



## COMMON FIXED POINT FOR TANGENTIAL MAPS OF GREGUS TYPE ON FUZZY METRIC SPACES

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**Abstract:** In this paper, we define extend the results of many others. We prove common fixed point theorems on tangential property for a Gregus type on pair of fuzzy metric spaces. We also deal on some coupled coincidence and common fixed point theorems.



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## 1 INTRODUCTION:

The notion of fuzzy sets introduced by Zadeh (1965, [4]) proved a turning point in the development of Mathematics. The study of fixed points for multi-valued contraction mappings using the Hausdorff metric was initiated by Markin (1973, [2]) and Nadler (1972, [3]).

Bhaskar and Lakshmikantham (2006, [1]) introduced the concepts of coupled fixed points and mixed monotone property and illustrated these results by proving the existence and uniqueness of the solution for a periodic boundary value problem.

## 2 Definitions and Preliminaries:

To set up our results in the next section we recall some definitions and facts.

**2.1 Definition:** A fuzzy set  $A$  in  $X$  is a function with domain  $X$  and values in  $[0, 1]$ .

**2.2 Definition:** A binary operation  $*$ :  $[0, 1] \times [0, 1] \rightarrow [0, 1]$  is a continuous t-norm if  $([0, 1], *)$  is a topological abelian monoid with unit 1 s.t.  $a * b \leq c * d$  whenever  $a \leq c$  and  $b \leq d$ ,  $\forall a, b, c, d \in [0, 1]$ .

Some examples are below:

- (i)  $*(a, b) = ab$ ,
- (ii)  $*(a, b) = \min\{a, b\}$ .

**2.3 Definition:** The 3-tuple  $(X, M, *)$  is called a fuzzy metric space if  $X$  is an arbitrary set,  $*$  is a continuous t-norm and  $M$  is a fuzzy set on  $X^2 \times [0, \infty)$  satisfying the following conditions:

- (FM-1)  $M(x, y, t) > 0$  and  $M(x, y, 0) = 0$
- (FM-2)  $M(x, y, t) = 1$  iff  $x = y$ ,
- (FM-3)  $M(x, y, t) = M(y, x, t)$ ,
- (FM-4)  $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$ ,
- (FM-5)  $M(x, y, \cdot) : (0, \infty) \rightarrow [0, 1]$  is continuous, for all  $x, y, z \in X$  and  $s, t > 0$ .

therefore,  $M(x, y, \cdot)$  is non-decreasing for all  $x, y \in X$

**2.4 Definition:** Let  $(X, M, *)$  be a fuzzy metric space.

- (i) A sequence  $\{x_n\}$  is said to be convergent to a point  $x \in X$  if  $\lim_{n \rightarrow \infty} M(x_n, x, t) = 1$  for all  $t > 0$ .
- (ii) A subset  $A \subseteq X$  is said to be closed if each convergent sequence  $\{x_n\}$  with  $x_n \in A$  and  $x_n \rightarrow x$ , we have  $x \in A$ .
- (iii) A subset  $A \subseteq X$  is said to be compact if each sequence in  $A$  has a convergent subsequence.

Throughout the paper  $X$  will represent the fuzzy metric space  $(X, M, *)$  and  $\kappa(X)$ , the set of compact subsets of  $X$ . For  $A, B \in \kappa(X)$  and for every  $t > 0$ , denote

$$M_{\min}^A(A, B, t) = \min\{\min_{a \in A} M(a, B, t), \min_{b \in B} M(A, b, t)\},$$

$$M^A(A, y, t) = \max\{M(x, y, t); x, y \in A\}$$

**Remark:** Obviously,  $M_{\min}^A(A, B, t) \leq M^A(a, B, t)$  whenever  $a \in A$  and  $M_{\min}^A(A, B, t) = 1 \iff A = B$ . Also  $M^A(A, y, t) = 1$  if  $y \in A$ .

## 3 Main Results

**3.1 Theorems:** Let  $A, B: X \rightarrow X$  and  $S, T: X \times X \rightarrow \kappa(X)$  be single and set-valued mappings satisfying the following conditions:

- (1) there exist contained coupled weak tangential points  $(z_1, z_2)$  to the mappings  $A$  and  $B$ .
- (2)  $(A, B)$  is tangential w. r. t.  $(S, T)$
- (3)  $\int_0^{M_T(S(x, y), T(u, v), t)} \psi(s) ds \geq \int_0^{m(x, y, u, v, t)} \psi(s) ds$

where

$$m(x, y, u, v, t)$$



$$= \emptyset \left[ \begin{array}{l} a \left( \begin{array}{l} M^\Delta(S(x, y), Ax, t) * M^\Delta(T(u, v), Bu, t) \\ + M^\Delta(S(x, y), Bu, \frac{t}{2}) * M^\Delta(T(u, v), Ax, \frac{t}{2}) \end{array} \right) \\ + (1 - 2a) \max \left\{ M^\Delta(S(x, y), Ax, t), M^\Delta(T(u, v), Bu, t), \right\} \\ \left. M_{\nabla}(S(x, y), Ax, t), M_{\nabla}(T(u, v), Bu, t) \right\} \end{array} \right]$$

