

APPROXIMATION OF THE QUADRATIC DOUBLE CENTRALIZERS AND QUADRATIC MULTIPLIERS ON NON-ARCHIMEDEAN BANACH ALGEBRAS

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ABSTRACT

In this paper, we establish stability of quadratic double centralizers and quadratic multipliers on non-Archimedean Banach algebras. We also prove the superstability of quadratic double centralizers on non-Archimedean Banach algebras which are weakly commutative and weakly without order, and of quadratic multipliers on non-Archimedean Banach algebras which are weakly without order.

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1. INTRODUCTION

The stability problem of functional equations originated from a question of Ulam[11] in 1940, concerning the stability of group homomorphisms. Let $(G_1, +)$ be a group and let $(G_2, +)$ be a metric group with the metric $d(.,.)$. Given $\epsilon > 0$, does there exist a $\delta > 0$, such that if a mapping $h:G_1 \rightarrow G_2$ satisfies the inequality $d(h(x, y), h(x) * h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \rightarrow G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$? In the other words, Under what condition does there exist a homomorphism near an approximate homomorphism? The concept of stability for functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. In 1941, D.H. Hyers [6] gave a first affirmative answer to the question of Ulam for Banach space. Let $f : E \rightarrow E'$ be a mapping between Banach spaces such that

$$\|f(x+y) - f(x) - f(y)\| \leq \delta$$

for all $x, y \in E$, and for some $\delta > 0$. Then there exists a unique additive mapping $T : E \rightarrow E'$ such that

$$\|f(x) - T(x)\| \leq \delta$$

for all $x \in E$. Moreover, if $f(tx)$ is continuous in t for each fixed $x \in E$, then T is linear. In 1950, T. Aoki [1] was the second author to treat this problem for additive mapping. Finally in 1978, Th. M. Rassias[8] proved the following Theorem: **Theorem** (Th. M. Rassias). Let $f : E \rightarrow E'$ be a mapping from a norm vector space E into a Banach space E' subject to the inequality

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p)$$

for all $x, y \in E$, where ϵ and p are constants with $\epsilon > 0$ and $p < 1$. Then there exists a unique additive mapping $T : E \rightarrow E'$ such that

$$\|f(x) - T(x)\| \leq \frac{2\epsilon}{2-2^p} \|x\|^p$$

for all $x \in E$. Also, if the function $t \rightarrow f(tx)$ from R into E' is continuous for each fixed $x \in E$, then T is linear. This stability phenomenon of this kind is called the Hyers-Ulam-Rassias stability. In 1991, Z. Gajda[3] answered the question for the case $p < 1$, which was raised by Rassias. In 1994, a generalization of the Rassias' theorem was obtained by Gavruta as follows [4]. The functional equation is called stable if any function satisfying that functional equation "approximately" is near to a true solution of functional equation. We say that a functional equation is superstable if every approximately solution is an exact solution of it.

Suppose that A is a Banach algebra. Recall that $A_l(A) := \{a \in A : aA = \{0\}\}$ is the left annihilator ideal and $A_r(A) := \{a \in A : Aa = \{0\}\}$ is the right annihilator ideal on A . A Banach algebra A is said to be strongly without order if $A_l(A) = A_r(A) = \{0\}$. We say that a Banach algebra A is quartic without order if $\{r \in A : \{ra^2; a \in A\} = \{0\}\} = \{0\} = \{r \in A : \{a^2r; a \in A\} = \{0\}\}$. It is not hard to see that if A is weakly without order then A is strongly without order.

A linear mapping $L : A \rightarrow A$ is said to be left centralizer on A if $L(ab) = L(a)b$ for all $a, b \in A$. Similarly, a linear mapping $R : A \rightarrow A$ that $R(ab) = aR(b)$ for all $a, b \in A$ is called right centralized on A . A double centralizer on A is a pair (L, R) , where L is a left centralizer, R is a right centralizer and $aL(b) = R(a)b$ for all $a, b \in A$. For example, (L_c, R_c) is a double

centralizer, where $L_c(a) := ca$ and $R_c(a) := ac$. The set $D(A)$ of all double centralizers equipped with the multiplication

$(L_1, R_1)(L_2, R_2) = (L_1L_2, R_2R_1)$ is an algebra. The notion of double centralizer was introduced by Hochschild[5] and by Johnson [7]. Johnson [7] proved that if A is an algebra satisfying $A_l(A) = A_r(A) = \{0\}$, and L, R are mappings on A

fulfilling $aL(b) = R(a)b$, $(a, b \in A)$, then (L, R) is a double centralizer. We can show that if $A^2 = A$ or $A_l(A) \cap A_r(A) = \{0\}$, then $L = R$ if and only if L and R are both left and right centralizer.

