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A NEW CONSTRUCTION OF MULTIWAVELETS WITH COMPOSITE DILATIONS

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ABSTRACT

Consider an affine system $\mathcal{A}_{AB}(\Psi)$ with composite dilations D_a, D_b, in which $a \in A, b \in B$, $A, B \subseteq GL_n(\mathbb{R})$ and $\Psi \in L^2(\mathbb{R}^n)$. It can be made an orthonormal AB-multiwavelet Ψ or a parsval frame AB-wavelet Ψ, by choosing appropriate sets A and B. In this paper, we constructe an orthonormal AB-multiwavelet that arises from AB-multiresolution analysis . Our construction is useful since the group B is shear group. More generally, we give a parsval frame AB-wavelet.

Indexing terms/Keywords

Wavelet with composite dilation; orthonormal basis; parsval frame; multiwavelet .

SUBJECT CLASSIFICATION

Mathematics Subject Classification: Primary 43A15, Secondary 42C40.

1 Introduction and Preliminaries

A collection of the form

$$\mathcal{A}_A(\Psi) = \{D_a T_k \Psi : a \in A, k \in \mathbb{Z}^n\}$$

is an *Affine system*. If $\mathcal{A}_A(\Psi)$ is an orthonromal basis or, more generally a parsval frame for $L^2(\mathbb{R}^n)$, then Ψ is called an A-*wavelet* or *parsval frame* A-*wavelet*, repectively. The Affine system $\mathcal{A}_A(\Psi)$ where $|\hat{\Psi}| = \chi_\Omega$, for some measurable set $\Omega \subseteq \hat{R}^n$, is called *minimally supported in frequency* (MSF) system. If Ψ is a parsval frame A-wavelet for $L^2(S)^\vee$, the corresponding function Ψ is called an MSF *wavelet* for $L^2(S)^\vee$, in which $L^2(S)^\vee = \{f \in L^2(\mathbb{R}^n) : supp\hat{f} \subseteq S\}$, for some measurable set $S \subseteq \hat{R}^n$. Fang and Wang in [6] introduce the MSF wavelets, which are studied also in [12], [13]. In particular, Dai and Larson in [2] consider a special kind of MSF wavelets Ψ, which satisfy $\hat{\Psi} = \chi_\Omega$ for some measurable sets Ω in $\hat{R}$. They prove that such a $\Psi(x)$ is a wavelet with dilation set $\mathcal{D} = \{2^n : n \in \mathbb{Z}\}$ and translation set $\mathcal{L} = \mathbb{Z}$ if and only if

1. The sets $\{\Omega + \lambda : \lambda \in \mathbb{Z}\}$ is a tiling of $\hat{R}$.
2. The sets $\{2^n \Omega : n \in \mathbb{Z}\}$ is a tiling of $\hat{R}$.

The result is later extended to higher dimensions in [3] for $\mathcal{L} = \mathbb{Z}^n$ and $\mathcal{D} = \{A^n : n \in \mathbb{Z}\}$, where A is any expanding $n \times n$ matrix. One can show in [10], [15], that Ψ is an orthonormal basis A-wavelet for $L^2(S)^\vee$ if and only if $\hat{R}^n = \cup_{k \in Z^n} (\Omega + k)$ and $S = \cup_{a \in A} (\Omega a^{-1})$ where the unions are disjoint up to a set of measure zero. Also this result explain in [16], for $L^2(\mathbb{R}^n)$. The construction and the study of orthonormal bases and parsval frames is of major importance in several areas of mathematics and applications, recently. The motivation for this study comes partly from signal processing, where such bases are useful in image compression and feature extraction. ([5], [8]).

To be more precise, we need to fix some notation. Throughout this paper, we shall consider the points $x \in \mathbb{R}^n$ to be column vectors, i.e., $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, and the points $\xi \in \hat{R}^n$ of the frequency domain to be row vectors, i.e., $\xi = (\xi_1, \ldots, \xi_n)$. A vector x multiplying a on the left is a row vector. Thus, $ax \in \mathbb{R}^n$ and $\xi a \in \hat{R}^n$.