(4) AAa = Aa, BBC = Bc, S(Aa, Ab) = T(Bc, Bd) and

$$AAb = Ab, BBd = Bd, S(Ab, Aa) = T(Bd, Bc) \text{ for } (a, b) \in C(A, S) \text{ and } (c, d) \in C(B, T),$$

(5) the pair (A, S) is weakly compatible, for all x, y, u, v ∈ X, 0 ≤ a < 1 and ϕ : [0, 1] → [0, 1] be a non-decreasing map such that ϕ(t) > t, t ≥ 0. Then A, B, S and T have a common coupled fixed point in X.

**Proof:** Since z<sub>1</sub>, z<sub>2</sub> ∈ A (x) ∩ B (X) so there exist points w<sub>1</sub>, w<sub>2</sub>, w<sub>1</sub>', w<sub>2</sub>' ∈ X such that z<sub>1</sub> = Aw<sub>1</sub> = Bw<sub>1</sub>', z<sub>2</sub> = Aw<sub>2</sub> = Bw<sub>2</sub>'. Again (A, B) is tangential w.r.t (S, T) so there exist sequences {x<sub>n</sub>} and {y<sub>n</sub>} in X such that

$$\lim_{n \rightarrow \infty} Ax_n = z_1 = \lim_{n \rightarrow \infty} By_n \in C \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \lim_{n \rightarrow \infty} S(x_n, y_n) \cap \lim_{n \rightarrow \infty} T(y_n, x_n)$$

$$\lim_{n \rightarrow \infty} Ay_n = z_2 = \lim_{n \rightarrow \infty} Bx_n \in D \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \lim_{n \rightarrow \infty} S(y_n, x_n) \cap \lim_{n \rightarrow \infty} T(x_n, y_n)$$

Now, we shall prove that

$$Aw_1 \in S(w_1, w_2), Aw_2 \in S(w_2, w_1), Bw_1' \in T(w_1', w_2')$$

and

$$Bw_2' \in T(w_2', w_1')$$

If not, putting x = x<sub>n</sub>, y = y<sub>n</sub>, u = w<sub>1</sub>' and v = w<sub>2</sub>' in (3), we get

$$m(x_n, y_n, w_1', w_2', t)$$

$$= \emptyset \left[ \begin{array}{l} a \left( \begin{array}{l} M^\Delta(S(x_n, y_n), Ax_n, t) * M^\Delta(T(w_1', w_2'), Bw_1', t) \\ + M^\Delta(S(x_n, y_n), Bw_1', \frac{t}{2}) * M^\Delta(T(w_1', w_2'), Ax_n, \frac{t}{2}) \end{array} \right) \\ + (1 - 2a) \max \left\{ M^\Delta(S(x_n, y_n), Ax_n, t), M^\Delta(T(w_1', w_2'), Bw_1', t), \right\} \\ \left. M_{\nabla}(S(x_n, y_n), Ax_n, t), M_{\nabla}(T(w_1', w_2'), Bw_1', t) \right\} \end{array} \right]$$

$$\int_0^{M_{\nabla}(S(x_n, y_n), T(w_1', w_2'), t)} \psi(s) ds \geq \int_0^{m(x_n, y_n, w_1', w_2', t)} \psi(s) ds$$

Letting n → ∞, we have

$$\int_0^{\lim_{n \rightarrow \infty} M_{\nabla}(S(x_n, y_n), T(w_1', w_2'), t)} \psi(s) ds \geq \int_0^{\lim_{n \rightarrow \infty} m(x_n, y_n, w_1', w_2', t)} \psi(s) ds,$$

but

$$\lim_{n \rightarrow \infty} m(x_n, y_n, w_1', w_2', t)$$

$$= \emptyset \left[ \begin{array}{l} a \left( \begin{array}{l} 1 * M^\Delta(T(w_1', w_2'), Bw_1', t) \\ + M^\Delta(z_1, Bw_1', \frac{t}{2}) * M^\Delta(T(w_1', w_2'), z_1, \frac{t}{2}) \end{array} \right) \\ + (1 - 2a) \max \left\{ 1, M^\Delta(T(w_1', w_2'), Bw_1', t), \right\} \\ \left. 1, M_{\nabla}(T(w_1', w_2'), Bw_1', t) \right\} \end{array} \right]$$