In particular, one of the important functional equations is the following functionalequation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{1.1}$$



which is called a quadratic functional equation. The function $f(x) = bx^2$ is a solution of this functional equation. Every solution of functional equation (1.1) is said to be a quadratic mapping. It is well known that a mapping f between real vector spaces is quadratic if and only if there exists a unique symmetric biadditive mapping B is given by

$$B(x, y) = \frac{1}{4}(f(x+y) - f(x-y))$$

The stability of quadratic functional equation (1.1) was proved by Skof [10] for mapping $f : E_1 \rightarrow E_2$ where E_1 is a normed space and E_2 is a Banach space. Cholewa[3] noticed that the theorem of Skof is still true if the relevant domain E_1 is replaced by an Abelian group.

A Banach algebra A is said to be *weakly commutative* if $(ab)^2 = a^2b^2$ for all $a, b \in A$. We can show that there is a Banach algebra weakly commutative that is not commutative (see Example 2.4 of the present paper).

Let K be a field. A non-Archimedean absolute value on K is a function $|\cdot| : K \rightarrow \mathbb{R}$ such that for any $a, b \in K$ we have

(i) $|a| \geq 0$ and equality holds if and only if $a = 0$,

(ii) $|ab| = |a||b|$,

(iii) $|a+b| \leq \max\{|a|, |b|\}$.

The condition (iii) is called the strict triangle inequality. By (ii), we have $|1| = |-1| = 1$. Thus, by induction, it follows from (iii) that $|n| \leq 1$ for each integer n . We always assume in addition that $|\cdot|$ is non-trivial, i.e., that there is an $a_0 \in K$ such that $|a_0| \notin \{0, 1\}$.

Let X be a linear space over a scalar field K with a non-Archimedean non-trivial Valuation $|\cdot|$. A function $\|\cdot\| : X \rightarrow \mathbb{R}$ is a non-Archimedean norm (valuation) if it satisfies the following conditions:

(NA1) $\|x\| = 0$ if and only if $x = 0$;

(NA2) $\|rx\| = |r|\|x\|$ for all $r \in K$ and $x \in X$;

(NA3) the strong triangle inequality (ultrametric); namely,

$$\|x+y\| \leq \max\{\|x\|, \|y\|\} \quad (x, y \in X).$$

Then $(X, \|\cdot\|)$ is called a non-Archimedean space.

It follows from (NA3) that

$$\|x_m + x_l\| \leq \max\{\|x_{j+1} - x_j\| : l \leq j \leq m-1\} \quad (m > l),$$

Therefore a sequence $\{x_m\}$ is Cauchy in X if and if $\{x_{m+1} - x_m\}$ converges to zero in non-Archimedean space. By a complete non-Archimedean space we mean on in which every Cauchy sequence is convergent. A non-Archimedean Banach algebra is a complete non-Archimedean algebra A which satisfies $\|ab\| \leq \|a\|\|b\|$ for all $a, b \in A$. For more detailed definitions of non-Archimedean Banach algebra, we can refer to [9].

2. MAIN RESULTS

In this section, let A be a non-Archimedean Banach algebra. We establish the stability of quadratic double centralizers.

Definition 2.1. A mapping $L : A \rightarrow A$ is a quadratic left centralizer if L satisfies the following properties:

1) L is a quadratic mapping,

2) L is a quadratic homogeneous, that is, $L(\lambda a) = |\lambda|^2 L(a)$ for all $a \in A$ and $\lambda \in C$,

3) $L(ab) = L(a)b^2$ for all $a, b \in A$.