The Fourier transform of f is defined as

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \xi x} dx,$$

where $\xi \in \hat{R}^n$, and the invers Fourier transform is

$$\breve{f}(x) = \int_{\hat{\mathbb{R}}^n} f(\xi) e^{2\pi i \xi x} d\xi.$$

Let $L^2(\mathbb{R}^n)$ be the space of all squar integrable functions on $\mathbb{R}^n$. It is well known that a countably family $\{e_j : j \in \mathrm{J}\}$ in $L^2(\mathbb{R}^n)$ is a frame if there exist constants $0 < \alpha \le \beta < \infty$ satisfing

$$\alpha \| f \|_2^2 \le \sum_{j \in \mathrm{J}} | \langle f, e_j \rangle |^2 \le \beta \| f \|_2^2$$

for all $f \in L^2(\mathbb{R}^n)$. The family $\{e_j\}_{j \in \mathrm{J}}$ is called a normalize tight frame or parsval frame if $\alpha = \beta = 1$. Therefore , if $\{e_j\}_{j \in \mathrm{J}}$ is a parsval frame in $L^2(\mathbb{R}^n)$, then

$$\| f \|_2^2 = \sum_{j \in \mathrm{J}} | \langle f, e_j \rangle |^2$$

for each $f \in L^2(\mathbb{R}^n)$. This is equivalent to reproducing formula

$$v = \sum_{j \in \mathrm{J}} \langle f, e_j \rangle e_j \qquad (1)$$

for all $f \in L^2(\mathbb{R}^n)$, where the series (1) converges in the norm of $L^2(\mathbb{R}^n)$. Equation (1) shows that a parsval frame provides a basis-like representation. In general, a parsval frame need not be a basis. For more detailes about frames see [4],[14].

For the reader's convenience we recall some basic concept of tiling set and packing set. The subspace L in $\mathbb{R}^n$ is a lattice if $\mathrm{L} = AZ^n$, where $A \in GL_n(\mathbb{R})$. Given a measurable set $\Omega \subseteq \mathbb{R}^n$ and a lattice L in $\mathbb{R}^n$, it to be said Ω *tiles* $\mathbb{R}^n$ by L translation, or Ω is a *fundamental domain* of L if the following properties hold :

1. $\cup_{l \in \Omega} (\Omega + l) = \mathbb{R}^n \quad a.e.,$
2. $\mu((\Omega + l) \cap (\Omega + l')) = 0$ for any $l \ne l' \in \mathrm{L}$.

It is called Ω *packs* $\mathbb{R}^n$ by L translation if only (ii) holds. Equivalently, Ω tiles $\mathbb{R}^n$ by L if and only if

$$\sum_{l \in \mathrm{L}} \chi_\Omega(x - l) = 1 \quad for \quad a.e. \quad x \in \mathbb{R}^n,$$

and Ω packs $\mathbb{R}^n$ by L if and only if

$$\sum_{l \in \mathrm{L}} \chi_\Omega(x - l) \le 1 \quad for \quad a.e. \quad x \in \mathbb{R}^n.$$

Clearly, $\mu(\Omega) = | detA |$ if Ω tiles by L, and $\mu(\Omega) \le | detA |$ if Ω packs by L. Furthermore, if Ω packs $\mathbb{R}^n$ by L and $\mu(\Omega) = | detA |$, then Ω necessarily tiles $\mathbb{R}^n$ by L. We refer the reader to [11] for more detailes about lattice tiling.

In general, Blanchard in [1] considers the deffinition of tiling sets, for an arbitrary group G. Let G be a group acting from right on a measurable set $S \subseteq \mathbb{R}^n$. Then Ω is a G*-tiling set* for S , if

1. $\cup_{g \in G} \Omega g = S \quad a.e.$
2. $\mu(\Omega g_1 \cap \Omega g_2) = 0$ for $g_1 \ne g_2 \in G$.

In this note, we construct an admissible wavelet Ψ, that it arise from AB-multiresolution analysis. Also, we give, more generally, a parsval frame for $L^2(\mathbb{R}^2)$.

2 Main Result

In this section our notation will be the same as before. We first recall an AB-affine system and AB-MRA. Then we construct some examples of AB-affine system, which are an orthononal basis or parsval frame of $L^2(\mathbb{R}^2)$.

