$\left|\sqrt{r^2\phi^2\sin^2\theta+C_3}\right|,\pm\sqrt{r^2l^2\sin^2\theta\sin^2\phi+C_4},\pm\sqrt{r^2\eta^2\sin^2\theta\sin^2\phi\sin^2 l+C_5}\Bigg\}$$

The deformation retract of the folded 6D Schwarzchild space Sc into the folded S_1 is defined as

$$\phi f:\left\{\pm\frac{4\pi}{3}\sqrt{\left(1-\frac{\mu}{r^3}\right)\mu^{\frac{2}{3}}\tau^2+C_o},\pm\left(\frac{1}{2}r^2+\frac{1}{3}\mu^{\frac{2}{3}}+\frac{1}{3}\mu^{\frac{1}{3}}\ln(1-\mu^{\frac{1}{3}})r-\frac{1}{3}\ln(1-\mu^{\frac{1}{3}})\mu^{\frac{2}{3}}\right.\right.$$

$$-\frac{1}{6}(\mu^{\frac{1}{3}}r+\frac{1}{2}\mu^{\frac{2}{3}})\ln(r^2+r\mu^{\frac{1}{3}}+\mu^{\frac{2}{3}})-\frac{1}{\sqrt{3}}(\mu^{\frac{2}{3}}+r\mu^{\frac{1}{3}})\tanh^{-1}\left(\frac{2r+\mu^{\frac{1}{3}}}{\sqrt{3}\mu^{\frac{1}{3}}}\right)+$$

$$\frac{1}{4}\mu^{\frac{2}{3}}\ln\left(\frac{4(r^2+r\mu^{\frac{1}{3}}+\mu^{\frac{2}{3}})}{3\mu^{\frac{2}{3}}}\right)+C_1\Bigg)^{\frac{1}{2}},\pm\sqrt{r^2\theta^2+C_2},\left|\pm\sqrt{r^2\phi^2\sin^2\theta+C_3}\right|,$$

$$\pm\sqrt{r^2l^2\sin^2\theta\sin^2\phi+C_4},\pm\sqrt{r^2\eta^2\sin^2\theta\sin^2\phi\sin^2 l+C_5}\Bigg\}\times I\to.$$

$$\left\{\pm\frac{4\pi}{3}\sqrt{\left(1-\frac{\mu}{r^3}\right)\mu^{\frac{2}{3}}\tau^2+C_o},\pm\left(\frac{1}{2}r^2+\frac{1}{3}\mu^{\frac{2}{3}}+\frac{1}{3}\mu^{\frac{1}{3}}\ln(1-\mu^{\frac{1}{3}})r-\frac{1}{3}\ln(1-\mu^{\frac{1}{3}})\mu^{\frac{2}{3}}\right.\right.$$

$$-\frac{1}{6}(\mu^{\frac{1}{3}}r+\frac{1}{2}\mu^{\frac{2}{3}})\ln(r^2+r\mu^{\frac{1}{3}}+\mu^{\frac{2}{3}})-\frac{1}{\sqrt{3}}(\mu^{\frac{2}{3}}+r\mu^{\frac{1}{3}})\tanh^{-1}\left(\frac{2r+\mu^{\frac{1}{3}}}{\sqrt{3}\mu^{\frac{1}{3}}}\right)$$

$$+\frac{1}{4}\mu^{\frac{2}{3}}\ln\left(\frac{4(r^2+r\mu^{\frac{1}{3}}+\mu^{\frac{2}{3}})}{3\mu^{\frac{2}{3}}}\right)+C_1\Bigg)^{\frac{1}{2}},\pm\sqrt{r^2\theta^2+C_2},\left|\sqrt{r^2\phi^2\sin^2\theta+C_3}\right|,$$

$$\pm\sqrt{r^2l^2\sin^2\theta\sin^2\phi+C_4},\pm\sqrt{r^2\eta^2\sin^2\theta\sin^2\phi\sin^2 l+C_5}\Bigg\} \quad (65)$$

with

$$\phi f(m,t)=\cos\frac{\pi t}{2}\left\{\pm\frac{4\pi}{3}\sqrt{\left(1-\frac{\mu}{r^3}\right)\mu^{\frac{2}{3}}\tau^2+C_o},\pm\left(\frac{1}{2}r^2+\frac{1}{3}\mu^{\frac{2}{3}}+\frac{1}{3}\mu^{\frac{1}{3}}\ln(1-\mu^{\frac{1}{3}})r-\right.\right.$$