Definition 2.2. A mapping $R : A \rightarrow A$ is a quadratic right centralizer if R satisfies the following properties:

1) R is a quadratic mapping,

2) R is quadratic homogeneous, that is, $R(\lambda a) = |\lambda|^2 R(a)$ for all $a \in A$ and $\lambda \in C$,

3) $R(ab) = a^2 R(b)$ for all $a, b \in A$.

Definition 2.3. A quadratic double centralizer of an algebra A is a pair (L, R) , where L is a quadratic left centralizer,

R is a quadratic right centralizer and $a^2 L(b) = R(a)b^2$ for all $a, b \in A$.

The following example introduces a quadratic double centralizer.

Example 2.4. Let $(A, \|\cdot\|)$ be a unital non-Archimedean Banach algebra. Let $B = A \times A \times A$. We define $\|a\| = \|a_1\| + \|a_2\| + \|a_3\|$ for all $a = (a_1, a_2, a_3)$ in B . It is not hard to see that $(B, \|\cdot\|)$ is a Banach space. For arbitrary elements $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$ in B , we define $ab = (0, a_1 b_3, 0)$. Since A is a non-Archimedean Banach algebra, we conclude that B is a non-Archimedean Banach algebra.

It is easy to see that $B^3 = \{abc : a, b, c \in B\} = \{0\}$. But $B^2 = \{ab : a, b \in B\}$ is not zero. Now we consider the mapping $T : B \rightarrow B$ defined by

$$T(a) = a^2 \quad (a \in B).$$

Then T is a quadratic mapping and quadratically homogeneous. Since $B^3 = \{0\}$, we get

$$T(ab) = (ab)^2 = 0 = a^2 b^2 = T(a)b^2 = a^2 T(b)$$

and

$$a^2 T(b) = a^2 b^2 = 0 = T(a)b^2$$

For all $a, b \in B$. Hence (T, T) is a quadratic double centralizer of B .

In the above example, B is a weakly commutative algebra, but it is not commutative.

Theorem 2.5. Suppose that $s \in \{-1, 1\}$ and that $f : A \rightarrow A$ is a mapping with $f(0) = 0$ for which there exist a mapping

$g : A \rightarrow A$ with $g(0) = 0$ and functions $\varphi_j : A \times A \times A \times A \rightarrow [0, \infty)$, $\psi_i : A \times A \rightarrow [0, \infty)$ ($1 \leq j \leq 2, 1 \leq i \leq 3$) such that

$$\tilde{\varphi}_j(a, b, c, d) := \sum_{k=0}^{\infty} \frac{\varphi_j(2^{sk} a, 2^{sk} b, 2^{sk} c, 2^{sk} d)}{|4|^{sk}} < \infty \quad (1 \leq j \leq 2), \quad (2.1)$$

$$\lim_{n \rightarrow \infty} \frac{\psi_i(2^{sn} a, b)}{|4|^{sn}} = 0 = \lim_{n \rightarrow \infty} \frac{\psi_i(a, 2^{sn} b)}{|4|^{sn}} \quad (1 \leq j \leq 3),$$

$$\begin{aligned} \left\| f(\lambda a + \lambda b + \lambda c) + f(\lambda a - \lambda b - \lambda c) - 2\lambda^2 f(a) - 2\lambda^2 f(b) - 2\lambda^2 f(c) \right\| &\leq \varphi_1(a, b, c, d) \\ \left\| g(\lambda a + \lambda b + \lambda c) + g(\lambda a - \lambda b - \lambda c) - 2\lambda^2 g(a) - 2\lambda^2 g(b) - 2\lambda^2 g(c) \right\| &\leq \varphi_2(a, b, c, d) \end{aligned} \quad (2.2)$$

$$\left\| f(ab) - f(a)b^2 \right\| \leq \psi_1(a, b) \quad (2.3)$$

$$\left\| g(ab) - a^2 g(b) \right\| \leq \psi_2(a, b)$$

$$\left\| a^2 f(b) - g(a)b^2 \right\| \leq \psi_3(a, b) \quad (2.4)$$

for all $a, b \in A$ and all $\lambda \in T = \{\lambda \in C : |\lambda| = 1\}$. Also, if for each fixed $a \in A$ the mappings $t \rightarrow f(ta)$ and $t \rightarrow g(ta)$ from R to A are continuous, then there exists a unique quadratic double centralizer (L, R) on A satisfying

$$\left\| f(a) - L(a) \right\| \leq \frac{1}{|4|} \tilde{\varphi}_1(a, a, 0, 0), \quad (2.5)$$

$$\left\| g(a) - R(a) \right\| \leq \frac{1}{|4|} \tilde{\varphi}_2(a, a, 0, 0), \quad (2.6)$$

for all $a \in A$.