$$= \emptyset \left[ \begin{array}{l} a \left( \begin{array}{l} M^\Delta(T(w_1', w_2'), Bw_1', t) \\ + M^\Delta(T(w_1', w_2'), Bw_1', t) \end{array} \right) \\ + (1 - 2a) \max \{ M^\Delta(T(w_1', w_2'), Bw_1', t) \} \end{array} \right]$$

$$= \emptyset [2a + (1 - 2a)] M^\Delta(T(w_1', w_2'), Bw_1', t)$$

$$\int_0^{M_{\nabla}(C, T(w_1', w_2'), t)} \psi(s) ds \geq \int_0^{\emptyset(M^\Delta(T(w_1', w_2'), Bw_1', t))} \psi(s) ds$$



Since,  $z_1 = Bw_1' \in C$ , we have

$$\begin{aligned} \int_0^{M^\Delta(Bw_1', T(w_1', w_2'), t)} \psi(s) ds &\geq \int_0^{M^\Delta(C, T(w_1', w_2'), kt)} \psi(s) ds \\ &\geq \int_0^{\emptyset(M^\Delta(T(w_1', w_2'), Bw_1', t))} \psi(s) ds \\ \Rightarrow \int_0^{M^\Delta(Bw_1', T(w_1', w_2'), t)} \psi(s) ds &\geq \int_0^{\emptyset(M^\Delta(Bw_1', T(w_1', w_2'), t))} \psi(s) ds \\ &\geq \int_0^{(M^\Delta(Bw_1', T(w_1', w_2'), t))} \psi(s) ds \end{aligned}$$

which is a contradiction. Hence  $Bw_1' \in T(w_1', w_2')$ .

Similarly, by putting  $x = y_n, y = x_n, u = w_2'$  and  $v = w_1'$  in (3), we get  $Bw_2' \in T(w_2', w_1')$ .

Again by taking  $x = w_1, y = w_2$ , and  $u = y_n$  and  $v = x_n$  in (3), we get

$m(w_1, w_2, y_n, x_n, t)$

$$\begin{aligned} &= \emptyset \left[ \begin{aligned} &a \left( M^\Delta(S(w_1, w_2), Aw_1, t) * M^\Delta(T(y_n, x_n), By_n, t) \right) \\ &+ M^\Delta \left( S(w_1, w_2), By_n, \frac{t}{2} \right) * M^\Delta \left( T(y_n, x_n), Aw_1, \frac{t}{2} \right) \end{aligned} \right] \\ &+ (1 - 2a) \max \left\{ M^\Delta(S(w_1, w_2), Aw_1, t), M^\Delta(T(y_n, x_n), By_n, t) \right\} \\ \int_0^{M^\Delta(S(w_1, w_2), T(y_n, x_n), t)} \psi(s) ds &\geq \int_0^{m(w_1, w_2, y_n, x_n, t)} \psi(s) ds, \text{ hence} \\ \int_0^{\lim_{n \rightarrow \infty} (M^\Delta(S(w_1, w_2), C, t))} \psi(s) ds &\geq \int_0^{\lim_{n \rightarrow \infty} m(w_1, w_2, y_n, x_n, t)} \psi(s) ds \\ &= \int_0^{\emptyset(M^\Delta(S(w_1, w_2), Aw_1, t))} \psi(s) ds \end{aligned}$$

As  $z_1 = Aw_1 \in C$ , we have

$$\begin{aligned} \int_0^{(M^\Delta(S(w_1, w_2), Aw_1, t))} \psi(s) ds &\geq \int_0^{M^\Delta(S(w_1, w_2), C, t)} \psi(s) ds \\ &\geq \int_0^{\emptyset(M^\Delta(S(w_1, w_2), Aw_1, t))} \psi(s) ds \\ &\geq \int_0^{M^\Delta(S(w_1, w_2), Aw_1, t)} \psi(s) ds \end{aligned}$$

Which is a contradiction. Hence  $Aw_1 \in S(w_1, w_2)$ .

Similarly, by putting  $x = w_2, y = w_1, u = x_n$  and  $v = y_n$  in (3),

we get  $z_2 = Aw_2 \in S(w_2, w_1)$ .