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$$\frac{1}{3}\ln(1-\mu^{\frac{1}{3}})\mu^{\frac{2}{3}}-\frac{1}{6}(\mu^{\frac{1}{3}}r+\frac{1}{2}\mu^{\frac{2}{3}})\ln(r^2+r\mu^{\frac{1}{3}}+\mu^{\frac{2}{3}})-\frac{1}{\sqrt{3}}(\mu^{\frac{2}{3}}+r\mu^{\frac{1}{3}})\tanh^{-1}\left(\frac{2r+\mu^{\frac{1}{3}}}{\sqrt{3}\mu^{\frac{1}{3}}}\right)$$

$$+\frac{1}{4}\mu^{\frac{2}{3}}\ln\left(\frac{4(r^2+r\mu^{\frac{1}{3}}+\mu^{\frac{2}{3}})}{3\mu^{\frac{2}{3}}}\right)+C_1\Bigg)^{\frac{1}{2}},\pm\sqrt{r^2\theta^2+C_2},\left|\sqrt{r^2\phi^2\sin^2\theta+C_3}\right|,$$

$$\pm\sqrt{r^2l^2\sin^2\theta\sin^2\phi+C_4},\pm\sqrt{r^2\eta^2\sin^2\theta\sin^2\phi\sin^2 l+C_5}\Big\}+\tan\frac{\pi t}{2}\Big\{\pm\frac{4\pi}{3}\sqrt{C_o},$$

$$\pm\left(\frac{1}{2}r^2+\frac{1}{3}\mu^{\frac{2}{3}}+\frac{1}{3}\mu^{\frac{1}{3}}\ln(1-\mu^{\frac{1}{3}})r-\frac{1}{3}\ln(1-\mu^{\frac{1}{3}})\mu^{\frac{2}{3}}-\frac{1}{6}(\mu^{\frac{1}{3}}r+\frac{1}{2}\mu^{\frac{2}{3}})\ln(r^2+r\mu^{\frac{1}{3}}+\mu^{\frac{2}{3}})-\right.$$

$$\frac{1}{\sqrt{3}}(\mu^{\frac{2}{3}}+r\mu^{\frac{1}{3}})\tanh^{-1}\left(\frac{2r+\mu^{\frac{1}{3}}}{\sqrt{3}\mu^{\frac{1}{3}}}\right)+\frac{1}{4}\mu^{\frac{2}{3}}\ln\left(\frac{4(r^2+r\mu^{\frac{1}{3}}+\mu^{\frac{2}{3}})}{3\mu^{\frac{2}{3}}}\right)+C_1\Bigg)^{\frac{1}{2}},\pm\sqrt{r^2\theta^2+C_2},$$

$$\left|\sqrt{r^2\phi^2\sin^2\theta+C_3}\right|,\pm\sqrt{r^2l^2\sin^2\theta\sin^2\phi+C_4},\pm\sqrt{r^2\eta^2\sin^2\theta\sin^2\phi\sin^2 l+C_5}\Big\} \qquad (66)$$

Also, for S_2, S_3, S_4 and S_5. Then the following theorem has been proved.

Theorem2:

Under the defined folding and any folding homeomorphic to this type of folding, the deformation retract of the folded 6D Schwarzchild space into the folded geodesics is different from the deformation retract of the 6D Schwarzchild space into the geodesics.

Now, let the folding be defined by

$$f^*: Sc \to Sc$$

where

$$f^*(x_o,x_1,x_2,x_3,x_4,x_5)=(x_o,|x_1|,x_2,x_3,x_4,x_5)$$

An isometric folding of the 6D Schwarzchild space into itself may be defined as:

$$f^*:\left\{\pm\frac{4\pi}{3}\sqrt{\left(1-\frac{\mu}{r^3}\right)\mu^{\frac{2}{3}}\tau^2+C_o},\left|\left(\frac{1}{2}r^2+\frac{1}{3}\mu^{\frac{2}{3}}+\frac{1}{3}\mu^{\frac{1}{3}}\ln(1-\mu^{\frac{1}{3}})r-\frac{1}{3}\ln(1-\mu^{\frac{1}{3}})\mu^{\frac{2}{3}}\right.\right.\right.$$

$$-\frac{1}{6}(\mu^{\frac{1}{3}}r+\frac{1}{2}\mu^{\frac{2}{3}})\ln(r^2+r\mu^{\frac{1}{3}}+\mu^{\frac{2}{3}})-\frac{1}{\sqrt{3}}(\mu^{\frac{2}{3}}+r\mu^{\frac{1}{3}})\tanh^{-1}\left(\frac{2r+\mu^{\frac{1}{3}}}{\sqrt{3}\mu^{\frac{1}{3}}}\right)$$