Let A and B be a countable subset of $GL_n(\mathbb{R})$. A collection of the form

$$\mathcal{A}_{AB}(\psi) = \{D_a D_b T_k \Psi : k \in \mathbb{Z}^n, a \in A, b \in B\},$$

is called **Affine systems with composite dilation**, or AB-**Affine system**, where $\Psi = \{\psi^1, \ldots, \psi^L\} \subset L^2(\mathbb{R}^n)$, and the operators T_k and D are called the translations and dilations, respectively, and defined as follows:

$$T_k f(x) = f(x-k),$$

and

$$D_a f(x) = |\det a|^{-1/2} f(a^{-1}x).$$

If $\mathcal{A}_{AB}(\psi)$ is an orthonormal basis (ON) or, more generally, a parsval frame (PF) for $L^2(\mathbb{R}^n)$, then Ψ is called an ON AB-*multiwavelet* or a *PF* AB-*multiwavelet*, respectively. Let $\mathcal{C} \subset GL_n(\mathbb{R})$ be a countable set contaning the identity matrix I and let $S \subset \hat{R}^n$ be a mesurable set. The set $\mathcal{C}$ is called S-*admissible* if tiling multiwavelets for $L^2(S)^\vee$ exist. In case $S = \hat{R}^n$, for simply $\mathcal{C}$ is called *admissible* (rather than $\hat{R}^n$ -admissible).

Associated with the Affine system with composite dilation, is the following generalization of the classical Multiresolution Analysis, that will be useful to construct more examples of AB multiwavelets, as well as examples with properties that are of great potentional in applications.

Let $B = \{b^j : j \in \mathbb{Z}\}$ be a collection of invertiable 2×2 matrices with $|\det b^j| = 1$, in which $b \in GL_n(\mathbb{R})$, and A be an invertiable 2×2 matrix with integer enteries. A sequence $\{V_i\}_{i \in \mathbb{Z}}$ of closed subspaces of $\mathbb{R}^n$ is called an AB- **Multiresolution Analysis** (AB-MRA) if the following holds :

1. $D_{b^j} T_k V_o = V_0$, for any $j \in \mathbb{Z}, k \in \mathbb{Z}^2$,
2. $V_i \subset V_{i+1}$, for each $i \in \mathbb{Z}$, where $V_i = D_a^{-i} V_o$,
3. $\cap_{i \in \mathbb{Z}} V_i = \{0\}$ and $\overline{\cup_{i \in \mathbb{Z}} V_i} = L^2(\mathbb{R}^2)$,
4. there exists $\phi \in L^2(\mathbb{R}^2)$ such that $\Phi_B = \{D_{b^j} T_k \phi : j \in \mathbb{Z}, k \in \mathbb{Z}^2\}$ is a semi- orthogonal Parsval frame for V_0 ; that is, Φ_B is a parsval frame for V_o and in addition, $D_{b^j} T_k \phi \perp D_{b^{j'}} T_{k'} \phi$ for any $j \neq j', j, j' \in \mathbb{Z}, k \neq k', k, k' \in \mathbb{Z}^2$.

The space V_0 is called an AB scaling space and the function ϕ is an AB scalling function for V_0. If in addition, Φ_B is an orthonormal basis, then ϕ is said an ON AB scaling function. (see [7], [8], [9]).

Now we need to explain a result by an elementary Fourier series argument.

Proposition 2.1 *Let* $I \subseteq \hat{R}^n$ *be a measurable set, that* $|I|<1$ *and* $\hat{\Psi} = \chi_I$*. Then, the collection* $\{F_k = M_k \hat{\Psi} : k \in Z^n\}$*, is a parsval frame for* $L^2(I)$*, in which* $M_k \hat{\Psi}(\xi) = e^{2\pi i \xi k} \hat{\Psi}(\xi)$*.*

Proof. First we show that $f = \sum_{k \in Z} \langle f, F_k \rangle F_k$, for each $f \in L^2(I)$. Indeed,

$$\| f - \sum_{k=-n}^{n} \langle f, F_k \rangle F_k \|_{L^2(I)}^2 = \int_I | f(x) - \sum_{k=-n}^{n} \langle f, F_k \rangle e^{2\pi i \xi k} \hat{\psi}(\xi) |^2 \, d\xi \leq \int_0^1 | f(x) - \sum_{k=-n}^{n} \langle f, e_k \rangle e^{2\pi i \xi k} |^2 \, d\xi \to 0$$

as $n \to \infty$

we consider, $\| F_k \|_{L^2(I)} = A$. So we have:

$$\| f \|_{L^2(I)}^2 = \langle f, f \rangle_{L^2(I)} = \langle \sum_{k \in Z} \langle f, F_k \rangle F_k, \sum_{k \in Z} \langle f, F_k \rangle F_k \rangle_{L^2(I)} = \sum_{k \in Z} | \langle f, F_k \rangle |^2 \| F_k \|_{L^2(I)}^2 = A \sum_{k \in Z} | \langle f, F_k \rangle |^2$$

After a normalization conclude that, the resteriction of the set $\{e^{2\pi i \xi k} : k \in Z^n\}$ to I, is a parsval frame for $L^2(I)$.