Proof: Let $s = 1$. Putting $a = b, c = d = 0$ and $\lambda = 1$ in (2.2), we have

$$\left\| f(2a) - 4f(a) \right\| \leq \varphi_1(a, a, 0, 0)$$

for all $a \in A$. One can use induction to show that

$$\left\| \frac{f(2^n a)}{|4|^n} - \frac{f(2^m a)}{|4|^m} \right\| \leq \frac{1}{|4|} \sum_{k=m}^{n-1} \frac{\varphi_1(2^k a, 2^k a, 0, 0)}{|4|^k} \quad (2.7)$$

for all $n > m \geq 0$ and all $a \in A$. It follows from (2.7) and (2.1) that sequence $\left\{ \frac{f(2^n a)}{|4|^n} \right\}$ is Cauchy. Since A is a non-

Archimedean Banach algebra, this sequence is convergent. Define

$$L(a) := \lim_{n \rightarrow \infty} \frac{f(2^n a)}{|4|^n}. \quad (2.8)$$



Replacing a and b by $2^n a$ and $2^n b$, respectively, in (2.2), we get

$$\left\| \frac{f(2^n(\lambda a + \lambda b))}{4^n} + \frac{f(2^n(\lambda a - \lambda b))}{4^n} - 2\lambda^2 \frac{f(2^n a)}{4^n} - 2\lambda^2 \frac{f(2^n b)}{4^n} \right\| \leq \frac{\varphi_1(2^n a, 2^n b, 0, 0)}{|4|^n}$$

Taking the limit as $n \rightarrow \infty$, we obtain

$$L(\lambda a + \lambda b) + L(\lambda a - \lambda b) = 2|\lambda|^2 L(a) + 2|\lambda|^2 L(b) \quad (2.9)$$

for all $a, b \in A$ and all $\lambda \in T$. Putting $\lambda = 1$ in (2.9), we obtain that L is a quadratic mapping. Setting $b := a$ in (2.9), we get

$$L(2\lambda a) = 4|\lambda|^2 L(a)$$

for all $a \in A, \lambda \in T$. But L is a quadratic mapping. So

$$L(\lambda a) = |\lambda|^2 L(a)$$

for all $a \in A$ and all $\lambda \in T$. Under the assumption that $f(ta)$ is continuous in $t \in R$ for each fixed $a \in A$, by the same reasoning as in the proof of [8], $L(\lambda a) = \lambda^2 L(a)$ for all $a \in A$ and all $\lambda \in R$. we obtain

$$L(\lambda a) = \lambda^2 L(a)$$

for all $a \in A$ and $\lambda \in C(\lambda \neq 0)$. This means that L is quadratic homogeneous. It follows from (2.3) and (2.8) that

$$\left\| L(ab) - L(a)b^2 \right\| = \lim_{n \rightarrow \infty} \frac{1}{|4|^n} \left\| f(2^n ab) - f(2^n a)b^2 \right\| \leq \lim_{n \rightarrow \infty} \frac{\varphi_1(2^n a, b)}{|4|^n} = 0$$

for all $a, b \in A$. Hence L is a quadratic left centralizer on A . Applying (2.7) with $m = 0$, we get $\|L(a) - f(a)\| \leq \frac{1}{|4|} \tilde{\varphi}_1(a, a, 0, 0)$

for all $a \in A$. It is well known that the quadratic mapping L satisfying (2.5) is unique. A similar argument gives us a unique quadratic right centralizer R defined by

$$\square R(a) := \lim_{n \rightarrow \infty} \frac{g(2^n a)}{4^n}$$

which satisfies (2.6). Now we let $a, b \in A$ arbitrarily. Since L is a quadratic homogeneous, it follows from (2.4) and (2.5) that