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$$+\frac{1}{4}\mu^{\frac{2}{3}}\ln\left(\frac{4(r^2+r\mu^{\frac{1}{3}}+\mu^{\frac{2}{3}})}{3\mu^{\frac{2}{3}}}\right)+C_1\Bigg)^{\frac{1}{2}}\Bigg|,\pm\sqrt{r^2\theta^2+C_2},\pm\sqrt{r^2\phi^2\sin^2\theta+C_3},$$

$$\pm\sqrt{r^2l^2\sin^2\theta\sin^2\phi+C_4},\pm\sqrt{r^2\eta^2\sin^2\theta\sin^2\phi\sin^2 l+C_5}\Bigg\}\to\Bigg\{\pm\frac{4\pi}{3}\sqrt{\left(1-\frac{\mu}{r^3}\right)\mu^{\frac{2}{3}}\tau^2+C_o}$$

$$,\Bigg|\Bigg(\frac{1}{2}r^2+\frac{1}{3}\mu^{\frac{2}{3}}+\frac{1}{3}\mu^{\frac{1}{3}}\ln(1-\mu^{\frac{1}{3}})r-\frac{1}{3}\ln(1-\mu^{\frac{1}{3}})\mu^{\frac{2}{3}}-\frac{1}{6}(\mu^{\frac{1}{3}}r+\frac{1}{2}\mu^{\frac{2}{3}})\ln(r^2+r\mu^{\frac{1}{3}}+\mu^{\frac{2}{3}})$$

$$-\frac{1}{\sqrt{3}}(\mu^{\frac{2}{3}}+r\mu^{\frac{1}{3}})\tanh^{-1}\left(\frac{2r+\mu^{\frac{1}{3}}}{\sqrt{3}\mu^{\frac{1}{3}}}\right)+\frac{1}{4}\mu^{\frac{2}{3}}\ln\left(\frac{4(r^2+r\mu^{\frac{1}{3}}+\mu^{\frac{2}{3}})}{3\mu^{\frac{2}{3}}}\right)+C_1\Bigg)^{\frac{1}{2}}\Bigg|,\pm\sqrt{r^2\theta^2+C_2}$$

$$,\pm\sqrt{r^2\phi^2\sin^2\theta+C_3},\pm\sqrt{r^2l^2\sin^2\theta\sin^2\phi+C_4},\pm\sqrt{r^2\eta^2\sin^2\theta\sin^2\phi\sin^2 l+C_5}\Bigg\} \quad (67)$$

The deformation retract of the folded 6D Schwarzchild space Sc into the folded S_1 is defined as:

$$ff^*:\Bigg\{\pm\frac{4\pi}{3}\sqrt{\left(1-\frac{\mu}{r^3}\right)\mu^{\frac{2}{3}}\tau^2+C_o},\Bigg|\Bigg(\frac{1}{2}r^2+\frac{1}{3}\mu^{\frac{2}{3}}+\frac{1}{3}\mu^{\frac{1}{3}}\ln(1-\mu^{\frac{1}{3}})r-\frac{1}{3}\ln(1-\mu^{\frac{1}{3}})\mu^{\frac{2}{3}}$$

$$-\frac{1}{6}(\mu^{\frac{1}{3}}r+\frac{1}{2}\mu^{\frac{2}{3}})\ln(r^2+r\mu^{\frac{1}{3}}+\mu^{\frac{2}{3}})-\frac{1}{\sqrt{3}}(\mu^{\frac{2}{3}}+r\mu^{\frac{1}{3}})\tanh^{-1}\left(\frac{2r+\mu^{\frac{1}{3}}}{\sqrt{3}\mu^{\frac{1}{3}}}\right)$$