We show that, there exists a relationship between an orthonormal basis and a fundamental domain. Also, there exists a relationship between a parsval frame and packing set. Therefore, we have the following:

Proposition 2.2 *Let* $\Omega \subseteq \hat{R}^n$*, be a measurable set and* $\hat{\psi} = \chi_\Omega$*, in* $L^2(\Omega)$*. Then, the collection* $\{(T_k \psi)^\wedge = e^{2\pi i \xi k} \chi_\Omega : k \in Z^n\}$ *is an orthonormal basis for* $L^2(\Omega)$ *if and only if* Ω *is a fundamental domain.*

Proof. Suppose that, the collection $\{(T_k \psi)^\wedge = e^{2\pi i \xi k} \chi_\Omega : k \in Z^n\}$, is an orthonormal basis for $L^2(\Omega)$. Then, $\| e^{2\pi i (.) k} \chi_\Omega(.) \|_2^2 = 1$. On one hand,

$$\| e^{2\pi i (.) k} \chi_\Omega(.) \|_2^2 = \int_{R^n} | e^{2\pi i \xi k} |^2 | \chi_\Omega(\xi) |^2 \, d\xi = \int_{R^n} \chi_\Omega(\xi) d\xi = \mu(\Omega).$$

Thus, $\mu(\Omega) = 1$. Therefore, Ω is a fundamental domain.

Conversely, assume that Ω is a fundamental domain. As, $\mu((\Omega + k) \cap (\Omega + k')) = 0$, conclude the measure of Ω cannot be larger than one. Thus, by proposition 2.1, the collection $\{e^{2\pi i \xi k} \chi_\Omega : k \in Z^n\}$, is a parsval frame for $L^2(\Omega)$. On one hand, Ω is a fundamental domain. So, the measure of Ω is exactly one. Then, $\| e^{2\pi i (.) k} \chi_\Omega \| = 1$. Therefore, the collection $\{(T_k \psi)^\wedge = e^{2\pi i \xi k} \chi_\Omega : k \in Z^n\}$ is an orthonormal basis for $L^2(\Omega)$.

Proposition 2.3 *Let* $\Omega \subseteq \hat{R}^n$*, be a measurable set and* $\hat{\psi} = \chi_\Omega$*, in* $L^2(\Omega)$*. Then, the collection* $\{(T_k \psi)^\wedge = e^{2\pi i k \xi} \chi_\Omega : k \in Z^n\}$ *is a parsval frame for* $L^2(\Omega)$ *if and only if* Ω *is a packing set by translation of* Z^n*, for* $\hat{R}^n$*. i.e.* $\mu((\Omega + k) \cap (\Omega + k')) = 0$ *for* $k \neq k' \in Z^n$*.*

Proof. First let us suppose Ω is a packing set by Z^n translation, for $\hat{R}^n$. So, the measure of the set Ω cannot be larger than one. Then, by the proposition 2.1, the collection $\{(T_k\psi)^\wedge = e^{2\pi ik\xi}\chi_\Omega : k \in Z^n\}$ is a parsval frame for $L^2(\Omega)$.

Conversely, suppose that $\{(T_k\psi)^\wedge = e^{2\pi ik\xi}\chi_\Omega : k \in Z^n\}$ is a parsval frame for $L^2(\Omega)$. Then, the measure of Ω, cannot be larger than one. Since, by contradiction, if $|\Omega| > 1$, then the collection $\{e^{2\pi i\xi k}\chi_\Omega(\xi) : k \in Z^n\}$ cannot be a parsval frame. Thus, Ω is a packing set by translation of Z^n, for $\hat{R}^n$.

We need to stating some basic properties of the translation and dilation operators, that will be used throughout this paper.