$$\begin{aligned} \left\| a^2 L(b) - R(a)b^2 \right\| &= \frac{1}{|4|^n} \left\| a^2 L(2^n b) - 4^n R(a)b^2 \right\| \\ &\leq \frac{1}{|4|^n} \left[\left\| a^2 L(2^n b) - a^2 f(2^n b) \right\| + \left\| a^2 f(2^n b) - g(a)(4^n b^2) \right\| \right. \\ &\quad \left. + \left\| 4^n g(a)b^2 - 4^n R(a)b^2 \right\| \right] \\ &\leq \frac{1}{|4|^{n+1}} \tilde{\varphi}_1(2^n b, 2^n a) \|a\|^2 + \frac{\varphi_3(a, 2^n b)}{|4|^n} + \|g(a) - R(a)\| \|b\|^2. \end{aligned}$$

The right hand side of the last inequality tends to $\|g(a) - R(a)\| \|b\|^2$ as $n \rightarrow \infty$.

By (2.6), we obtain

$$\left\| a^2 L(b) - R(a)b^2 \right\| = \frac{1}{|4|} \tilde{\varphi}_2(a, a, 0, 0) \|b\|^2.$$

Since R is a quadratic mapping, we thus obtain

$$\begin{aligned} \left\| a^2 L(b) - R(a)b^2 \right\| &= \frac{1}{|4|^n} \left\| 4^n a^2 L(b) - R(2^n a)b^2 \right\| \\ &\leq \frac{1}{|4|} \tilde{\varphi}_2(2^n a, 2^n a, 0, 0) \|a\|^2 \\ &= \frac{1}{|4|} \sum_{k=n}^{\infty} \frac{\varphi_2(2^k a, 2^k a)}{|4|^k} \|b\|^2. \end{aligned}$$

Passing to the limit as $n \rightarrow \infty$, we conclude $a^2 L(b) = R(a)b^2$. Thus (L, R) is a quadratic double centralizer. The proof for $s = -1$ is similar to $s = 1$.

Corollary 2.6. Suppose that $f : A \rightarrow A$ is a mapping for which there exist a mapping $g : A \rightarrow A$ and constants $\varepsilon > 0$ and $0 \leq p \neq 2$ such that

$$\begin{aligned} \left\| f(\lambda a + \lambda b + \lambda c) + f(\lambda a - \lambda b - \lambda c) - 2\lambda^2 f(a) - 2\lambda^2 f(b) - 2\lambda^2 f(c) \right\| &\leq \varepsilon(\|a\|^p + \|b\|^p, \|c\|^p + \|d\|^p), \\ \left\| g(\lambda a + \lambda b + \lambda c) + g(\lambda a - \lambda b - \lambda c) - 2\lambda^2 g(a) - 2\lambda^2 g(b) - 2\lambda^2 g(c) \right\| &\leq \varepsilon(\|a\|^p + \|b\|^p, \|c\|^p + \|d\|^p), \\ \left\| f(ab) - f(a)b^2 \right\| &\leq \varepsilon \|a\|^p \|b\|^p, \\ \left\| g(ab) - a^2 g(b) \right\| &\leq \varepsilon \|a\|^p \|b\|^p, \\ \left\| a^2 f(b) - g(a)b^2 \right\| &\leq \varepsilon \|a\|^p \|b\|^p \end{aligned}$$

for all $a, b \in A$ and all $\lambda \in T$. Also, if for each fixed $a \in A$ the mappings $t \rightarrow f(ta)$ and $t \rightarrow g(ta)$ from R to A are continuous, then there exists a unique quadratic double centralizer (L, R) on A satisfying

$$\begin{aligned} \left\| f(a) - L(a) \right\| &\leq \frac{2\varepsilon}{|4|-|2|^p} \|a\|^p, \\ \left\| g(a) - R(a) \right\| &\leq \frac{2\varepsilon}{|4|-|2|^p} \|a\|^p \end{aligned}$$

for all $a \in A$.

Proof: For $j = 1, 2$, putting $\varphi_j(a, b) = \varepsilon(\|a\|^p + \|b\|^p + \|c\|^p + \|d\|^p)$ and for $i = 1, 2, 3$ putting $\psi_i(a, b) = \varepsilon\|a\|^p\|b\|^p$ in Theorem 2.5, we get the desired results.

3. STABILITY OF QUADRATIC MULTIPLIERS

Throughout this section, assume that A is a non-Archimedean Banach algebra.