$$+\frac{1}{4}\mu^{\frac{2}{3}}\ln\left(\frac{4(r^2+r\mu^{\frac{1}{3}}+\mu^{\frac{2}{3}})}{3\mu^{\frac{2}{3}}}\right)+C_1\Bigg)^{\frac{1}{2}}\Bigg|,\pm\sqrt{r^2\theta^2+C_2},\pm\sqrt{r^2\phi^2\sin^2\theta+C_3},$$

$$\pm\sqrt{r^2l^2\sin^2\theta\sin^2\phi+C_4},\pm\sqrt{r^2\eta^2\sin^2\theta\sin^2\phi\sin^2 l+C_5}\Bigg\}\times I\to\Bigg\{\pm\frac{4\pi}{3}\sqrt{\left(1-\frac{\mu}{r^3}\right)\mu^{\frac{2}{3}}\tau^2+C_o}$$

$$,\Bigg|\Bigg(\frac{1}{2}r^2+\frac{1}{3}\mu^{\frac{2}{3}}+\frac{1}{3}\mu^{\frac{1}{3}}\ln(1-\mu^{\frac{1}{3}})r-\frac{1}{3}\ln(1-\mu^{\frac{1}{3}})\mu^{\frac{2}{3}}-\frac{1}{6}(\mu^{\frac{1}{3}}r+\frac{1}{2}\mu^{\frac{2}{3}})\ln(r^2+r\mu^{\frac{1}{3}}+\mu^{\frac{2}{3}})$$

$$-\frac{1}{\sqrt{3}}(\mu^{\frac{2}{3}}+r\mu^{\frac{1}{3}})\tanh^{-1}\left(\frac{2r+\mu^{\frac{1}{3}}}{\sqrt{3}\mu^{\frac{1}{3}}}\right)+\frac{1}{4}\mu^{\frac{2}{3}}\ln\left(\frac{4(r^2+r\mu^{\frac{1}{3}}+\mu^{\frac{2}{3}})}{3\mu^{\frac{2}{3}}}\right)+C_1\Bigg)^{\frac{1}{2}}\Bigg|,\pm\sqrt{r^2\theta^2+C_2}$$

$$,\pm\sqrt{r^2\phi^2\sin^2\theta+C_3},\pm\sqrt{r^2l^2\sin^2\theta\sin^2\phi+C_4},\pm\sqrt{r^2\eta^2\sin^2\theta\sin^2\phi\sin^2 l+C_5}\Big\} \quad (68)$$

with

$$ff^*(m,t)=(1-t)\Bigg\{\pm\frac{4\pi}{3}\sqrt{\left(1-\frac{\mu}{r^3}\right)\mu^{\frac{2}{3}}\tau^2+C_o},\Bigg|\Bigg(\frac{1}{2}r^2+\frac{1}{3}\mu^{\frac{2}{3}}+\frac{1}{3}\mu^{\frac{1}{3}}\ln(1-\mu^{\frac{1}{3}})r-\frac{1}{3}\ln(1-\mu^{\frac{1}{3}})\mu^{\frac{2}{3}}$$

$$-\frac{1}{6}(\mu^{\frac{1}{3}}r+\frac{1}{2}\mu^{\frac{2}{3}})\ln(r^2+r\mu^{\frac{1}{3}}+\mu^{\frac{2}{3}})-\frac{1}{\sqrt{3}}(\mu^{\frac{2}{3}}+r\mu^{\frac{1}{3}})\tanh^{-1}\left(\frac{2r+\mu^{\frac{1}{3}}}{\sqrt{3}\mu^{\frac{1}{3}}}\right)$$

$$+\frac{1}{4}\mu^{\frac{2}{3}}\ln\left(\frac{4(r^2+r\mu^{\frac{1}{3}}+\mu^{\frac{2}{3}})}{3\mu^{\frac{2}{3}}}\right)+C_1\Bigg)^{\frac{1}{2}}\Bigg|,\pm\sqrt{r^2\theta^2+C_2},\pm\sqrt{r^2\phi^2\sin^2\theta+C_3},$$

$$\pm\sqrt{r^2l^2\sin^2\theta\sin^2\phi+C_4},\pm\sqrt{r^2\eta^2\sin^2\theta\sin^2\phi\sin^2 l+C_5}\Big\}+\sin\frac{\pi t}{2}\Bigg\{\pm\frac{4\pi}{3}\sqrt{\left(1-\frac{\mu}{r^3}\right)\mu^{\frac{2}{3}}\tau^2+C_o}$$