Proposition 2.4 *Let*

$$G = \{U = D_a T_k : (a,k) \in GL_n(R) \times R^n\}.$$

G is a subgroup of the group of unitary operators on $L^2(R^n)$. We consider $\hat{U}\hat{f} = (Uf)^\wedge$. Then we have:

1. $D_a T_k = T_{ak} D_a$,
2. $D_{a_1} D_{a_2} = D_{a_1 a_2}$, for each $a_1, a_2 \in GL_n(R)$,
3. for $U = D_a T_k$, then $\hat{U} = D_{a^{-1}} M_{-k}$, where $D_{a^{-1}}\hat{f}(\xi) = |\det a|^{1/2}\hat{f}(\xi a)$,
4. $\hat{D}_a L^2(S) = L^2(Sa^{-1})$, for measurrable set $S \subset \hat{R}^n$, and $L^2(S) = \{\hat{f} \in L^2(\hat{R}^n) : supp\hat{f} \subseteq S\}$.

In the sequal we costruct an orthonormal AB-multiwavelet that arises from AB-multiresolution analysis. Also, we give a parsval frame AB-wavelet.

Example 2.5 *Let* $a = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, *and* $b = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$. *Let* $G = \{(b^j, k) : j \in Z, k \in Z^2\}$, *in which* $b^j = \begin{pmatrix} j+1 & j \\ -j & -j+1 \end{pmatrix}$. *Then* G *is a group with group multiplication:*

$$(b^l, m)(b^j, k) = (b^{l+j}, k + b^{-j}m). \qquad (2)$$

The identity element of this group is $(I, 0)$, so we have $(b^j, k)^{-1} = (b^{-j}, -b^j k)$. The multiplication (2) is consistent with the operation that maps $x \in R^2$ into $b^j(x+k) \in R^2$. This is clarified by introducing the unitary representation π of G, acting on $L^2(R^2)$, defined by

$$(\pi(b^j, k)f)(x) = f((b^j, k)^{-1}x) = f(b^{-j}x - k) = (D_b^j T_k f)(x), \qquad (3)$$

for $f \in L^2(R^2)$. The observation that

$$(D_b^l T_m)(D_b^j T_k) = (D_b^{l+j} T_{k + b^{-j}m}),$$

where $l, j \in \mathrm{Z}, k, m \in \mathrm{Z}^2$, shows how the group operation (2) is associated with the unitary representation (3).

Let $S_0 = \{\xi = (\xi_1, \xi_2) \in \hat{R}^2 : |\xi_2 - \xi_1| \leq 1\}$ and define

$$V_0 = L^2(S_0)^{\vee} = \{f \in L^2(\mathrm{R}^2) : supp\hat{f} \subset S_0\}.$$

For all $j \in \mathrm{Z}$ and $k \in \mathrm{Z}^2$, we have

$$(\pi(b^j, k)f)^{\wedge}(\xi) = (D_b^j T_k f)^{\wedge}(\xi) = e^{-2\pi i \xi b^j k} \hat{f}(\xi b^j), \tag{4}$$

and, $\xi b^j = (\xi_1, \xi_2) b^j = (j\xi_1 + \xi_1 - j\xi_2, j\xi_1 - j\xi_2 + \xi_2)$. Then the action of b^j maps the bias strip domain S_0 into itself. So the condition (i) of AB-MRA has been proved. Thus the space V_0 is invariant under the action of the operators $\pi(b^j, k)$.

Let

$$S_i = S_0 a^i = \{\xi = (\xi_1, \xi_2) \in \hat{R}^2 : |\xi_2 - \xi_1| \leq 2^i\},$$

and

$$V_i = \{f \in L^2(\mathrm{R}^2) : supp\hat{f} \subset S_i\}.$$

We can see that the space $\{V_i\}_{i\in \mathrm{Z}}$ satisfy the following properties :

$$(1) V_i \subset V_{i+1}, i \in \mathrm{Z};\ (2) D_a^{-i} V_o = V_i;\ (3) \cap_{i\in \mathrm{Z}} V_i = \{0\};\ (4) \overline{\cup_{i\in \mathrm{Z}} V_i} = L^2(\mathrm{R}^2).$$