Definition 3.1. We say that a mapping $T : A \rightarrow A$ is a quadratic multiplier if T satisfies the following properties:

- 1) T is a quadratic mapping,
- 2) T is quadratic homogeneous, that is, $T(\lambda a) = \lambda^2 T(a)$ for all $a \in A$ and $\lambda \in C$,
- 3) $a^2 T(b) = T(a)b^2$ for all $a, b \in A$.

Example 2.4 introduces a quadratic multiplier. We investigate the stability of quadratic multipliers.

Theorem 3.2. Suppose that $s \in \{-1, 1\}$ and that $f : A \rightarrow A$ is a mapping with $f(0) = 0$ for which there exist functions, $\varphi : A \times A \times A \times A \rightarrow [0, \infty)$, $\psi : A \times A \rightarrow [0, \infty)$ such that

$$\tilde{\varphi}(a, b, c, d) := \sum_{k=0}^{\infty} \frac{\varphi(2^{sk} a, 2^{sk} b, 2^{sk} c, 2^{sk} d)}{|4|^{sk}} < \infty, \quad (3.1)$$

$$\lim_{n \rightarrow \infty} \frac{\psi(2^{sn} a, b)}{|4|^{sn}} = 0 = \lim_{n \rightarrow \infty} \frac{\psi(a, 2^{sn} b)}{|4|^{sn}},$$

$$\begin{aligned} \left\| f(\lambda a + \lambda b + \lambda c) + f(\lambda a - \lambda b - \lambda c) - 2\lambda^2 f(a) - 2\lambda^2 f(b) - 2\lambda^2 f(c) \right\| &\leq \varphi(a, b, c, d), \\ \left\| a^2 f(b) - f(a)b^2 \right\| &\leq \psi(a, b) \end{aligned} \quad (3.2)$$

for all $a, b \in A$ and all $\lambda \in T$. Also, if for each fixed $a \in A$ the mappings $t \rightarrow f(ta)$ from R to A are continuous, then there exists a unique quadratic multiplier T on A satisfying

$$\left\| f(a) - T(a) \right\| \leq \frac{1}{|4|} \tilde{\varphi}(a, a, 0, 0), \quad (3.3)$$

for all $a \in A$.

Proof. Let $s = 1$. Putting $c = d = 0$. By the same reasoning as in the proof of Theorem 2.5, there exists a unique quadratic mapping $T : A \rightarrow A$ defined by

$$T(a) := \lim_{n \rightarrow \infty} \frac{f(2^n a)}{4^n}$$

with satisfying $T(\lambda a) = |\lambda|^2 T(a)$ for all $a \in A$ and all $\lambda \in C$. Also, $\left\| f(a) - T(a) \right\| \leq \frac{1}{|4|} \tilde{\varphi}(a, a, 0, 0)$ for all $a \in A$. Let $a, b \in A$ be arbitrarily. Then T is quadratic homogeneous.



By using (3.2) and (3.3), we have

$$\begin{aligned} \|a^2T(b)-T(a)b^2\| &= \frac{1}{|4|^n} \|a^2T(2^n b)-4^n T(a)b^2\| \\ &\leq \frac{1}{|4|^n} [\|a^2T(2^n b)-a^2f(2^n b)\| + \|a^2f(2^n b)-f(a)(4^n b^2)\| \\ &\quad + \|4^n f(a)b^2-4^n T(a)b^2\|] \\ &\leq \frac{1}{|4|^{n+1}} \tilde{\varphi}(2^n b, 2^n a, 0, 0) \|a\|^2 + \frac{\psi(a, 2^n b)}{|4|^n} + \frac{1}{|4|} \tilde{\varphi}(a, a, 0, 0) \|b\|^2. \end{aligned}$$

It follows from (3.1) that

$$\|a^2T(b)-T(a)b^2\| = \frac{1}{|4|} \tilde{\varphi}(a, a, 0, 0) \|b\|^2.$$

Finally, we obtain

$$\begin{aligned} \|a^2T(b)-T(a)b^2\| &= \frac{1}{|4|^n} \|4^n a^2T(b)-T(2^n a)b^2\| \\ &\leq \frac{1}{|4|} \tilde{\varphi}_2(2^n a, 2^n a, 0, 0) \|b\|^2 \\ &= \frac{1}{|4|} \sum_{k=n}^{\infty} \frac{\varphi(2^k a, 2^k a, 0, 0)}{|4|^k} \|b\|^2 \\ &\rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

So $a^2T(b) = T(a)b^2$. Hence T is a quadratic multiplier.