Hence  $(w_1, w_2) \in C(A, S)$  and  $(w_1', w_2') \in C(B, T)$ . Now (4), gives

$$AAw_1 = Aw_1, BBw_1' = Bw_1' \text{ and } S(Aw_1, Aw_2) = T(Bw_1', Bw_2')$$

$$AAw_2 = Aw_2, BBw_2' = Bw_2' \text{ and } S(Aw_2', Aw_1') = T(Bw_2', Bw_1').$$

But we have  $z_1 = Aw_1 = Bw_1', z_2 = Aw_2 = Bw_2'$ . This gives,

$$Az_1 = z_1 = Bz_1 \text{ and } S(z_1, z_2) = T(z_1, z_2)$$

$$Az_2 = z_2 = Bz_2 \text{ and } S(z_2, z_1) = T(z_2, z_1)$$

Also, weak compatibility of  $(A, S)$  gives  $AS(w_1, w_2) \in S(Aw_1, Aw_2)$

$$\emptyset z_1 = Bz_1 = Az_1 \in AS(w_1, w_2) \in S(Aw_1, Aw_2) = S(z_1, z_2) = T(z_1, z_2).$$

Similarly, we can have  $z_2 = Bz_2 = Az_2 \in S(z_2, z_1) = T(z_2, z_1)$ . Hence  $(z_1, z_2)$  is a common coupled fixed point of the mappings  $A, B, S$  and  $T$ .

**Example:** Let  $X = R$  and  $a * b = ab$  and  $M(x, y, t) = \frac{t}{t+d(x,y)}$  then  $(X, M)$  is a fuzzy metric space. Define  $A, B: X \rightarrow X$  and  $S, T: X \times X \rightarrow \kappa(X)$  by setting



$$Ax = \begin{cases} 2x - 1, & x \leq 1 \\ 3, & x > 1 \end{cases}, \quad Bx = \begin{cases} 2 - x, & 1 \leq x \leq 2 \\ 3, & \text{otherwise} \end{cases} \quad \text{and}$$

$$S(x, y) = \begin{cases} [x + y - 4, x + y + 2], & \text{if } x, y \in [0, 3] \\ [x - 1, y - 1], & \text{otherwise} \end{cases}$$

$$T(x, y) = \begin{cases} [2x - y + 1, 3x + y], & x < y \\ [x - 2y - 1, x + 3], & x \geq y \end{cases}$$

Consider the sequences,  $\{x_n\} = 1 - \frac{1}{n}$  and  $\{y_n\} = 1 + \frac{1}{n}$ , then

$$\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} By_n \rightarrow 1 \in \lim_{n \rightarrow \infty} S(x_n, y_n) \cap \lim_{n \rightarrow \infty} T(y_n, x_n)$$

$$\lim_{n \rightarrow \infty} Ay_n = \lim_{n \rightarrow \infty} Bx_n \rightarrow 3 \in \lim_{n \rightarrow \infty} S(y_n, x_n) \cap \lim_{n \rightarrow \infty} T(x_n, y_n)$$

This shows that (A, B) is tangential w.r.t (S, T)

$$A_1 = B_1 = 1 \in [0, 6] = S(1, 3) \cap T(3, 1)$$

Also,

$$A_3 = B_3 = 3 \in [0, 6] = S(3, 1) \cap T(1, 3)$$

Hence all the conditions of above theorems are satisfied and (1, 3) is a coupled fixed point of the maps A, B, S and T.

**3.2 Theorem:** Let  $A, B : X \rightarrow X$  and  $S, T : X \times X \rightarrow X$  be single and set-valued mappings satisfying the following conditions:

- (1) there exist contained coupled weak tangential points  $(z_1, z_2)$  to the mappings A and B.
- (2) (A, B) is tangential with respect to (S, T).

$$(3) \int_0^{M_{\nabla}(S(x, y), T(u, v), t)} \psi(s) ds \geq \int_0^{m(x, y, u, v, t)} \psi(s) ds \quad \text{where}$$

$m(x, y, u, v, t)$

$$= \left[ \begin{array}{cc} a \left( M^{\Delta}(S(x, y), Ax, t), M^{\Delta}(T(u, v), Bu, t) \right) & \\ + a \left( M^{\Delta} \left( S(x, y), Bu, \frac{t}{2} \right), M^{\Delta} \left( T(u, v), Ax, \frac{t}{2} \right) \right) & \\ + (1 - 2a) \max \left\{ M^{\Delta}(S(x, y), Ax, t) * M^{\Delta}(T(u, v), Bu, t), \right. & \\ \left. M_{\nabla}(S(x, y), Ax, t), M_{\nabla}(T(u, v), Bu, t) \right\} & \end{array} \right]$$

- a.  $AAa = Aa = BBc, S(Aa, Ab) = T(Bc, Bd)$  and  $AAb = Ab = BBd, S(Ab, Aa) = T(Bd, Bc)$  for  $(a, b) \in C(A, S)$  and  $(c, d) \in C(B, T)$
- b. the pair (A,S) is weakly compatible,

for all  $x, y, u, v \in X, 0 < a < 1$  and  $\phi: [0, 1] \rightarrow [0, \infty)$  non-decreasing map such that  $\phi(t) > t, t > 0$  then A, B, S and T have a common coupled fixed point in X.

Proof The result follows directly from theorem (3.1).

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