consider $A = \{a^i : i \in \mathrm{Z}\}$, $B = \{b^j : j \in \mathrm{Z}\}$, and $U = U_1 \cup U_2$, where U_1 is a triangle with vertices at $(0,0), (-1,0), (0,1)$, and $U_2 = \{\xi \in \hat{R}^2 : -\xi \in U_1\}$. Define φ by $\hat{\varphi}(\xi) = \chi_U(\xi)$. A simple computation shows that U is a fundamental domain of Z^2 and a B-tiling region for S_0, too. That is, $\hat{R}^2 = \cup_{k\in \mathrm{Z}^2}(U + k)$ and $S_0 = \cup_{j\in \mathrm{Z}}(Ub^j)$, where the unions are disjoint up to a set of measure zero. Therefore, $\Phi_B = \{D_b T_k \varphi : b \in B, k \in \mathrm{Z}^2\}$ is an orthonormal basis of V_0 and φ is a scaling function of V_0. Since the dilation operator D_a^i is a unitary, thus the collection $\{D_a^i D_b^j T_k \varphi : j \in \mathrm{Z}, k \in \mathrm{Z}^2\}$, $i \in \mathrm{Z}$ is an orthonormal basis of V_i. Thus $\{V_i\}_{i\in \mathrm{Z}}$ is an AB-MRA with scaling function φ.

In order to have an orthonormal wavelet system, we must be obtained an orthogonal complement of V_0 in V_1. Let W_0 be an orthogonal complement V_0 in V_1, that is, $V_1 = V_0 \oplus W_0$. By the standard MRA wavelet construction, if we find an orthogonal basis for W_0 , then we have a wavelet system. Since $V_0 = L^2(S_0)^{\vee}$ and $V_1 = L^2(S_1)^{\vee}$ so we have

$$L^2(S_1)^{\vee} = L^2(S_0 a)^{\vee} = L^2((S_0 a \setminus S_0) \cup S_0)^{\vee} = L^2(S_0 a \setminus S_0)^{\vee} \oplus L^2(S_0)^{\vee}.$$

Then we define $W_0 = L^2(S_0 a \setminus S_0)^{\vee} = L^2(S_1 \setminus S_0)^{\vee}$. We set :

$$R_0 := S_1 \setminus S_0 = \{\xi = (\xi_1, \xi_2) \in \hat{\mathrm{R}}^2 : 1 \leq |\xi_2 - \xi_1| \leq 2\},$$

then

$$W_0 = \{ f \in L^2(\mathrm{R}^2) : supp\hat{f} \subset R_0 \}.$$

We shall now explain how to construct an AB-multiwavelet generated by three mutually orthogonal functions ψ^1, ψ^2, ψ^3 of norm 1. To do this, define the following subsets of $R_0 = S_1 \setminus S_0$:

$$E_1 = E_1^+ \cup E_1^-, E_2 = E_2^+ \cup E_2^-, E_3 = E_3^+ \cup E_3^-,$$

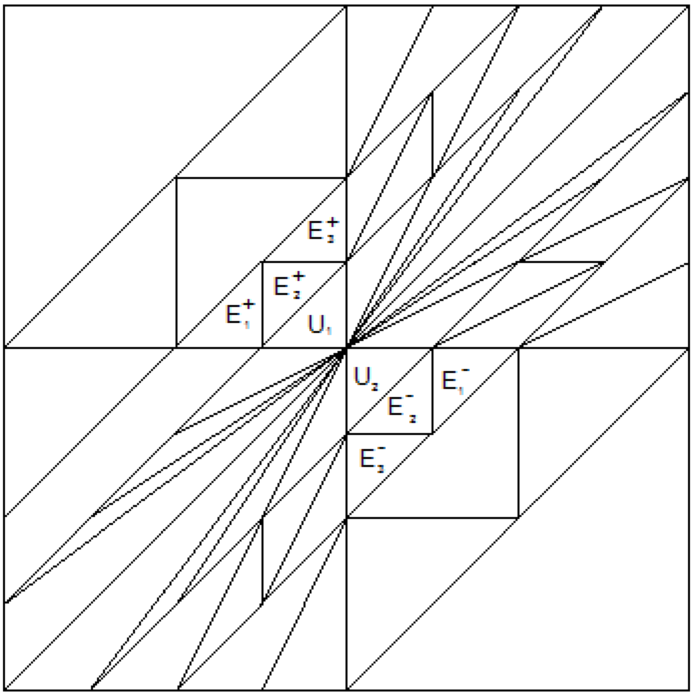
where

$$E_1^+ = \{\xi = (\xi_1, \xi_2) \in \hat{\mathrm{R}}^2 : -2 \le \xi_1 \le -1, 0 \le \xi_2 \le \xi_1 + 2\},$$

$$E_2^+ = \{\xi = (\xi_1, \xi_2) \in \hat{\mathrm{R}}^2 : -1 \le \xi_1 \le 0, \xi_1 + 1 \le \xi_2 \le 1\},$$