The proof for $s = -1$ is similar.

Corollary 3.3. Suppose that $f : A \rightarrow A$ is a mapping for which there exist nonnegative real numbers ε and p with $p \neq 2$ such that

$$\begin{aligned} \|f(\lambda a + \lambda b + \lambda c) + f(\lambda a - \lambda b - \lambda c) - 2\lambda^2 f(a) - 2\lambda^2 f(b) - 2\lambda^2 f(c)\| &\leq \varepsilon(\|a\|^p + \|b\|^p), \\ \|a^2 f(b) - f(a)b^2\| &\leq \varepsilon \|a\|^p \|b\|^p \end{aligned}$$

for all $a, b \in A$ and all $\lambda \in T$. Also, if for each fixed $a \in A$ the mappings $t \rightarrow f(ta)$ from R to A are continuous, then there exists a unique quadratic multiplier T on A satisfying

$$\|f(a) - T(a)\| \leq \frac{2\varepsilon}{|4|-2^p} \|a\|^p$$

for all $a \in A$.

Proof: Putting $\varphi(a, b) = \varepsilon(\|a\|^p + \|b\|^p + \|c\|^p + \|d\|^p)$ and $\psi(a, b) = \varepsilon \|a\|^p \|b\|^p$ in Theorem 3.2, we get the desired results.

4. SUPERSTABILITY OF QUADRATIC DOUBLE CENTRALIZERS

In this section, we prove the superstability of quadratic double centralizers on non-Archimedean Banach algebras which are weakly without order and weakly commutative.

Theorem 4.1. Suppose that A is a non-Archimedean Banach algebra weakly without order and weakly commutative and $s \in \{-1, 1\}$. Let $L, R : A \rightarrow A$ are mappings for which there exists a function $\psi : A \times A \rightarrow [0, \infty)$ such that

$$\begin{aligned} \lim_{n \rightarrow \infty} n^{-2s} \psi(n^s x, y) = 0 &= \lim_{n \rightarrow \infty} n^{-2s} \psi(x, n^s y) \\ \|x^2 L(y) - R(y)^2\| &\leq \psi(x, y) \end{aligned}$$

for all $x, y \in A$. Then (L, R) is a quadratic double centralizer.

Proof: We first show that L is a quadratic homogeneous. To do this, pick $\lambda \in C$ and $x, y \in A$. We have

$$\begin{aligned}
\left\| n^{2s} z^2 (L(\lambda x) - \lambda^2 L(x)) \right\| &= \left\| n^{2s} z^2 L(\lambda x) - \lambda^2 n^{2s} z^2 L(x) \right\| \\
&\leq \left\| n^{2s} z^2 L(\lambda x) - R(n^s z)(\lambda x)^2 \right\| + \left\| \lambda^2 R(n^s z)x^2 - \lambda^2 n^{2s} z^2 L(x) \right\| \\
&\leq \psi(n^s z, \lambda x) + |\lambda|^2 \psi(n^s z, x).
\end{aligned}$$

So

$$\left\| z^2 (L(\lambda x) - \lambda^2 L(x)) \right\| \leq |n|^{-2s} \psi(n^s z, \lambda x) + |\lambda|^2 |n|^{-2s} \psi(n^s z, x).$$

Since A is weakly without order, we conclude that $L(\lambda x) = |\lambda|^2 L(x)$. The quadraticity of L follows from

$$\begin{aligned}
&\left\| z^2 (L(x+y) + L(x-y) - 2L(x) - 2L(y)) \right\| \\
&= |n|^{-2s} \left\| n^{2s} z^2 L(x+y) + n^{2s} z^2 L(x-y) - 2n^{2s} z^2 L(x) - 2n^{2s} z^2 L(y) \right\| \\
&\leq |n|^{-2s} \left[\left\| n^{2s} z^2 L(x+y) - R(n^s z)(x+y)^2 \right\| + \left\| n^{2s} z^2 L(x-y) - R(n^s z)(x-y)^2 \right\| \right] \\
&+ |2| \left[\left\| R(n^s z)x^2 - n^{2s} z^2 L(x) \right\| + \left\| R(n^s z)y^2 - n^{2s} z^2 L(y) \right\| \right] \\
&\leq |n|^{-2s} [\psi(n^s z, x+y) + \psi(n^s z, x-y) + |2| \psi(n^s z, x) + |2| \psi(n^s z, y)]
\end{aligned}$$

for all $x, y \in A$.