$$,\Bigg|\Bigg(\frac{1}{2}r^2+\frac{1}{3}\mu^{\frac{2}{3}}+\frac{1}{3}\mu^{\frac{1}{3}}\ln(1-\mu^{\frac{1}{3}})r-\frac{1}{3}\ln(1-\mu^{\frac{1}{3}})\mu^{\frac{2}{3}}-\frac{1}{6}(\mu^{\frac{1}{3}}r+\frac{1}{2}\mu^{\frac{2}{3}})\ln(r^2+r\mu^{\frac{1}{3}}+\mu^{\frac{2}{3}})$$

$$-\frac{1}{\sqrt{3}}(\mu^{\frac{2}{3}}+r\mu^{\frac{1}{3}})\tanh^{-1}\left(\frac{2r+\mu^{\frac{1}{3}}}{\sqrt{3}\mu^{\frac{1}{3}}}\right)+\frac{1}{4}\mu^{\frac{2}{3}}\ln\left(\frac{4(r^2+r\mu^{\frac{1}{3}}+\mu^{\frac{2}{3}})}{3\mu^{\frac{2}{3}}}\right)+C_1\Bigg)^{\frac{1}{2}}\Bigg|,\pm\sqrt{r^2\theta^2+C_2}$$

$$,\pm\sqrt{r^2\phi^2\sin^2\theta+C_3},\pm\sqrt{r^2l^2\sin^2\theta\sin^2\phi+C_4},\pm\sqrt{r^2\eta^2\sin^2\theta\sin^2\phi\sin^2 l+C_5}\Big\} \quad (69)$$

Also, for S_2, S_3, S_4 and S_5. then the following theorem has been proved.

Theorem3:

Under the defined folding and any folding homeomorphic to this type of folding, the deformation retract of the folded 6D Schwarzchild space into the folded geodesics is the same as the deformation retract of the 6D Schwarzchild space into the geodesics.

4 Energy Conservation

For the case of $(\theta=\phi=l=\pi/2)$, the equations to be solved are

$$\left(1-\frac{\mu}{r^3}\right)\dot{t}=k \quad (70)$$

$$c^2 = -c^2\left(1-\frac{\mu}{r^3}\right)\dot{t}^2 + \left(1-\frac{\mu}{r^3}\right)^{-1}\dot{r}^2 + r^2\dot{\eta}^2 \tag{71}$$

$$r^2\dot{\eta} = h \tag{72}$$

making use of (70) and (72) in (71) and after some manipulations we get

$$\frac{1}{2}\dot{r}^2 + \frac{h^2}{2r^2}\left(1-\frac{\mu}{r^3}\right) + \frac{c^2\mu}{2r^3} = c^2(1-k^2) \tag{73}$$

For the potential in higher dimensions, we recall the familiar Newton law in $n+4$ dimensions [24]

$$V_{n+4};\ \frac{G_{n+4}M}{r_n^{n+1}} \tag{74}$$

So, for 5 dimensions the potential is inversely proportional to r^3:

$$V_6;\ \frac{G_6 M}{r^3} \tag{75}$$

The term

$$\frac{h^2}{2r^2}\left(1-\frac{\mu}{r^3}\right) + \frac{c^2\mu}{2r^3} \tag{76}$$

represents the 6D potential and the equation (73) is the Energy conservation formula in 5D. Equation (73) is the $6D$ analogy of the 4D one for the case of 4D Schwarzchild field

$$\frac{1}{2}\dot{r}^2 + \frac{h^2}{2r}\left(1-\frac{\mu}{r}\right) - \frac{c^2\mu}{2r} = c^2(k^2-1) \tag{77}$$

which expresses energy conservation in 4D with potential

$$V_4 = \frac{h^2}{2r^2}\left(1-\frac{\mu}{r}\right) - \frac{c^2\mu}{2r} \tag{78}$$

5 Conclusion

The deformation retract of the six dimensional Schwarzchild space has been investigated by making use of Lagrangian equations. The retraction of this space into itself and into geodesics has been presented. The deformation retraction of the six dimensional Schwarzchild space is six dimensional space-time geodesics which found to be a great circle. The folding of this space has been discussed and it was found that this folding, and any folding homeomorphic to that folding, have the same or different deformation retract of the six dimensional Schwarzchild space into a geodesic. A relation for energy conservation in 6D, similar to the one for 4D, has been derived.

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