$$E_3^+ = \{\xi = (\xi_1, \xi_2) \in \hat{\mathrm{R}}^2 : -1 \le \xi_1 \le 0, 1 \le \xi_2 \le \xi_1 + 2\},$$

and $E_l^- = \{\xi \in \hat{\mathrm{R}}^2 : -\xi \in E_l^+\}, l = 1,2,3.$



We then define $\psi^l, l = 1,2,3$, by setting $\hat{\psi}^l = \chi_{E_l}, l = 1,2,3.$ Notice that each set E_l is a fundamental domain of Z^2, that is, the function $\{e^{-2\pi i \xi k} : k \in \mathrm{Z}^2\}$, restricted to E_l form an orthonormal basis of $L^2(E_l)$. It follows that the collection $\{e^{-2\pi i \xi k}\hat{\psi}^l(\xi) : k \in \mathrm{Z}^2\}$ is an orthonormal basis of $L^2(E_l), l = 1,2,3$. A simple direct calculation shows that the sets $\{E_l b^{-j} : j \in \mathrm{Z}, l = 1,2,3\}$ are a partition of R_0, that is,

$$\cup_{l=1}^{3} \cup_{j \in \mathrm{Z}} E_l b^{-j} = R_0, \quad (5)$$

where the union is disjoint.

But the dilations D_b^{-j} are unitary operators. Hence they maps an orthonormal basis into an orthonormal basis. Thus for each $j \in \mathbb{Z}$, the set $\{e^{-2\pi i \xi k}\hat{\psi}^l(\xi b^j) : k \in \mathbb{Z}^2\}$ is an orthonormal basis for $L^2(E_l b^{-j})$. It follows from (5), that

$$L^2(R_0) = \oplus_{l=1}^{3} \oplus_{j\in \mathbb{Z}} L^2(E_l b^{-j}). \tag{6}$$

Since, for each fixed $j \in \mathbb{Z}$, b^j maps $\mathbb{Z}^2$ into itself, the collection $\{e^{-2\pi i \xi k}\hat{\psi}^l(\xi b^j) : k \in \mathbb{Z}^2\}$ is equal to the collection $\{e^{-2\pi i \xi b^j k}\hat{\psi}^l(\xi b^j) : k \in \mathbb{Z}^2\}$. It follows from (6), that the collection

$$\{e^{-2\pi i \xi k}\hat{\psi}^l(\xi b^j) : k \in \mathbb{Z}^2, j \in \mathbb{Z}, l = 1,2,3\} = \{e^{-2\pi i \xi b^j k}\hat{\psi}^l(\xi b^j) : k \in \mathbb{Z}^2, j \in \mathbb{Z}, l = 1,2,3\}$$

is an orthonormal basis of $L^2(R_0)$. Thus, by taking the inverse Fourier transform, we have that $\{D_b^j T_k \psi^l : j \in \mathbb{Z}, k \in \mathbb{Z}^2, l = 1,2,3\}$ is an orthonormal basis of $W_0 = L^2(R_0)^\vee$. In order to obtain the desired ON AB-affine system for $L^2(\mathbb{R}^2)$, we apply the dilations $D_a^i, i \in \mathbb{Z}$ to the orthonormal basis. The dilations operators D_a^i, for each $i \in \mathbb{Z}$, maps R_0 into R_i, in which

$$R_i = R_0 a^i = \{\xi = (\xi_1, \xi_2) \in \hat{\mathbb{R}}^2 : 2^i \le |\xi_2 - \xi_1| \le 2^{i+1}\},$$

and we have $\cup_{i\in \mathbb{Z}} R_i = \hat{\mathbb{R}}^2$, where the unions are disjoint. Using the unitary operators D_a^i, for each $i \in \mathbb{Z}$, thus the set $\{D_a^i \pi(b^j, k)\psi^l : k \in \mathbb{Z}^2, j \in \mathbb{Z}, l = 1,2,3\}$ is an orthonormal basis of $L^2(R_i)^\vee = W_i$. Since the spaces $L^2(R_i)$ (and thus the spaces W_i) are mutually orthogonal, it follows that the system

$$\{D_a^i \pi(b^j, k)\psi^l : k \in \mathbb{Z}^2, i, j \in \mathbb{Z}, l = 1,2,3\} = \{D_a^i D_b^j T_k \psi^l : k \in \mathbb{Z}^2, i, j \in \mathbb{Z}, l = 1,2,3\}$$

is an orthonormal basis of $L^2(\mathbb{R}^2) = \oplus_{i\in \mathbb{Z}} W_i$, that is, $\Psi = \{\psi^1, \psi^2, \psi^3\}$ is an ON AB-multiwavelet.