Finally, since A is a quadratic commutative non-Archimedean Banach algebra, we have

$$\begin{aligned}
\left\| z^2 (L(xy) - L(x)y^2) \right\| &= |n|^{-2s} \left\| n^{2s} z^2 L(xy) - n^{2s} z^2 L(x)y^2 \right\| \\
&\leq |n|^{-2s} \left[\left\| n^{2s} z^2 L(xy) - R(n^s z)(xy)^2 \right\| \right. \\
&\quad \left. + \left\| R(n^s z)x^2 y^2 - n^{2s} z^2 L(x)y^2 \right\| \right] \\
&\leq |n|^{-2s} [\psi(n^s z, xy) + \psi(n^s z, x)] |y|^2
\end{aligned}$$

for all $x, y \in A$. So $L(xy) = L(x)y^2$. Thus L is a quadratic left centralizer. One can similarly prove that R is a quadratic right centralizer. Since L is quadratic homogeneous, $L(x) = |n|^{-2s} L(n^s x)$ for all $n \in N$ and $x \in A$. Thus

$$\begin{aligned}
\left\| x^2 (L(y) - R(x)y^2) \right\| &= |n|^{-2s} \left\| x^2 L(n^s y) - R(x)(n^s y)^2 \right\| \\
&\leq |n|^{-2s} \psi(x, n^s y)
\end{aligned}$$

and hence by (4.1) we infer that $x^2 L(y) = R(x)y^2$ for all $x, y \in A$. Thus (L, R) is a quadratic centralizer.

Corollary 4.2. Suppose A is a non-Archimedean Banach algebra weakly without order and weakly commutative and $L, R : A \rightarrow A$ are mappings for which there exist a nonnegative real number ε and a real number p either greater than 2 or less than 2, such that

$$\left\| x^2 T(y) - R(x)y^2 \right\| \leq \varepsilon \|x\|^p \|y\|^p$$

for all $x, y \in A$. Then (L, R) is a quadratic double centralizer.

Proof: Using Theorem 4.1 with $\psi(x, y) \leq \varepsilon \|x\|^p \|y\|^p$ we get the desired result.

5. SUPERSTABILITY OF QUADRATIC MULTIPLIERS

In this section, we prove the superstability of quadratic multipliers on non-Archimedean Banach algebras which are weakly without order.

Theorem 5.1. Suppose that A is a Banach algebra with weakly without order and $s \in \{-1, 1\}$. Let $T : A \rightarrow A$ are mappings for which there exists a function $\psi : A \times A \rightarrow [0, \infty)$ such that

$$\begin{aligned}
\lim_{n \rightarrow \infty} |n|^{-2s} \psi(n^s x, y) &= 0 = \lim_{n \rightarrow \infty} |n|^{-2s} \psi(x, n^s y) \\
\left\| x^2 L(y) - R(y)^2 \right\| &\leq \psi(x, y)
\end{aligned}$$



for all $x, y \in A$. Then (L, R) is a quadratic multiplier.

Proof: By the same reasoning as in the proof of Theorem 4.1, putting $L = R = T$, we can show that the mapping T is a quadratic multiplier.

Corollary 5.2. Suppose that A is a weakly without order non-Archimedean Banach algebra and that $T : A \rightarrow A$ is a mapping for which there exist a nonnegative real number ε and a real number p either greater than 2 or less than 2, such that

$$\|x^2 T(y) - T(x) y^2\| \leq \varepsilon \|x\|^p \|y\|^p$$

for all $x, y \in A$. Then T is a quadratic multiplier.

Proof: Using Theorem 5.1 with $\psi(x, y) = \varepsilon \|x\|^p \|y\|^p$, we get the result.

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