The number of generators of this AB-multiwavelet is fixed. Infact, by the next proposition, if we could replace Ψ by $\Phi = \{\phi^1, \ldots \phi^L\}$, then $L = 3$.

Proposition 2.6 *([8], [9]): Let G be a countable set and, for each $u \in G$, let T_u be a unitary operator acting on a Hilbert space $\mathbb{H}$. Assume that, for each T_u, there is a unique $u^* \in G$ such that $T_{u^*} = T_u^*$. Suppose $\Phi = \{\phi^1, \ldots, \phi^N\}$, $\Psi = \{\psi^1, \ldots, \psi^M\} \subset \mathbb{H}$, where $N, M \in \mathbb{N} \cup \{\infty\}$. If $\{T_u \psi^k : u \in G, 1 \le k \le N\}$ and $\{T_u \psi^i : u \in G, 1 \le i \le M\}$ are each orthonormal basis for $\mathbb{H}$, then $N = M$.*

The following result establishes the number of generators needed to obtain an orthonormal MRA AB-wavelet.

Theorem 2.7 *([8], [9]): Let $\Psi = \{\psi^1, \ldots, \psi^L\}$ be an orthonormal MRA AB-multiwavelet for $L^2(\mathbb{R}^n)$, and let $N = |B/aBa^{-1}|$ (= the order of quotient group B/aBa^{-1}). Assume that $|\det a| \in \mathbb{N}$. Then $L = N|\det a| - 1$.*

By using this theorem, we can calculate the number of AB-multiwavelet.

Remark 2.8 *In example (2.5), the set* B *is considred as* $B = \{b^j : j \in Z\}$ *in which,* $b^j = \begin{pmatrix} j+1 & j \\ -j & -j+1 \end{pmatrix}$. *By a simple calculation, we get* $ab^j a^{-1} = b^j$, *thus,* $aBa^{-1} = <b>$ *and it is clearly* $B = <b>$. *Then* $B/aBa^{-1}; I_{2\times 2}$, *thus* $N = |B/aBa^{-1}| = 1$. *Therefore,* $L = N|deta| - 1 = 1.4 - 1 = 3$.

Now we give a parsval frame wavelet with composite dilation from AB-MRA with a single generator.

Example 2.9 *Let* $F = F_1 \cup F_2$, *where* F_1 *is a trapezoid with vertices* $(-1,0)$, $(-\frac{1}{2},0)$, $(0,\frac{1}{2})$, $(0,1)$, *and* $F_2 = \{\xi = (\xi_1,\xi_2) \in \hat{R}{}^2 : -\xi \in F_1\}$. *Suppose that* S_i, A *and* B *are defined in Example (2.5), and let* $H := S_0 \setminus S_{-1} = \{\xi = (\xi_1,\xi_2) \in \hat{R}{}^2 : \frac{1}{2} \le |\xi_2 - \xi_1| \le 1\}$. *A simple direct computation shows that* $H = \cup_{j\in Z} Fb^j$, *where the union is disjoint. It follows from the Plancherel theorem (using the fact that* F *is contained inside a fundamental domain) that the function* $\chi_F(\xi)$ *satisfies* $\sum_{k\in Z^2} |\langle \hat{f}, e^{2\pi i(.)k}\chi_F \rangle|^2 = \| \hat{f} \|^2$, *for all* $\hat{f} \in L^2(F)$, *and the collection*

$$\{D_b^j e^{2\pi i\xi k}\chi_F(\xi) : k \in Z^2, j \in Z\}$$

is a parsval frame of $L^2(H)$. Similarly to the construction above, we have $\hat{R}{}^2 = \cup_{i\in Z} Ha^i$, where the union is disjoint. Define ψ by setting $\overset{\wedge}{\psi} = \chi_F$. It follows that the system

$$\{D_a^i D_b^j T_k \psi : i, j \in Z, k \in Z^2\},$$

is a parsval frame of $L^2(R^2)$. That is to say the function ψ, is a parsval frame wavelet with composite dilations.

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